Chapter 4

Implicit and Explicit Algorithms for Split Common Fixed Point Problems

4.1 Introduction

In the recent past, several split type feasibility problems from nonlinear analysis have been studied because of their applications in science, engineering, medical sciences, etc. One of the split type problems is the split common fixed point problem (in short, SCFPP) which is to find a fixed point of an operator such that its image under the bounded linear operator is a fixed point of another operator, that is,

$$\text{find } x^* \in \mathrm{Fix}(T) \text{ such that } Ax^* \in \mathrm{Fix}(S), \tag{4.1.1}$$

where Fix(T) and Fix(S) denote the set of fixed points of the operators $T : H_1 \to H_1$ and $S : H_2 \to H_2$, respectively, such that $\mathrm{Fix}(T) \neq \emptyset$ and $\mathrm{Fix}(S) \neq \emptyset$ and $A : H_1 \to H_2$ is a bounded linear operator. We denote by Ω the set of solutions of SCFPP (4.1.1) and assume that $\Omega \neq \emptyset$. The SCFPP (4.1.1) was introduced by Censor and Segal [49]. They proposed the following parallel algorithm for solving problem (4.1.1) for a class of directed operators in finite dimensional spaces.

Algorithm 4.1.1. Initialization: *Take arbitrary* $x_1 \in \mathbb{R}^N$.
Iterative Step: *For a given current* $x_n \in \mathbb{R}^N$, *compute*

$$x_{n+1} = T(x_n - \gamma A^\top (I - S)Ax_n), \quad n \in \mathbb{N}, \tag{4.1.2}$$

where $\gamma \in \left(0, \frac{1}{L}\right)$ *with* L *being the spectral radius of the operator* $A^\top A$ *and* $A^\top$ *is the transpose of matrix* A.
LAST STEP*: Update* $n := n + 1$.

Note that the class of directed operators includes the metric projections. Hence Algorithm 4.1.1 recovers Byrne's CQ algorithm [31]. Also notice that the underlying space in Algorithm 4.1.1 is a finite-dimensional space $\mathbb{R}^N$. Later, Cui and Wang [70], Moudafi [128, 129] and Kraikaew and Saejung [112] proposed different kinds of algorithms for solving problem (4.1.1) in the setting of Hilbert spaces. Moudafi [128] considered the following algorithm and proved its weak convergence for demicontractive operators.

Algorithm 4.1.2. INITIALIZATION: *Take arbitrary* $x_1 \in H_1$.
ITERATIVE STEP: *For a given current* $x_n \in H_1$*, compute*

$$\begin{aligned} u_n &= x_n + \gamma A^*(S - I)Ax_n, \\ x_{n+1} &= (1 - \alpha_n)u_n + \alpha_n T u_n, \quad n \in \mathbb{N}, \end{aligned} \tag{4.1.3}$$

where $\gamma \in \left(0, \frac{1-\mu}{L}\right)$ *with* L *being the spectral radius of the operator* A^*A*,* A^* *is the adjoint operator of* A*,* $\alpha_n \in (0, 1)$ *and* μ *is demicontractive constant.*
LAST STEP*: Update* $n := n + 1$.

Notice that Moudafi [128] only studied weak convergence (see [128, Theorem 2.1]). It is well known that the strong convergence theorem is always more convenient to use. For this purpose, Kraikaew and Saejung [112] slightly modified the Algorithm 4.1.2 to obtain a strong convergence which is as follow.

Algorithm 4.1.3. INITIALIZATION: *Take arbitrary* $x_1 \in H_1$.
ITERATIVE STEP: *For a given current* $x_n \in H_1$*, compute*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T(x_n + \gamma A^*(S - I)Ax_n), \quad n \in \mathbb{N}, \tag{4.1.4}$$

where $\gamma \in \left(0, \frac{1}{L}\right)$ *with* L *being the spectral radius of the operator* A^*A*,* $\alpha_n \in (0, 1)$*,* $u \in H_1$ *is a fixed chosen point,* T *is strongly quasi-nonexpansive and* S *is quasi-nonexpansive.*
LAST STEP*: Update* $n := n + 1$.

They established its strong convergence (see [112, Theorem 3.2]). One can also find strong convergence scheme of problem (4.1.1) in [94]. Subsequently, problem (4.1.1) has been studied to the case of quasi-nonexpansive operators in [129], quasi-pseudo-contractive operators in [194], and finitely many directed operators in [183], in the setting of infinite-dimensional Hilbert spaces.

Motivated by the work in [196], in this chapter, we propose two methods for solving SCFPP (4.1.1). First, we introduce an implicit algorithm and consequently by discretizing the implicit algorithm, we obtain an explicit algorithm. Under some mild conditions, we show the strong convergence of the presented algorithms to a solution of SCFPP (4.1.1) which also solves the following variational inequality problem defined over the set of solutions of SCFPP (4.1.1):

$$\text{Find } x^* \in \Omega \text{ such that } \langle \sigma f(x^*) - Bx^*, z - x^* \rangle \leq 0, \quad \text{for all } z \in \Omega, \tag{4.1.5}$$

where $f, B : H_1 \to H_1$ are (nonlinear) operators and $\sigma > 0$. At the end of this chapter, we apply our iterative algorithms to solve some convex and nonlinear problems, namely, variational problems and equilibrium problems.

