Chapter 5

Iterative Methods for Generalized Split Feasibility Problems in Banach Spaces

5.1 Introduction and Formulations

Let X_1 and X_2 be uniformly convex and uniformly smooth real Banach spaces. Let $M : X_1 \rightrightarrows X_1^*$ be a maximal monotone set-valued operator such that $M^{-1}(0) \neq \emptyset$, $S : X_2 \to X_2$ be a nonexpansive mapping such that $\text{Fix}(S) \neq \emptyset$ and $A : X_1 \to X_2$ be a bounded linear operator. We consider the following generalized split feasibility problem in the setting of Banach spaces:

$$\text{Find } x^* \in \text{Fix}(V) \cap M^{-1}(0) \text{ such that } Ax^* \in \text{Fix}(S), \tag{5.1.1}$$

where $V : C \to C$ is a mapping such that $\text{Fix}(V) \neq \emptyset$ and C is a nonempty closed convex subset of X_1. If we consider $V \equiv I$ the identity mapping, then problem (5.1.1) reduces to the following generalized split feasibility problem:

$$\text{Find } x^* \in M^{-1}(0) \text{ such that } Ax^* \in \text{Fix}(S). \tag{5.1.2}$$

We denote by Υ and Φ the solution set of problem (5.1.1) and (5.1.2), respectively, and assume that $\Upsilon \neq \emptyset$ and $\Phi \neq \emptyset$.

When $X_1 = H_1$ is a real Hilbert space and $X_2 = H_2$ is another real Hilbert space, then problems (5.1.1) and (5.1.2) are considered and studied by Takahashi et al. [173].

In this chapter, we propose iterative algorithms for finding the approximate solutions of problems (5.1.1) and (5.1.2) in the setting of Banach spaces. We study the weak convergence of proposed algorithms under some suitable conditions. At the end, we derive some algorithms and convergence results for some problems from nonlinear analysis, namely, split feasibility problems, equilibrium problems, etc.

5.2 Algorithms and Convergence Result

Rest of the chapter, unless otherwise specified, we assume that X_1 and X_2 are uniformly convex and 2-uniformly smooth real Banach spaces having smoothness constant κ satisfying $0 < \kappa \leq \frac{1}{\sqrt{2}}$, $A : X_1 \to X_2$ is a bounded linear operator, and J_1 and J_2 are duality mappings on X_1 and X_2 , respectively, such that J_1 is weakly sequentially continuous.

We propose the following algorithm to solve the problem (5.1.1).

Algorithm 5.2.1. *Choose arbitrary $x_1 \in C$ and $\beta_n \in (0,1)$, compute*

$$x_{n+1} = J_1^{-1}\left(\beta_n J_1(x_n) + (1-\beta_n)J_1 V J_\lambda^M \left(J_1^{-1}\left(J_1(x_n) - \gamma A^* J_2(I-S)Ax_n\right)\right)\right), \; n \in \mathbb{N}, \tag{5.2.1}$$

where $0 < c \leq \beta_n \leq d < 1$, $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$ and $\lambda > 0$.

We also propose the following algorithm to solve the problem (5.1.2).

Algorithm 5.2.2. *Choose arbitrary $x_1 \in X_1$ and compute*

$$x_{n+1} = J_\lambda^M \left(J_1^{-1}\left(J_1(x_n) - \gamma A^* J_2(I-S)Ax_n\right)\right), \quad n \in \mathbb{N}, \tag{5.2.2}$$

where $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$ and $\lambda > 0$.

We first establish the weak convergence of the sequence generated by Algorithm 5.2.2 to a solution of problem (5.1.2).

Theorem 5.2.1. *Let $M : X_1 \rightrightarrows X_1^*$ be a maximal monotone operator such that $M^{-1}(0) \neq \emptyset$. Let J_λ^M be a relative resolvent operator of M for $\lambda > 0$, and $S : X_2 \to X_2$ be a given nonexpansive mapping such that* $\mathrm{Fix}(S) \neq \emptyset$. *Then the sequence $\{x_n\}$ generated by Algorithm 5.2.2 converges weakly to an element $z \in \Phi$.*