4.2 Algorithms and Convergence Results

For solving the problems (4.1.1) and (4.1.5), we assume the following assumptions:

Assumption 4.2.1. (a) *$f : H_1 \to H_1$ is a k-contraction mapping;*

(b) *$T : H_1 \to H_1$ is a firmly nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$*;*

(c) *$B : H_1 \to H_1$ is a self-adjoint, strongly positive bounded linear operator with coefficient $\alpha > 0$;*

(d) *$S : H_2 \to H_2$ is a nonexpansive mapping such that* $\mathrm{Fix}(S) \neq \emptyset$*;*

(e) *$A : H_1 \to H_2$ is a bounded linear operator.*

We now propose the following algorithms for finding a solution of SCFPP (4.1.1) that is also a solution of (4.1.5).

Algorithm 4.2.1. *For any $t \in \left(0, \frac{1}{\alpha - \sigma k}\right)$, define a net $\{x_t\} \subset H_1$ in the following way:*

$$x_t = T[I - \gamma A^*(I - S)A]T[t\sigma f(x_t) + (I - tB)x_t], \tag{4.2.1}$$

where $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$, $\alpha > 0$, $\sigma > 0$, $0 < k < 1$ and $\alpha - \sigma k > 0$.

Algorithm 4.2.2. *Choose arbitrary $x_1 \in H_1$, and compute*

$$x_{n+1} = T[I - \gamma A^*(I - S)A]T[\alpha_n \sigma f(x_n) + (I - \alpha_n B)x_n], \quad n \in \mathbb{N}, \tag{4.2.2}$$

where $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$, $\sigma > 0$ and $\alpha_n \in [0, 1]$.

Remark 4.2.1. It is well known that the metric projection is firmly nonexpansive, and hence nonexpansive. Hence, Algorithms 4.2.1 and 4.2.2 are more general than [196, Algorithm 3.1] and [196, Algorithm 3.4], respectively.

We now show that the net $\{x_t\}$ and the sequence $\{x_n\}$ generated by Algorithms 4.2.1 and 4.2.2 converges to a unique solution x^* of SCFPP (4.1.1), which also solves the variational inequality problem (4.1.5).

Theorem 4.2.1. *Under the Assumptions (a)-(e), the net $\{x_t\}$ defined by (4.2.1) converges to a unique solution x^* of split common fixed point problem (4.1.1), which also solves the variational inequality problem (4.1.5).*

Proof. Let us define $U := I - \gamma A^*(I - S)A$, then by Lemma 2.1.9(iii), U is averaged mapping and hence nonexpansive. Since T is firmly nonexpansive, therefore it is nonexpansive, and also composition of nonexpansive mappings is also nonexpansive, so we have, TUT is nonexpansive.

Define a mapping $W_t := T[I - \gamma A^*(I - S)A]T[t\sigma f + (I - tB)]$. Clearly, W_t is a self-mapping on H_1. For any $x,\ y \in H_1$, we have

$$\begin{aligned} \|W_t x - W_t y\| &= \|TUT[t\sigma f(x) + (I - tB)x] - TUT[t\sigma f(y) + (I - tB)y]\| \\ &\leq \|t\sigma f(x) + (I - tB)x - (t\sigma f(y) + (I - tB)y)\| \\ &= \|t\sigma(f(x) - f(y)) + (I - tB)(x - y)\| \\ &\leq \|t\sigma(f(x) - f(y))\| + \|(I - tB)(x - y)\| \\ &\leq t\sigma k\|x - y\| + (1 - t\alpha)\|x - y\| \\ &= [1 - (\alpha - \sigma k)t]\|x - y\|. \end{aligned} \tag{4.2.3}$$

Therefore, W_t is a contraction when $t \in \left(0, \frac{1}{\alpha - \sigma k}\right)$. So, W_t has a unique fixed point in H_1, denoted by x_t, that is, $x_t := TUT[tf(x_t) + (I - tB)x_t]$. From (4.2.3), it is clear that the net $\{x_t\}$ defined by (4.2.1) is well-defined.

Let $p \in \Omega$ then $p \in \text{Fix}(T)$ and $Ap \in \text{Fix}(S)$. From the definition of U, we have $p \in \text{Fix}(U)$. It follows that

$$\begin{aligned} \|x_t - p\| &= \|TUT[t\sigma f(x_t) + (I - tB)x_t] - TUTp\| \\ &\leq \|t\sigma f(x_t) + (I - tB)x_t - p\| \\ &= \|t\sigma f(x_t) - t\sigma f(p) + (I - tB)x_t - (I - tB)p - p + (I - tB)p + t\sigma f(p)\| \\ &= \|t\sigma(f(x_t) - f(p)) + (I - tB)(x_t - p) + t(\sigma f(p) - Bp)\| \\ &\leq t\sigma\|f(x_t) - f(p)\| + \|(I - tB)(x_t - p)\| + t\|\sigma f(p) - Bp\| \\ &\leq t\sigma k\|x_t - p\| + (1 - t\alpha)\|x_t - p\| + t\|\sigma f(p) - Bp\| \\ &= [1 - (\alpha - \sigma k)t]\|x_t - p\| + t\|\sigma f(p) - Bp\|. \end{aligned}$$

Hence,

$$\|x_t - p\| \leq \frac{1}{(\alpha - \sigma k)}\|\sigma f(p) - Bp\|.$$

Therefore, $\{x_t\}$ is bounded and hence $\{f(x_t)\}$, $\{Ux_t\}$ and $\{Bx_t\}$ are also bounded. From (4.2.1), we observe that

$$\begin{aligned}\|x_t - TUTx_t\| &= \|TUT[t\sigma f(x_t) + (I - tB)x_t] - TUTx_t\| \\ &\leq t\|\sigma f(x_t) - Bx_t\| \to 0 \text{ as } t \to 0. \end{aligned} \tag{4.2.4}$$

We now show that $\{x_t\}$ is relatively norm-compact as $t \to 0+$. Assume that $\{t_n\} \subset \left(0, \frac{1}{\alpha - \sigma k}\right)$ is such that $t_n \to 0+$ as $n \to \infty$. In particular, from (4.2.4), we have

$$\|x_{t_n} - TUTx_{t_n}\| \leq t_n\|\sigma f(x_{t_n}) - Bx_{t_n}\| \to 0 \text{ as } t_n \to 0.$$

Put $x_n := x_{t_n}$ and $y_n := y_{t_n}$, we have

$$\|x_n - TUTx_n\| \to 0 \text{ as } n \to \infty. \tag{4.2.5}$$

Setting $y_{t_n} := t_n\sigma f(x_{t_n}) + (I - t_nB)x_{t_n}$, we get

$$\begin{aligned}\|y_{t_n} - p\| &= \|t_n\sigma f(x_{t_n}) + (I - t_nB)x_{t_n} - p\| \\ &= \|(x_{t_n} - p) + t_n(\sigma f(x_{t_n}) - Bx_{t_n})\| \\ &\leq \|x_{t_n} - p\| + t_n\|\sigma f(x_{t_n}) - Bx_{t_n}\|. \end{aligned} \tag{4.2.6}$$

Also,

$$\begin{aligned}\|x_{t_n} - y_{t_n}\| &= \|x_{t_n} - (t_n\sigma f(x_{t_n}) + (I - t_nB)x_{t_n})\| \\ &= t_n\|\sigma f(x_{t_n}) - Bx_{t_n}\| \to 0 \text{ as } t_n \to 0. \end{aligned} \tag{4.2.7}$$

So,

$$\|x_n - y_n\| \to 0 \;\text{ as } n \to \infty. \tag{4.2.8}$$

Since $\{x_n\}$ is a bounded sequence in a Hilbert space, so it has a weakly convergent subsequence [36]. Therefore, there exists $x^* \in H_1$ such that $x_{n_i} \rightharpoonup x^*$. By Opial's condition, we have $x_n \rightharpoonup x^*$.

Indeed, let there exist another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup y^*$ where $y^* \neq x^*$.

$$\begin{aligned}\lim_{n\to\infty}\|x_n - y^*\| &= \lim_{j\to\infty}\|x_{n_j} - y^*\| < \lim_{j\to\infty}\|x_{n_j} - x^*\| \\ &= \lim_{n\to\infty}\|x_n - x^*\| = \lim_{i\to\infty}\|x_{n_i} - x^*\| \\ &< \lim_{i\to\infty}\|x_{n_i} - y^*\| = \lim_{n\to\infty}\|x_n - y^*\|,\end{aligned}$$

which is a contradiction, and hence every subsequence of $\{x_n\}$ converges weakly to x^*. This implies that $x_n \rightharpoonup x^*$.

In view of (4.2.8), we have for all $f \in H_1$,

$$\begin{aligned}\|f(y_n) - f(x^*)\| &= \|f(y_n) - f(x_n) + f(x_n) - f(x^*)\| \\ &\leq \|f(y_n) - f(x_n)\| + \|f(x_n) - f(x^*)\| \\ &\leq \|f\|\|y_n - x_n\| + \|f(x_n) - f(x^*)\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{4.2.9}$$

So, from (4.2.9), we have $y_n \rightharpoonup x^*$. In view of demiclosed principle and (4.2.5), we have $TUTx^* = x^*$. By Proposition 2.1.1 (v), we have $Tx^* = x^*$ and $Ux^* = x^*$, and hence $S(Ax^*) = Ax^*$. Thus $x^* \in \text{Fix}(T)$ and $Ax^* \in \text{Fix}(S)$. This shows that x^* is a solution of split common fixed point problem.

Next we show that x^* also solves the variational inequality problem (4.1.5). Since T is firmly nonexpansive with a fixed point p, we have

$$\langle Ty_{t_n} - y_{t_n}, Ty_{t_n} - p\rangle \leq 0. \tag{4.2.10}$$

Since every firmly nonexpansive mapping with a fixed point is firmly quasi-nonexpansive, so in view of firmly quasi-nonexpansiveness of T and (4.2.6), we have

$$\begin{aligned}\|x_{t_n} - p\| = \|TUTy_{t_n} - p\| &\leq \|Ty_{t_n} - p\| \\ &\leq \|y_{t_n} - p\| - \|Ty_{t_n} - y_{t_n}\| \\ &\leq \|x_{t_n} - p\| + t_n\|\sigma f(x_{t_n}) - Bx_{t_n}\| - \|Ty_{t_n} - y_{t_n}\|. \end{aligned} \tag{4.2.11}$$

This implies that

$$\|Ty_{t_n} - y_{t_n}\| \leq t_n\|\sigma f(x_{t_n}) - Bx_{t_n}\| \to 0 \text{ as } t_n \to 0.$$

Thus,

$$\|Ty_n - y_n\| \to 0 \text{ as } n \to \infty. \tag{4.2.12}$$

Since $y_n \rightharpoonup x^*$, so in view of (4.2.12) we have for all $f \in H_1$,

$$\begin{aligned}\|f(Ty_n) - f(x^*)\| &= \|f(Ty_n) - f(y_n) + f(y_n) - f(x^*)\| \\ &\leq \|f(Ty_n) - f(y_n)\| + \|f(y_n) - f(x^*)\| \\ &\leq \|f\|\|Ty_n - y_n\| + \|f(y_n) - f(x^*)\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{4.2.13}$$