Proof. Let $p \in \Phi$. Then, $J_\lambda^M p = p$ and $S(Ap) = Ap$. Let

$$y_n = J_1^{-1}(J_1(x_n) - \gamma A^* J_2(I-S)Ax_n).$$

In view of equation (2.4.2) and Lemma 2.4.6, we have

$$\begin{aligned}
\phi(p, y_n) &= \phi(p, J_1^{-1}(J_1(x_n) - \gamma A^* J_2(I-S)Ax_n)) \\
&= \|p\|^2 - 2\langle p, J_1(x_n) - \gamma A^* J_2(I-S)Ax_n\rangle \\
&\quad + \|J_1(x_n) - \gamma A^* J_2(I-S)Ax_n\|^2 \\
&= \|p\|^2 - 2\langle p, J_1(x_n) - \gamma A^* J_2(I-S)Ax_n\rangle \\
&\quad + \|x_n - \gamma J_1^{-1} A^* J_2(I-S)Ax_n\|^2 \\
&\leq \|p\|^2 - 2\langle p, J_1(x_n)\rangle + 2\gamma\langle Ap, J_2(I-S)Ax_n\rangle \\
&\quad + \|\gamma J_1^{-1} A^* J_2(I-S)Ax_n\|^2 - 2\langle x_n, \gamma A^* J_2(I-S)Ax_n\rangle + 2\|\kappa x_n\|^2 \\
&\leq \|p\|^2 - 2\langle p, J_1(x_n)\rangle + 2\gamma\langle Ap, J_2(I-S)Ax_n\rangle \\
&\quad + \gamma^2\|A\|^2\|(I-S)Ax_n\|^2 - 2\gamma\langle Ax_n, J_2(I-S)Ax_n\rangle + \|x_n\|^2 \\
&\leq \phi(p, x_n) + \gamma^2\|A\|^2\|(I-S)Ax_n\|^2 \\
&\quad + 2\gamma\langle Ap - Ax_n, J_2(I-S)Ax_n\rangle. \qquad (5.2.3)
\end{aligned}$$

From nonexpansiveness of S and Lemma 2.4.6, we have

$$\begin{aligned}
&\langle Ap - Ax_n, J_2(I-S)Ax_n\rangle \\
&= \langle Ap - S(Ax_n), J_2(I-S)Ax_n\rangle - \|(I-S)Ax_n\|^2 \\
&\leq \frac{1}{2}\|(I-S)Ax_n\|^2 + \frac{1}{2}\|Ap - S(Ax_n)\|^2 \\
&\quad - \frac{1}{2}\|Ax_n - Ap\|^2 - \|(I-S)Ax_n\|^2 \\
&= -\frac{1}{2}\|(I-S)Ax_n\|^2 + \frac{1}{2}\|S(Ax_n) - Ap\|^2 - \frac{1}{2}\|Ax_n - Ap\|^2 \\
&= -\frac{1}{2}\|(I-S)Ax_n\|^2, \qquad (5.2.4)
\end{aligned}$$

that is,

$$2\gamma\langle Ap - Ax_n, J_2(I-S)Ax_n\rangle \leq -\gamma\|(I-S)Ax_n\|^2. \qquad (5.2.5)$$

Notice that $\gamma \in (0, 1/\|A\|^2)$ and making use of inequality (5.2.5) in (5.2.3), we have

$$\begin{aligned}
\phi(p, y_n) &\leq \phi(p, x_n) + \gamma^2\|A\|^2\|(I-S)Ax_n\|^2 - \gamma\|(I-S)Ax_n\|^2 \\
&= \phi(p, x_n) - \gamma(1 - \gamma\|A\|^2)\|(I-S)Ax_n\|^2. \qquad (5.2.6)
\end{aligned}$$

In view of relative nonexpansiveness of J_λ^M and (5.2.6), we have

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, J_\lambda^M y_n) \leq \phi(p, y_n) \\
&\leq \phi(p, x_n) - \gamma(1 - \gamma\|A\|^2)\|(I-S)Ax_n\|^2 \qquad (5.2.7) \\
&\leq \phi(p, x_n). \qquad (5.2.8)
\end{aligned}$$

Hence, from (5.2.8), the sequence $\phi(p, x_n)$ is a decreasing sequence, and from (2.4.3), it is lower bounded by 0. Consequently, it converges to some finite limit, so $\lim_{n\to\infty} \phi(p, x_n)$ exists and, in particular $\phi(p, x_n)$ is bounded. Then, by (2.4.3), $\{x_n\}$ is also bounded. Again by the fact that $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$ and by passing to the limit in (5.2.7), we obtain

$$\gamma(1 - \gamma\|A\|^2) \lim_{n\to\infty} \left(\|(I - S)Ax_n\|^2\right) \le \lim_{n\to\infty} \left(\phi(p, x_n) - \phi(p, x_{n+1})\right),$$

so, we have

$$\lim_{n\to\infty} \|(I - S)Ax_n\| = 0. \tag{5.2.9}$$

Now, consider

$$\begin{aligned}
\phi(x_n, y_n) &= \phi(x_n, J_1^{-1}(J_1(x_n) - \gamma A^* J_2(I - S)Ax_n) \\
&= \|x_n\|^2 - 2\langle x_n, J_1(x_n) - \gamma A^* J_2(I - S)Ax_n\rangle \\
&\quad + \|J_1(x_n) - \gamma A^* J_2(I - S)Ax_n\|^2 \\
&\le \|x_n\|^2 - 2\langle x_n, J_1(x_n)\rangle + 2\gamma\langle Ax_n, J_2(I - S)Ax_n\rangle \\
&\quad + \|x_n\|^2 - 2\gamma\langle Ax_n, J_2(I - S)Ax_n\rangle + \gamma^2\|A\|^2\|(I - S)Ax_n\|^2 \\
&\le \phi(x_n, x_n) + \gamma^2\|A\|^2\|(I - S)Ax_n\|^2.
\end{aligned} \tag{5.2.10}$$

In view of (2.4.4), (5.2.9) and (5.2.10), we have

$$\lim_{n\to\infty} \phi(x_n, y_n) = 0. \tag{5.2.11}$$

By Lemma 2.4.8, we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{5.2.12}$$