Thus, $Ty_n \rightharpoonup x^*$. Also in view of (4.2.10), one may notice that

$$\begin{aligned}
&\|Ty_{t_n} - p\|^2 \\
&= \langle Ty_{t_n} - p, Ty_{t_n} - p\rangle \\
&= \langle Ty_{t_n} - y_{t_n}, Ty_{t_n} - p\rangle + \langle y_{t_n} - p, Ty_{t_n} - p\rangle \\
&\leq \langle y_{t_n} - p, Ty_{t_n} - p\rangle \\
&= \langle t_n\sigma f(x_{t_n}) + (I - t_nB)x_{t_n} - p, Ty_{t_n} - p\rangle \\
&= \langle t_n\sigma f(x_{t_n}) - t_n\sigma f(p) + (I - t_nB)x_{t_n} - (I - t_nB)p \\
&\quad + (I - t_nB)p - p + t_n\sigma f(p), Ty_{t_n} - p\rangle \\
&= \langle t_n\sigma(f(x_{t_n}) - f(p)) + (I - t_nB)(x_{t_n} - p), Ty_{t_n} - p\rangle \\
&\quad + t_n\langle \sigma f(p) - Bp, Ty_{t_n} - p\rangle \\
&\leq (\|t_n\sigma(f(x_{t_n}) - f(p)) + (I - t_nB)(x_t - p)\|)\|Ty_{t_n} - p\| \\
&\quad + t_n\langle \sigma f(p) - Bp, Ty_{t_n} - p\rangle \\
&\leq (\|t_n\sigma(f(x_{t_n}) - f(p))\| + \|(I - t_nB)(x_{t_n} - p)\|)\|Ty_{t_n} - p\| \\
&\quad + t_n\langle \sigma f(p) - Bp, Ty_{t_n} - p\rangle \\
&\leq (t_n\sigma k\|x_{t_n} - p\| + (1 - t_n\alpha)\|x_{t_n} - p\|)\|Ty_{t_n} - p\| + t_n\langle \sigma f(p) - Bp, Ty_{t_n} - p\rangle \\
&\leq ((1 - t_n(\alpha - \sigma k))\|x_{t_n} - p\|)\|Ty_{t_n} - p\| + t_n\langle \sigma f(p) - Bp, Ty_{t_n} - p\rangle \\
&\leq \left(\frac{1 - t_n(\alpha - \sigma k)}{2}\right)\|x_{t_n} - p\|^2 + \frac{1}{2}\|Ty_{t_n} - p\|^2 + t_n\langle \sigma f(p) - Bp, Ty_{t_n} - p\rangle.
\end{aligned}$$

Therefore,

$$\|Ty_{t_n} - p\|^2 \leq (1 - t_n(\alpha - \sigma k))\|x_{t_n} - p\|^2 + 2t_n\langle \sigma f(p) - Bp, Ty_{t_n} - p\rangle. \quad (4.2.14)$$

Consequently,

$$\begin{aligned}
\|x_{t_n} - p\|^2 &= \|TUTy_{t_n} - TUTp\|^2 \leq \|Ty_{t_n} - p\|^2 \\
&\leq (1 - t_n(\alpha - \sigma k))\|x_{t_n} - p\|^2 + 2t_n\langle \sigma f(p) - Bp, Ty_{t_n} - p\rangle.
\end{aligned}$$

So,

$$\|x_{t_n} - p\|^2 \leq \frac{2}{\alpha - \sigma k}\langle \sigma f(p) - Bp, Ty_{t_n} - p\rangle,$$

that is,

$$\|x_n - p\|^2 \leq \frac{2}{\alpha - \sigma k}\langle \sigma f(p) - Bp, Ty_n - p\rangle. \quad (4.2.15)$$

Since $p \in \Omega$ was arbitrary, inequality (4.2.15) holds for all $p \in \Omega$. Since $x^* \in \Omega$, from (4.2.15), we have

$$\|x_n - x^*\|^2 \leq \frac{2}{\alpha - \sigma k}\langle \sigma f(x^*) - Bx^*, Ty_n - x^*\rangle.$$

Since $Ty_n \rightharpoonup x^*$, we have $x_n \to x^*$, that is, $x_{t_n} \to x^*$. Thus, the bounded net $\{x_t\}$ has a convergent subnet $\{x_{t_n}\}$ that converges to x^*. Therefore, the net $\{x_t\}$ is relative norm-compact as $t \to 0+$. Also letting $n \to \infty$ in (4.2.15), we obtain

$$0 \leq \|x^* - p\|^2 \leq \frac{2}{\alpha - \sigma k}\langle \sigma f(p) - Bp, x^* - p\rangle.$$

This implies that

$$\langle (\sigma f - B)p, x^* - p\rangle \geq 0 \quad \text{or} \quad \langle (B - \sigma f)p, p - x^*\rangle \geq 0.$$

Since $(B - \sigma f)$ is continuous. Thus by the Lemma 2.5.1, we have

$$\langle (B - \sigma f)x^*, p - x^*\rangle \geq 0. \tag{4.2.16}$$

Since p was arbitrary, we have

$$\langle (B - \sigma f)x^*, z - x^*\rangle \geq 0, \quad \text{for all} \quad z \in \Omega.$$

By [119, Lemma 2.3], $(B - \sigma f)$ is strongly monotone, and hence, again by Lemma 2.5.1, the solution of the variational inequality (4.2.16) is unique. Therefore, each cluster point of $\{x_t\}$ (as $t \to 0+$) equals x^*. Therefore, $x_t \to x^*$. This completes the proof. □

Remark 4.2.2. If we consider $T = P_C$, $S = P_Q$, $\text{Fix}(T) = C$ and $\text{Fix}(S) = Q$, then Theorem 4.2.1 generalizes [196, Theorem 3.1]. Furthermore, if $B \equiv I$ (respectively, $f \equiv 0$), then Theorem 4.2.1, generalizes [196, Corollary 3.2] (respectively, [196, Corollary 3.3]).