In view of relative nonexpansiveness of J_λ^M and (5.2.2), we have

$$\begin{aligned}
0 &\le \phi(p, y_n) - \phi(p, J_\lambda^M y_n) \\
&\le \phi(p, x_n) - \phi(p, x_{n+1}) \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{5.2.13}$$

From (5.2.6), we obtain the boundedness of $\phi(p, y_n)$. Again in view of (2.4.3), we have the boundedness of $\{y_n\}$. Thus, by strongly relative nonexpansiveness of J_λ^M, and from (5.2.13), we have

$$\lim_{n\to\infty} \phi(J_\lambda^M y_n, y_n) = 0, \tag{5.2.14}$$

so by Lemma 2.4.8, we have

$$\lim_{n\to\infty} \|J_\lambda^M y_n - y_n\| = 0. \tag{5.2.15}$$

Since X_1 is uniformly convex, it is reflexive [61, Milman-Pettis'theorem, Theorem 1.17]. Therefore, X_1 is reflexive and by the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in X_1$ (see [84, property 1.8]).

Now we show that $x_n \rightharpoonup z$. In order to show this, we have to show that every subsequence of x_n converges weakly to z. Assume to the contrary that there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup y \in X_1$ where $z \neq y$. Since J_1 is weakly sequentially continuous, so $J_1 x_{n_i} \overset{*}{\rightharpoonup} J_1 z$ and $J_1 x_{n_j} \overset{*}{\rightharpoonup} J_1 y$,

$$\begin{aligned}\langle y - z, J_1 z\rangle &= \lim_{i\to\infty} \langle y - z, J_1 x_{n_i}\rangle \\ &= \lim_{n\to\infty} \langle y - z, J_1 x_n\rangle \\ &= \lim_{j\to\infty} \langle y - z, J_1 x_{n_j}\rangle = \langle y - z, J_1 y\rangle.\end{aligned}$$

Thus, we obtain $\langle z - y, J_1 z - J_1 y\rangle = 0$. Since X_1 is uniformly convex, by [7, Theorem 2.14] it is strictly convex. Then by Lemma 2.4.4, we have $z = y$. Thus we have shown that every subsequence of $\{x_n\}$ converges weakly to z. This implies that $x_n \rightharpoonup z$. Since A is bounded linear operator, so $Ax_n \rightharpoonup Az$. Thus by (5.2.9) and using the fact that S is demiclosed at 0, we have $S(Az) = Az$. From (5.2.12), we have $y_n \rightharpoonup z$ as follow: For all $f \in X_1^*$,

$$\begin{aligned}\|f(y_n) - f(z)\| &= \|f(y_n) - f(x_n) + f(x_n) - f(z)\| \\ &\leq \|f(y_n) - f(x_n)\| + \|f(x_n) - f(z)\| \\ &\leq \|f\|\|y_n - x_n\| + \|f(x_n) - f(z)\| \to 0 \text{ as } n \to \infty. \qquad (5.2.16)\end{aligned}$$

Thus $y_n \rightharpoonup z$. Notice (5.2.15) and by the relative nonexpansiveness of J_λ^M, we have $J_\lambda^M z = z$. Thus we have shown that $x_n \rightharpoonup z$ such that $z \in M^{-1}(0)$ and $Az \in \text{Fix}(S)$. This completes the proof. □

Theorem 5.2.2. *Let $M : X_1 \rightrightarrows X_1^*$ be a maximal monotone operator such that $D(M) \subseteq C$ and $M^{-1}(0) \neq \emptyset$. Let J_λ^M be the relative resolvent operator of M for $\lambda > 0$, and $S : X_2 \to X_2$ be a given nonexpansive mapping such that* $\text{Fix}(S) \neq \emptyset$. *Let $V : C \to C$ be a generalized nonspreading mapping such that* $\text{Fix}(V) \neq \emptyset$. *Then the sequence $\{x_n\}$ generated by Algorithm 5.2.1 converges weakly to an element $z \in \Upsilon$, which is identified as the strong limit of the projection of $\{x_n\}$ onto Υ, that is, $z = \lim_{n\to\infty} P_\Upsilon x_n$.*

Proof. Let $y_n = J_1^{-1}(J_1(x_n) - \gamma A^* J_2(I - S)Ax_n))$ and $z_n = J_\lambda^M y_n$. Then, (5.2.1), takes the following form

$$x_{n+1} = J_1^{-1}(\beta_n J_1(x_n) + (1 - \beta_n) J_1 V z_n).$$

Let $p \in \Upsilon$. Then, $J_\lambda^M p = p$, $Vp = p$ and $S(Ap) = Ap$. It follow that

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, J_1^{-1}(\beta_n J_1(x_n) + (1-\beta_n)J_1(Vz_n))) \\
&= \|p\|^2 - 2\langle p, \beta_n J_1 x_n + (1-\beta_n)J_1(Vz_n)\rangle \\
&\quad + \|\beta_n J_1(x_n) + (1-\beta_n)J_1(V(z_n))\|^2 \\
&\le \|p\|^2 - 2\beta_n\langle p, J_1 x_n\rangle - 2(1-\beta_n)\langle p, J_1(Vz_n)\rangle \\
&\quad + \beta_n\|x_n\|^2 + (1-\beta_n)\|Vz_n\|^2 \\
&\le \beta_n(\|p\|^2 - 2\langle p, J_1 x_n\rangle + \|x_n\|^2) \\
&\quad + (1-\beta_n)(\|p\|^2 - 2\langle p, J_1 V z_n\rangle + \|Vz_n\|^2) \\
&= \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, Vz_n) \\
&\le \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, z_n) \\
&= \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, J_\lambda^M y_n) \\
&\le \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, y_n) \\
&\le \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, x_n) \\
&\quad - (1-\beta_n)^2\gamma(1-\gamma\|A\|^2)\|(I-S)Ax_n\|^2 \\
&= \phi(p, x_n) - (1-\beta_n)^2\gamma(1-\gamma\|A\|^2)\|(I-S)Ax_n\|^2 \qquad (5.2.17) \\
&\le \phi(p, x_n). \qquad (5.2.18)
\end{aligned}$$

Hence, from (5.2.18), the sequence $\phi(p, x_n)$ is decreasing, and from (2.4.3) it is lower bounded by 0. Consequently, it converges to some finite limit, so $\lim_{n\to\infty}\phi(p, x_n)$ exists and, in particular $\phi(p, x_n)$ is bounded. Then by (2.4.3), $\{x_n\}$ is also bounded. Again by the fact that $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$, $0 < c \le \beta_n \le d < 1$, and by passing to the limit in (5.2.17), we obtain

$$(1-\beta_n)^2\gamma(1-\gamma\|A\|^2)\lim_{n\to\infty}\left(\|(I-S)Ax_n\|^2\right) \le \lim_{n\to\infty}\left(\phi(p, x_n) - \phi(p, x_{n+1})\right),$$

so, we have

$$\lim_{n\to\infty}\|(I-S)Ax_n\| = 0. \qquad (5.2.19)$$

Since J_λ^M and V are relative nonexpansive, from (5.2.6), we have

$$\phi(p, Vz_n) \le \phi(p, z_n) = \phi(p, J_\lambda^M y_n) \le \phi(p, y_n) \le \phi(p, x_n).$$

Hence boundedness of $\phi(p, x_n)$ implies the boundedness of $\phi(p, J_\lambda^M y_n)$ and $\phi(p, Vz_n)$. Thus from (2.4.3), $\{J_\lambda^M y_n\}$ and $\{Vz_n\}$ are bounded. Put

$$r = \sup_{n\in\mathbb{N}\cup\{0\}}\{\|J_1(x_n)\|, \|J_1(J_\lambda^M(y_n))\|, \|J_1(Vz_n)\|\}.$$

Since X_1 is uniformly smooth Banach space, X_1^* is a uniformly convex Banach space [61]. So by Lemma 2.4.3, we have

$$\begin{aligned}
&\phi(p, x_{n+1})\\
&= \phi(p, J_1^{-1}(\beta_n J_1(x_n) + (1-\beta_n)J_1(Vz_n)))\\
&= \|p\|^2 - 2\langle p, \beta_n J_1 x_n + (1-\beta_n)J_1(Vz_n)\rangle + \|\beta_n J_1(x_n) + (1-\beta_n)J_1(V(z_n))\|^2\\
&\leq \|p\|^2 - 2\beta_n\langle p, J_1 x_n\rangle - 2(1-\beta_n)\langle p, J_1(Vz_n)\rangle + \beta_n\|x_n\|^2\\
&\quad + (1-\beta_n)\|Vz_n\|^2 - \beta_n(1-\beta_n)g(\|J_1(x_n) - J_1 Vz_n\|)\\
&\leq \beta_n(\|p\|^2 - 2\langle p, J_1 x_n\rangle + \|x_n\|^2) + (1-\beta_n)(\|p\|^2 - 2\langle p, J_1 Vz_n\rangle + \|Vz_n\|^2)\\
&\quad - \beta_n(1-\beta_n)g(\|J_1(x_n) - J_1 Vz_n\|)\\
&= \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, Vz_n) - \beta_n(1-\beta_n)g(\|J_1(x_n) - J_1(Vz_n)\|)\\
&\leq \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, z_n) - \beta_n(1-\beta_n)g(\|J_1(x_n) - J_1 Vz_n\|)\\
&= \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, J_\lambda^M y_n) - \beta_n(1-\beta_n)g(\|J_1(x_n) - J_1 Vz_n\|)\\
&\leq \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, y_n) - \beta_n(1-\beta_n)g(\|J_1(x_n) - J_1 Vz_n\|)\\
&\leq \beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, x_n) - (1-\beta_n)^2\gamma(1-\gamma\|A\|^2)\|(I-S)Ax_n\|^2\\
&\quad - \beta_n(1-\beta_n)g(\|J_1(x_n) - J_1 Vz_n\|)\\
&= \phi(p, x_n) - (1-\beta_n)^2\gamma(1-\gamma\|A\|^2)\|(I-S)Ax_n\|^2\\
&\quad - \beta_n(1-\beta_n)g(\|J_1(x_n) - J_1 Vz_n\|),
\end{aligned}$$