We now prove the strong convergence of the sequence $\{x_n\}$ generated by Algorithm 4.2.2 to a solution of SCFPP.

Theorem 4.2.2. *Let σ be a given constant such that $0 < \alpha - \sigma k < 1$, and assume that the sequence $\{\alpha_n\}$ satisfies the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$*;*

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$*.*

Then under the Assumptions (a)-(e), the sequence $\{x_n\}$ defined by (4.2.2) converges to a point $x^* \in \Omega$, which solves the variational inequality inequality (4.1.5).

Proof. Let $p \in \Omega$. In view of definition of U, (4.2.2) becomes

$$x_{n+1} = TUT[\alpha_n \sigma f(x_n) + (I - \alpha_n B)x_n].$$

Consider

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&= \|TUT[\alpha_n \sigma f(x_n) + (I - \alpha_n B)x_n] - TUTp\| \\
&\leq \|\alpha_n \sigma f(x_n) + (I - \alpha_n B)x_n - p\| \\
&= \|\alpha_n \sigma f(x_n) - \alpha_n \sigma f(p) + (I - \alpha_n B)x_n - (I - \alpha_n B)p + (I - \alpha_n B)p - p + \alpha_n \sigma f(p)\| \\
&= \|\alpha_n \sigma(f(x_n) - f(p)) + (I - \alpha_n B)(x_n - p) + \alpha_n(\sigma f(p) - Bp)\| \\
&\leq \alpha_n \sigma \|f(x_n) - f(p)\| + \|(I - \alpha_n B)(x_n - p)\| + \alpha_n \|\sigma f(p) - Bp\| \\
&\leq \alpha_n \sigma k \|x_n - p\| + (1 - \alpha_n \alpha)\|x_n - p\| + \alpha_n \|\sigma f(p) - Bp\| \\
&= [1 - (\alpha - \sigma k)\alpha_n]\|x_n - p\| + \alpha_n \|\sigma f(p) - Bp\| \\
&= [1 - (\alpha - \sigma k)\alpha_n]\|x_n - p\| + (\alpha - \sigma k)\alpha_n \|\sigma f(p) - Bp\|/(\alpha - \sigma k) \\
&\leq \max\{\|x_n - p\|, \|\sigma f(p) - Bp\|/(\alpha - \sigma k)\} \\
&\leq \max\{\|x_1 - p\|, \|\sigma f(p) - Bp\|/(\alpha - \sigma k)\}.
\end{aligned}$$

This implies that $\{x_n\}$ is a bounded sequence and so are $\{f(x_n)\}$, $\{Ux_n\}$ and $\{Bx_n\}$. Also, every firmly nonexpansive mapping is averaged, and thus T is averaged. Since composition of averaged mapping is averaged, TU is averaged. Hence, there exists $\beta_1 \in (0, 1)$ such that $TU = (1 - \beta_1)I + \beta_1 V_1$ where V_1 is nonexpansive mapping.

Set

$$y_n := T(\alpha_n \sigma f(x_n) + (I - \alpha_n B)x_n),$$

and

$$z_n := \alpha_n \sigma f(x_n) + (I - \alpha_n B)x_n,$$

for all n. Since T is averaged, therefore there exists $\beta_2 \in (0, 1)$ such that $T = (1 - \beta_2)I + \beta_2 V_2$ where V_2 is nonexpansive mapping. It follows that

$$\begin{aligned}
y_n &= Tz_n \\
&= ((1 - \beta_2)I + \beta_2 V_2)z_n \\
&= ((1 - \beta_2)I + \beta_2 V_2)(\alpha_n \sigma f(x_n) + (I - \alpha_n B)x_n) \\
&= ((1 - \beta_2)(\alpha_n \sigma f(x_n) + (I - \alpha_n B)x_n) + \beta_2 V_2(\alpha_n \sigma f(x_n) + (I - \alpha_n B)x_n) \\
&= ((1 - \beta_2)(x_n + \alpha_n(\sigma f(x_n) - Bx_n)) + \beta_2 V_2 z_n \\
&= (1 - \beta_2)x_n + (1 - \beta_2)\alpha_n(\sigma f(x_n) - Bx_n) + \beta_2 V_2 z_n \\
&= (1 - \beta_2)x_n + \beta_2 \left[\frac{(1 - \beta_2)}{\beta_2}\alpha_n(\sigma f(x_n) - Bx_n) + V_2 z_n\right] \\
&= (1 - \beta_2)x_n + \beta_2 q_n, \qquad (4.2.17)
\end{aligned}$$

where,

$$q_n = \frac{(1-\beta_2)}{\beta_2}\alpha_n(\sigma f(x_n) - Bx_n) + V_2 z_n. \tag{4.2.18}$$