and

$$(1-\beta_n)^2\gamma(1-\gamma\|A\|^2)\|(I-S)Ax_n\|^2 + \beta_n(1-\beta_n)^2 g(\|J_1(x_n) - J_1 Vzn)\|) \\ \leq \phi(p, x_n) - \phi(p, x_{n+1}), \quad (5.2.20)$$

using the fact $0 < c \leq \beta_n \leq d < 1$, $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$ and by (5.2.19). Passing to the limit in (5.2.20), we have

$$\lim_{n\to\infty} g(\|J_1(x_n) - J_1(Vz_n)\|) = 0. \quad (5.2.21)$$

Since $g : [0,\infty) \to [0,\infty)$ is a continuous, strictly increasing, and convex function with $g(0) = 0$, therefore

$$\lim_{n\to\infty} \|J_1(x_n) - J_1(Vz_n)\| = 0. \quad (5.2.22)$$

Since X_1 is uniformly convex and uniformly smooth, it is a smooth, strictly convex and reflexive Banach space. Then J_1 is a single-valued bijection. In this case, the duality mapping J_1^* from X_1^* onto $X_1^{**} = X_1$ coincides with the inverse of the duality mapping J_1 from X_1 onto X_1^*, that is, $J_1^* = J_1^{-1}$. Since X_1 is uniformly convex, X_1^*

is uniformly smooth (see [61]). Therefore, by uniformly smoothness of X_1^*, J_1^{-1} is uniformly norm-to-norm continuous on bounded sets (see [61, 167]). Thus, we obtain

$$\lim_{n\to\infty}\|x_n - Vz_n\| = \lim_{n\to\infty}\left\|J_1^{-1}(J_1(x_n)) - J_1^{-1}(J_1(Vz_n))\right\| = 0. \tag{5.2.23}$$

Also, we have assumed that $y_n = J_1^{-1}(J_1(x_n) - \gamma A^* J_2(I-S)Ax_n))$, so we have

$$\begin{aligned}
\phi(x_n, y_n) &= \phi(x_n, J_1^{-1}(J_1(x_n) - \gamma A^* J_2(I-S)Ax_n) \\
&= \|x_n\|^2 - 2\langle x_n, J_1(x_n) - \gamma A^* J_2(I-S)Ax_n\rangle \\
&\quad + \|J_1(x_n) - \gamma A^* J_2(I-S)Ax_n\|^2 \\
&\leq \|x_n\|^2 - 2\langle x_n, J_1(x_n)\rangle + 2\gamma\langle Ax_n, J_2(I-S)Ax_n\rangle \\
&\quad + \|x_n\|^2 - 2\gamma\langle Ax_n, J_2(I-S)Ax_n\rangle + \gamma^2\|A\|^2\|(I-S)Ax_n\|^2 \\
&\leq \phi(x_n, x_n) + \gamma^2\|A\|^2\|(I-S)Ax_n\|^2.
\end{aligned} \tag{5.2.24}$$

In view of (2.4.4), (5.2.19) and (5.2.24), we have

$$\lim_{n\to\infty}\phi(x_n, y_n) = 0. \tag{5.2.25}$$

Thus in view of (5.2.25), boundedness of $\{x_n\}$ and Lemma 2.4.2, we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{5.2.26}$$

Consequently, we have

$$\lim_{n\to\infty}\|y_n - Vz_n\| = 0, \tag{5.2.27}$$

that is,

$$\lim_{n\to\infty}\|y_n - VJ_\lambda^M y_n\| = 0. \tag{5.2.28}$$

Since X_1 is uniformly convex, it is reflexive. By the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in X_1$, also J_1 is weakly sequentially continuous, therefore $x_n \rightharpoonup z$, and so $Ax_n \rightharpoonup Az$. Thus from (5.2.19) and knowing the fact that S is demiclosed at 0, we have $S(Az) = Az$. In view of (5.2.26), we have $y_n \rightharpoonup z$. Further notice that $V : C \to C$ is relative nonexpansive mapping and $J_\lambda^M : X_1 \to D(M)$ is strongly nonexpansive such that $D(M) \subseteq C$. Hence in view of [14, Lemma 3.2 and Lemma 3.3] (see also [13, Lemma 2.4]) and [13, Remark 2.5], we have $VJ_\lambda^M : X_1 \to C$ is relative nonexpansive mapping such that $\mathrm{Fix}(VJ_\lambda^M) = \mathrm{Fix}(V) \cap \mathrm{Fix}(J_\lambda^M)$. Since, $y_n \rightharpoonup z$, from (5.2.28) and by relative nonexpansiveness of VJ_λ^M, we have $z \in \mathrm{Fix}(VJ_\lambda^M)$. Thus, we have $J_\lambda^M z = z$ and $Vz = z$. Hence we have shown that $z \in \Upsilon$, where $\Upsilon = \mathrm{Fix}(V) \cap M^{-1}(0) \cap A^{-1}\mathrm{Fix}(S)$. In view of Lemma 2.4.12, $\mathrm{Fix}(V)$ is closed and convex. M is maximal monotone set-valued mapping, so $M^{-1}(0)$ is closed and convex. Since S is nonexpansive, so $\mathrm{Fix}(S)$ is closed and convex. Since A is linear and continuous, therefore $A^{-1}\text{Fix}(S)$ is closed and convex. Thus Υ is closed convex subspace of X_1. Now we have to show that $z = \lim_{n\to\infty} \Pi_\Upsilon x_n$. Let $u_n = \Pi_\Upsilon x_n$, for each $n \in \mathbb{N}$. Then $u_n \in \Upsilon$ and $u_{n+1} = \Pi_\Upsilon x_{n+1}$. Since inequality (5.2.18) holds for each $p \in \Upsilon$, we have