Further, we have

$$\begin{aligned}
&\|q_{n+1} - q_n\| = \\
&\left\|\frac{(1-\beta_2)}{\beta_2}\alpha_{n+1}(\sigma f(x_{n+1}) - Bx_{n+1}) + V_2 z_{n+1} - \frac{(1-\beta_2)}{\beta_2}\alpha_n(\sigma f(x_n) - Bx_n) - V_2 z_n\right\| \\
&\leq \|V_2 z_{n+1} - V_2 z_n\| + \frac{(1-\beta_2)}{\beta_2}\left[\alpha_{n+1}(\sigma f(x_{n+1}) - Bx_{n+1}) - \alpha_n(\sigma f(x_n) - Bx_n)\right] \\
&\leq \|z_{n+1} - z_n\| + \frac{(1-\beta_2)}{\beta_2}\left[\alpha_{n+1}(\sigma f(x_{n+1}) - Bx_{n+1}) - \alpha_n(\sigma f(x_n) - Bx_n)\right].
\end{aligned} \tag{4.2.19}$$

In view of (4.2.2) and (4.2.17), we have

$$\begin{aligned}
x_{n+1} &= TUy_n \\
&= ((1-\beta_1)I + \beta_1 V_1)y_n \\
&= (1-\beta_1)y_n + \beta_1 V_1 y_n \\
&= (1-\beta_1)[(1-\beta_2)x_n + \beta_2 q_n] + \beta_1 V_1 y_n \\
&= (1-\beta_1)(1-\beta_2)x_n + (1-\beta_1)\beta_2 q_n + \beta_1 V_1 y_n \\
&= (1-(\beta_1 + \beta_2 - \beta_1\beta_2))x_n + (1-\beta_1)\beta_2 q_n + \beta_1 V_1 y_n \\
&= (1-\beta_3)x_n + \beta_3\left[\frac{(1-\beta_1)\beta_2}{\beta_3}q_n + \frac{\beta_1}{\beta_3}V_1 y_n\right] \\
&= (1-\beta_3)x_n + \beta_3 p_n,
\end{aligned} \tag{4.2.20}$$

where $\beta_3 = \beta_1 + \beta_2 - \beta_1\beta_2$ and $p_n = \frac{(1-\beta_1)\beta_2}{\beta_3}q_n + \frac{\beta_1}{\beta_3}V_1 y_n$. So, in view of (4.2.19), we have

$$
\begin{aligned}
&\|p_{n+1} - p_n\| = \\
&= \left\| \frac{(1-\beta_1)\beta_2}{\beta_3} q_{n+1} + \frac{\beta_1}{\beta_3} V_1 y_{n+1} - \frac{(1-\beta_1)\beta_2}{\beta_3} q_n - \frac{\beta_1}{\beta_3} V_1 y_n \right\| \\
&= \left\| \frac{(1-\beta_1)\beta_2}{\beta_3} (q_{n+1} - q_n) + \frac{\beta_1}{\beta_3} (V_1 y_{n+1} - V_1 y_n) \right\| \\
&\leq \frac{(1-\beta_1)\beta_2}{\beta_3} \|q_{n+1} - q_n\| + \frac{\beta_1}{\beta_3} \|V_1 y_{n+1} - V_1 y_n\| \\
&\leq \frac{(1-\beta_1)\beta_2}{\beta_3} \|q_{n+1} - q_n\| + \frac{\beta_1}{\beta_3} \|y_{n+1} - y_n\| \\
&\leq \frac{(1-\beta_1)\beta_2}{\beta_3} \|q_{n+1} - q_n\| + \frac{\beta_1}{\beta_3} \|T z_{n+1} - T z_n\| \\
&\leq \frac{(1-\beta_1)\beta_2}{\beta_3} \|q_{n+1} - q_n\| + \frac{\beta_1}{\beta_3} \|z_{n+1} - z_n\| \\
&\leq \frac{(1-\beta_1)\beta_2}{\beta_3} \|z_{n+1} - z_n\| \\
&\quad + \frac{(1-\beta_1)(1-\beta_2)}{\beta_3} (\alpha_{n+1}(\sigma f(x_{n+1}) - B x_{n+1}) - \alpha_n(\sigma f(x_n) - B x_n)) \\
&\quad + \frac{\beta_1}{\beta_3} \|z_{n+1} - z_n\| \\
&\leq \frac{\beta_1 + \beta_2 - \beta_1\beta_2}{\beta_3} \|z_{n+1} - z_n\| \\
&\quad + \frac{(1-\beta_3)}{\beta_3} (\alpha_{n+1}(\sigma f(x_{n+1}) - B x_{n+1}) - \alpha_n(\sigma f(x_n) - B x_n)) \\
&\leq \|z_{n+1} - z_n\| + \frac{(1-\beta_3)}{\beta_3} (\alpha_{n+1}(\sigma f(x_{n+1}) - B x_{n+1}) - \alpha_n(\sigma f(x_n) - B x_n)) \\
&= \|(\alpha_{n+1} \sigma f(x_{n+1}) + (I - \alpha_{n+1} B) x_{n+1}) - (\alpha_n \sigma f(x_n) + (I - \alpha_n B) x_n)\| \\
&\quad + \frac{(1-\beta_3)}{\beta_3} (\alpha_{n+1}(\sigma f(x_{n+1}) - B x_{n+1}) - \alpha_n(\sigma f(x_n) - B x_n)) \\
&= \|(x_{n+1} - x_n) + (\alpha_{n+1}(\sigma f(x_{n+1}) - B x_{n+1}) - (\alpha_n(\sigma f(x_n) - B x_n)\| \\
&\quad + \frac{(1-\beta_3)}{\beta_3} (\alpha_{n+1}(\sigma f(x_{n+1}) - B x_{n+1}) - \alpha_n(\sigma f(x_n) - B x_n)) \\
&\leq \|x_{n+1} - x_n\| + (\alpha_{n+1}(\sigma f(x_{n+1}) - B x_{n+1}) - \alpha_n(\sigma f(x_n) - B x_n)) \\
&\quad + \frac{(1-\beta_3)}{\beta_3} (\alpha_{n+1}(\sigma f(x_{n+1}) - B x_{n+1}) - \alpha_n(\sigma f(x_n) - B x_n)).
\end{aligned}
$$

This implies that

$$\|p_{n+1} - p_n\| - \|x_{n+1} - x_n\| \leq (\alpha_{n+1}(\sigma f(x_{n+1}) - Bx_{n+1}) - \alpha_n(\sigma f(x_n) - Bx_n)) + \frac{(1-\beta_3)}{\beta_3}(\alpha_{n+1}(\sigma f(x_{n+1}) - Bx_{n+1}) - \alpha_n(\sigma f(x_n) - Bx_n)), \quad (4.2.21)$$

and

$$\limsup_{n\to\infty}(\|p_{n+1} - p_n\| - \|x_{n+1} - x_n\|) \leq 0. \quad (4.2.22)$$

Therefore, from equation (4.2.20), inequality (4.2.22) and Lemma 2.1.5, we have

$$\lim_{n\to\infty} \|p_n - x_n\| = 0. \quad (4.2.23)$$

In view of (4.2.20) and (4.2.23), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \beta_3 \lim_{n\to\infty} \|p_n - x_n\| = 0, \quad (4.2.24)$$

and

$$\lim_{n\to\infty} \|z_n - x_n\| = \lim_{n\to\infty} \alpha_n \|\sigma f(x_n) - Bx_n\| = 0. \quad (4.2.25)$$

From (4.2.24) and (4.2.25), we have

$$\begin{aligned} \|TUTz_n - z_n\| &= \|TUTz_n - x_n + x_n - z_n\| \\ &\leq \|TUTz_n - x_n\| + \|x_n - z_n\| \\ &\leq \|x_{n+1} - x_n\| + \|x_n - z_n\| \quad \to 0 \text{ as } n \to \infty. \quad (4.2.26) \end{aligned}$$

We can also have

$$|\|TUTz_n - p\| - \|z_n - p\|| \leq \|TUTz_n - z_n\|.$$

Taking limit both the sides and by using (4.2.26), we obtain

$$\begin{aligned} \lim_{n\to\infty} |(\|TUTz_n - p\| - \|z_n - p\|)| &= 0 \\ \left|\lim_{n\to\infty} (\|TUTz_n - p\| - \|z_n - p\|)\right| &= 0 \\ \lim_{n\to\infty} (\|TUTz_n - p\| - \|z_n - p\|) &= 0. \quad (4.2.27) \end{aligned}$$

By using nonexpansiveness of TU and T, we have

$$\|TUTz_n - p\| \leq \|Tz_n - p\| \leq \|z_n - p\|,$$

and therefore,

$$\|TUTz_n - p\| - \|z_n - p\| \leq \|Tz_n - p\| - \|z_n - p\| \leq 0.$$

Thus, in view of (4.2.27), we have

$$\lim_{n\to\infty}(\|Tz_n - p\| - \|z_n - p\|) = 0.$$

Since T is firmly nonexpansive and hence strongly nonexpansiveness [36], therefore

$$\lim_{n\to\infty}\|Tz_n - z_n\| = 0,$$

that is,

$$\lim_{n\to\infty}\|y_n - z_n\| = 0. \tag{4.2.28}$$

Now we show that $x_n \to x^*$, where x^* is required solution of (4.1.5). Notice that

$$\begin{aligned}
&\|y_n - x^*\|^2 \\
&= \langle y_n - x^*, y_n - x^*\rangle \\
&= \langle y_n - z_n, y_n - x^*\rangle + \langle z_n - x^*, y_n - x^*\rangle \\
&\le \langle z_n - x^*, y_n - x^*\rangle \\
&= \langle \alpha_n\sigma f(x_n) + (I - \alpha_n B)x_n - x^*, y_n - x^*\rangle \\
&= \langle \alpha_n\sigma f(x_n) - \alpha_n\sigma f(x^*) + \alpha_n\sigma f(x^*) + (I - \alpha_n B)x_n - x^* - (I - \alpha_n B)x^* \\
&\quad + (I - \alpha_n B)x^*, y_n - x^*\rangle \\
&= \langle \alpha_n\sigma(f(x_n) - f(x^*)) + (I - \alpha_n B)(x_n - x^*) + \alpha_n(\sigma f(x^*) - Bx^*), y_n - x^*\rangle \\
&= \langle \alpha_n\sigma(f(x_n) - f(x^*)) + (I - \alpha_n B)(x_n - x^*), y_n - x^*\rangle \\
&\quad + \alpha_n\langle \sigma f(x^*) - Bx^*, y_n - x^*\rangle \\
&\le \|\alpha_n\sigma(f(x_n) - f(x^*)) + (I - \alpha_n B)(x_n - x^*)\|\|y_n - x^*\| \\
&\quad + \alpha_n\langle \sigma f(x^*) - Bx^*, y_n - x^*\rangle \\
&= (\alpha_n\sigma\|f(x_n) - f(x^*)\| + \|(I - \alpha_n B)(x_n - x^*)\|)\|y_n - x^*\| \\
&\quad + \alpha_n\langle \sigma f(x^*) - Bx^*, y_n - x^*\rangle \\
&\le (\alpha_n\sigma k\|x_n - x^*\| + (1 - \alpha_n\alpha)\|x_n - x^*\|)\|y_n - x^*\| + \alpha_n\langle \sigma f(x^*) - Bx^*, y_n - x^*\rangle \\
&= ((1 - \alpha_n(\alpha - \sigma k))\|x_n - x^*\|)\|y_n - x^*\| + \alpha_n\langle \sigma f(x^*) - Bx^*, y_n - x^*\rangle \\
&= \left(\frac{1 - \alpha_n(\alpha - \sigma k)}{2}\right)\|x_n - x^*\|^2 + \frac{1}{2}\|y_n - x^*\|^2 + \alpha_n\langle \sigma f(x^*) - Bx^*, y_n - x^*\rangle.
\end{aligned}$$