$$\phi(u_n, x_{n+1}) \leq \phi(u_n, x_n). \tag{5.2.29}$$

From Lemma 2.4.7 (ii), we have

$$\phi(u_n, \Pi_\Upsilon x_{n+1}) + \phi(\Pi_\Upsilon x_{n+1}, x_{n+1}) \leq \phi(u_n, x_{n+1}),$$

which implies that

$$\phi(\Pi_\Upsilon x_{n+1}, x_{n+1}) \leq \phi(u_n, x_{n+1}) - \phi(u_n, \Pi_\Upsilon x_{n+1}). \tag{5.2.30}$$

Since $\phi(u_n, \Pi_\Upsilon x_{n+1}) \geq 0$, we have

$$\phi(u_{n+1}, x_{n+1}) \leq \phi(u_n, x_{n+1})$$

and hence from (5.2.29), we have

$$\phi(u_{n+1}, x_{n+1}) \leq \phi(u_n, x_n).$$

So, $\phi(u_n, x_n)$ is a decreasing sequence. Since $\phi(u_n, x_n)$ is bounded below by 0, it is convergent. Also, in view of (5.2.29) and (5.2.30), we have

$$\begin{aligned}\phi(u_n, u_{n+1}) &\leq \phi(u_n, x_{n+1}) - \phi(u_{n+1}, x_{n+1})\\ &\leq \phi(u_n, x_n) - \phi(u_{n+1}, x_{n+1}).\end{aligned}$$

By induction, we have

$$\phi(u_n, u_{n+m}) \leq \phi(u_n, x_n) - \phi(u_{n+m}, x_{n+m}) \quad \text{for each } m \in \mathbb{N}.$$

Using Lemma 2.4.9, we have, for $m,\ n$ with $n > m$,

$$g(\|u_m - u_n\|) \leq \phi(u_m, u_n) \leq \phi(u_m, x_m) - \phi(u_n, x_n),$$

and hence

$$\lim_{n\to\infty} g(\|u_n - u_m\|) = 0.$$

Then the properties of g yields that

$$\lim_{n\to\infty} \|u_n - u_m\| = 0,$$

this implies that $\{u_n\}$ is a Cauchy sequence in Υ. Since X_1 is complete and Υ is closed, therefore Υ is complete. Hence $\{u_n\}$ converges strongly to some point $u \in \Upsilon$. Now we will show that $u = z$. Since $u_n = \Pi_\Upsilon x_n$, so from Lemma 2.4.5 (i), we have

$$\langle u_n - z, J_1 x_n - J_1 u_n \rangle \geq 0, \quad \text{for each } z \in \Upsilon. \tag{5.2.31}$$

Also, we know that $\{u_n\}$ converges strongly to some $u \in \Upsilon$ and J_1 is weakly sequentially continuous. Letting $n \to \infty$ in (5.2.31), we have

$$\langle u - z, J_1 z - J_1 u \rangle \geq 0,$$

that is,

$$\langle u - z, J_1 u - J_1 z \rangle \leq 0.$$

Also monotonicity of J implies that

$$\langle u - z, J_1 u - J_1 z \rangle \geq 0.$$

Thus,

$$\langle u - z, J_1 u - J_1 z \rangle = 0.$$

By using the strict convexity of X_1 and Lemma 2.4.4, we obtain that $u = z$. Therefore, $\{x_n\}$ converges weakly to $z = \lim\limits_{n\to\infty} \Pi_\Upsilon x_n$. This completes the proof. □

When $V \equiv I$ the identity operator in Theorem 5.2.2, we have the following Corollary.

Corollary 5.2.3. *Let $M : X_1 \rightrightarrows X_1^*$ be a maximal monotone operator such that $M^{-1}(0) \neq \emptyset$, J_λ^M be the relative resolvent operator of M for $\lambda > 0$, and $S : X_2 \to X_2$ be a given nonexpansive mapping such that* Fix$(S) \neq \emptyset$. *Then, the sequence $\{x_n\}$ generated by the following algorithm, for any $x_1 \in X_1$*

$$x_{n+1} = J_1^{-1}\left(\beta_n J_1(x_n) + (1-\beta_n) J_1 J_\lambda^M\left(J_1^{-1}\left(J_1(x_n) - \gamma A^* J_2 (I - S) A x_n\right)\right)\right), \quad n \in \mathbb{N}, \tag{5.2.32}$$

$0 < c \leq \beta_n \leq d < 1$ and $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$ converges weakly to an element $z \in \Phi$, which is identified as the strong limit of the projection of $\{x_n\}$ onto Φ that is, $z = \lim\limits_{n\to\infty} P_\Phi x_n$.

5.3 Applications

Let K be a nonempty closed convex subset of a smooth strictly convex and reflexive Banach space X. Let i_K be the indicator function for $K \subseteq X$, that is, $i_K(x) = 0$ if $x \in K$ and ∞ otherwise. Then $i_K : X \to (-\infty, \infty]$ is a proper lower semicontinuous convex function. Rockafellar's maximal monotonicity theorem [156, 157] ensures that the subdifferential $\partial i_K \subset X \times X^*$ of i_K is maximal monotone. In this case, it is known that ∂i_K is reduced to the normality operator N_K for K, that is,

$$N_K(x) = \{x^* \in X^* : \langle y - x, x^* \rangle \le 0 \text{ for all } y \in K\}.$$

Indeed, for any $x \in K$,

$$\begin{aligned} \partial i_K(x) &\Leftrightarrow \{x^* \in X^* : i_K(x) + \langle y - x, x^* \rangle \le i_K(y) \text{ for all } y \in X\} \\ &\Leftrightarrow \{x^* \in X^* : \langle y - x, x^* \rangle \le 0 \text{ for all } y \in K\} \\ &\Leftrightarrow N_K(x). \end{aligned}$$

We also know that Π_K is the resolvent of N_K. In fact, $\Pi_K = (J + 2^{-1}N_K)^{-1}J$ (see [110]).

Remark 5.3.1. If P_K is a metric projection of X onto K, then from [14], [168] and [169], we have

$$\langle P_K x - P_K y, J(x - P_K x) - J(y - P_K y) \rangle \ge 0, \quad \text{for all } x, y \in K.$$

We also have that if x_n is a sequence in X such that $x_n \rightharpoonup p$ and $\|x_n - P_K x_n\| \to 0$, then $p = P_K p$, that is, $p \in K$. In fact, assume that $x_n \rightharpoonup p$ and $\|x_n - P_K x_n\| \to 0$. It is clear that $P_K x_n \rightharpoonup p$ and $\|J(x_n - P_K x_n)\| = \|x_n - P_K x_n\| \to 0$. Since P_K is the metric projection of X onto K, we have

$$\langle P_K x_n - P_K p, J(x_n - P_K x_n) - J(p - P_K p) \rangle \ge 0, \quad \text{for all } x, y \in K.$$

Then,

$$-\|p - P_K p\|^2 = \langle p - P_K p, -J(p - P_K p) \rangle \ge 0, \quad \text{for all } x, y \in K.$$

and hence $p = P_K p$.

5.3.1 Split Feasibility Problem

Let C be a nonempty closed convex subset of X_1. Consider $M = \partial i_C$ and $S = P_Q$, where P_Q is metric projection onto a nonempty closed convex subset Q of X_2. Then, we have $J_\lambda^M = \Pi_C$ and $\text{Fix}(S) = Q$. The generalized split feasibility problem (5.1.2) reduces to the following split feasibility problem in the setting of Banach spaces:

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q. \tag{5.3.1}$$

Further, in this setting, algorithm (5.2.2) reduces to the following algorithm: For any $x_1 = x \in X_1$,

$$x_{n+1} = \Pi_C \left(J_1^{-1} \left(J_1(x_n) - \gamma A^* J_2(I - P_Q)Ax_n \right) \right), \quad n \in \mathbb{N}. \tag{5.3.2}$$

Let Φ_1 denote the solution set of (5.3.1), that is, $\Phi_1 = \{x \in C : Ax \in Q\}$. As a consequence of Theorem 5.2.1, we have the following weak convergence result for algorithm (5.3.2).

Theorem 5.3.1. *Let C and Q be nonempty closed convex subsets of X_1 and X_2, respectively. $A : X_1 \to X_2$ be a bounded linear operator, and $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$. Then the sequence $\{x_n\}$ generated by (5.3.2) converges weakly to an element $z \in \Phi_1$, provided that $\Phi_1 \neq \emptyset$.*

Proof. Using Lemma 2.4.5, Remark 5.3.1 and from the proof of Theorem 5.2.1, we obtain the desired result. □

As a consequence of Theorem 5.2.2, we have the following result.

Theorem 5.3.2. *Let C be a nonempty closed convex subset of X_1, $A : X_1 \to X_2$ be a bounded linear operator, and $S : X_2 \to X_2$ be a given nonexpansive mapping such that $\mathrm{Fix}(S) \neq \emptyset$. Let $V : C \to C$ be a nonspreading mapping such that $\mathrm{Fix}(V) \neq \emptyset$. For any $x_1 \in C$, define*

$$x_{n+1} = J_1^{-1} \left(\beta_n J_1(x_n) + (1 - \beta_n) J_1 V \Pi_C \left(J_1^{-1}(J_1(x_n) - \gamma A^* J_2(I - S)Ax_n) \right) \right), \ n \in \mathbb{N}, \tag{5.3.3}$$

where $\beta_n \in (0, 1)$ such that $0 < c \leq \beta_n \leq d < 1$ and $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$. Then the sequence $\{x_n\}$ generated by (5.3.3) converges to an element $z \in \Phi_2$, where $\Phi_2 = \{z \in \mathrm{Fix}(V) : Az \in \mathrm{Fix}(S)\}$.

Proof. A generalized nonspreading mapping $V : C \to C$ is nonspreading. Also set of fixed points of nonspreading mappings V are closed and convex [111]. Furthermore, putting $M = \partial i_C$ in Theorem 5.2.2, we have that $J_\lambda^M = \Pi_C$ for all $\lambda > 0$. Since Π_C is strongly relative nonexpansive [110, Lemma 2.4 and Theorem 5.2], thus the desired result follow from proof of Theorem 5.2.2. □

5.3.2 Equilibrium Problem

Now, we apply our results to the equilibrium problems.

The following result is a special case of a result by Aoyama et al. [12, Theorem 3.5].

Lemma 5.3.3. *Let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying conditions (A1)–(A4) given in section 2.6. Let $A_F : X \rightrightarrows X^*$ be a set-valued mapping defined by*

$$A_F(x) = \begin{cases} x^* \in X^* : F(x,y) \geq \langle y - x, x^* \rangle \text{ for all } y \in K, & \text{if } x \in K, \\ \emptyset, & \text{if } x \notin K. \end{cases} \tag{5.3.4}$$

Then A_F is a maximal monotone operator with $D(A_F) \subseteq K$ and $\mathrm{EP}(F) = A_F^{-1}(0)$. Furthermore, for $r > 0$, the resolvent T_r^F of F coincides with the resolvent $(J + rA_F)^{-1}J$ of A_F, that is,

$$T_r^F(x) = (J + rA_F)^{-1}J(x). \tag{5.3.5}$$

As a consequence of Theorem 5.2.2, we have the following results.

Theorem 5.3.4. *Let C be a nonempty closed convex subset of X_1, $F : C \times C \to \mathbb{R}$ satisfy the conditions (A1)–(A4) given in section 2.6, and T_λ^F denote the resolvent of A_F (as defined in (5.3.5)) of index $\lambda > 0$. Let $A : X_1 \to X_2$ be a bounded linear operator and $S : X_2 \to X_2$ be a given nonexpansive mapping such that* $\mathrm{Fix}(S) \neq \emptyset$. *For any $x_1 \in X_1$, define*

$$x_{n+1} = J_1^{-1}\left(\beta_n J_1(x_n) + (1-\beta_n)J_1 T_\lambda^F \left(J_1^{-1}(J_1(x_n) - \gamma A^* J_2(I - S)Ax_n)\right)\right), \ n \in \mathbb{N}, \tag{5.3.6}$$

where $\beta_n \in (0,1)$ such that $0 < c \leq \beta_n \leq d < 1$ and $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$. Then the sequence $\{x_n\}$ generated by (5.3.6) converges to an element $z \in \Xi$, where $\Xi = \{z \in \mathrm{EP}(F) : Az \in \mathrm{Fix}(S)\}$.

Proof. Putting $V = I$ and $M = A_F$ in Theorem 5.2.2, we have that $J_\lambda^M = T_\lambda^F$ for all $\lambda > 0$. Since T_λ^F is firmly nonexpansive type, so by [110, Theorem 5.2], it is strongly relative nonexpansive. Thus the result follow from proof of Theorem 5.2.2. □

Theorem 5.3.5. *Let C be a nonempty closed convex subset of X_1. Let $F : C \times C \to \mathbb{R}$ satisfy the conditions (A1)–(A4) given in section 2.6, and T_λ^F denote the resolvent of A_F (as defined in (5.3.5)) of index $\lambda > 0$. Let $A : X_1 \to X_2$ be a bounded linear operator, $S : X_2 \to X_2$ be a given nonexpansive mapping such that* $\mathrm{Fix}(S) \neq \emptyset$, *and $V : C \to C$ be a generalized nonspreading mapping such that* $\mathrm{Fix}(V) \neq \emptyset$. *For any $x_1 \in C$, define*

$$x_{n+1} = J_1^{-1}\left(\beta_n J_1(x_n) + (1-\beta_n)J_1 V T_\lambda^F \left(J_1^{-1}(J_1(x_n) - \gamma A^* J_2(I - S)Ax_n)\right)\right), \ n \in \mathbb{N}, \tag{5.3.7}$$

where $\beta_n \in (0,1)$ such that $0 < c \leq \beta_n \leq d < 1$ and $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$. Then the sequence $\{x_n\}$ generated by (5.3.7) converges to an element $z \in \{z \in \mathrm{EP}(F) \cap \mathrm{Fix}(V) : \ Az \in \mathrm{Fix}(S)\}$.

Proof. Putting $M = A_F$ in Theorem 5.2.2, we have that $J_\lambda^M = T_\lambda^F$ for all $\lambda > 0$. Thus we have the desired result of Theorem 5.3.5. □