Hence,

$$\|y_n - x^*\|^2 \le (1 - \alpha_n(\alpha - \sigma k))\|x_n - x^*\|^2 + 2\alpha_n\langle \sigma f(x^*) - Bx^*, y_n - x^*\rangle. \tag{4.2.29}$$

Consequently,

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|TUy_n - x^*\|^2 \\
&\leq \|y_n - x^*\|^2 \\
&\leq (1 - \alpha_n(\alpha - \sigma k))\|Ty_n - x^*\|^2 + 2\alpha_n\langle \sigma f(x^*) - Bx^*, y_n - x^*\rangle \\
&\leq (1 - \alpha_n(\alpha - \sigma k))\|y_n - x^*\|^2 + 2\alpha_n\langle \sigma f(x^*) - Bx^*, y_n - x^*\rangle \\
&\leq (1 - \alpha_n(\alpha - \sigma k))\|x_n - x^*\|^2 + (\alpha - \sigma k)\frac{2\alpha_n}{(\alpha - \sigma k)}\langle \sigma f(x^*) - Bx^*, y_n - x^*\rangle.
\end{aligned} \tag{4.2.30}$$

So, in order to apply Lemma 2.1.3, we have to show that

$$\limsup_{n\to\infty}\langle \sigma f(x^*) - Bx^*, y_n - x^*\rangle \leq 0.$$

Since $\{x_n\}$ is a bounded sequence, so it has a weakly convergent subsequence, say $\{x_{n_i}\}$, such that $x_{n_i} \rightharpoonup z \in H_1$. In view of (4.2.25), we have $z_{n_i} \rightharpoonup z$. Again from (4.2.26) and demiclosed principle we have $TUTz = z$. As T and U are averaged mapping. By Proposition 2.1.1 (v), we have $z \in \mathrm{Fix}(T)$ and $z \in \mathrm{Fix}(U)$ and hence by Lemma 2.1.9 (iiib) $Az \in \mathrm{Fix}(S)$. Thus $z \in \Omega$. Since $z_{n_i} \rightharpoonup z$, therefore from (4.2.28), we have $y_{n_i} \rightharpoonup z$. Consider

$$\begin{aligned}
\limsup_{n\to\infty}\langle \sigma f(x^*) - Bx^*, y_n - x^*\rangle &= \lim_{i\to\infty}\langle \sigma f(x^*) - Bx^*, y_{n_i} - x^*\rangle \\
&= \langle \sigma f(x^*) - Bx^*, z - x^*\rangle \leq 0.
\end{aligned} \tag{4.2.31}$$

By the Lemma 2.1.3 and taking into account inequality (4.2.31) in (4.2.30), we have $x_n \to x^*$. This completes the proof. □

Remark 4.2.3. If we consider $T = P_C$, $S = P_Q$, $\mathrm{Fix}(T) = C$ and $\mathrm{Fix}(S) = Q$, then Theorem 4.2.2 generalizes [196, Theorem 3.5]. Furthermore, if $B \equiv I$ (respectively, $f \equiv 0$), then Theorem 4.2.2, generalizes [196, Corollary 3.7] (respectively, [196, Corollary 3.9]).

4.3 Applications

We now pay attention to applying our iterative algorithms to some problems from convex and nonlinear analysis.

Variational problems via resolvent mappings: For a given maximal monotone operator $M_1 : H_1 \rightrightarrows H_1$, it is well known that its associated resolvent mapping

$J_\lambda^{M_1} = (I + \lambda M_1)$ is firmly-nonexpansive and $0 \in M_1(x) \Leftrightarrow J_\lambda^{M_1}(x) = x$; See, for example [167, 198]. This means that zeroes of M_1 are exactly fixed points of its resolvent mapping. Let $T = J_\lambda^{M_1}$ and $S = J_\lambda^{M_2}$, where $M_2 : H_2 \rightrightarrows H_2$ is another maximal monotone operator. We consider the problem of finding $x^* \in \Omega_1$ such that

$$\langle \sigma f(x^*) - Bx^*, z - x^* \rangle \leq 0, \quad \text{for all} \quad z \in \Omega_1, \tag{4.3.1}$$

where $\Omega_1 = M_1^{-1}(0) \cap A^{-1}(M_2^{-1}(0))$. Under these restrictions, Algorithms 4.2.1 and 4.2.2 reduces to the following Algorithms 4.3.1 and 4.3.2, respectively.

Algorithm 4.3.1. *For any $t \in \left(0, \frac{1}{\alpha - \sigma k}\right)$, define a net $\{x_t\} \subset H_1$ in the following way:*

$$x_t = J_\lambda^{M_1}[I - \gamma A^*(I - J_\lambda^{M_2})A]J_\lambda^{M_1}[t\sigma f(x_t) + (I - tB)x_t], \tag{4.3.2}$$

where $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$, $\alpha > 0$, $\sigma > 0$, $0 < k < 1$ and $\alpha - \sigma k > 0$.

Algorithm 4.3.2. *Choose arbitrary $x_1 \in H_1$, compute*

$$x_{n+1} = J_\lambda^{M_1}[I - \gamma A^*(I - J_\lambda^{M_2})A]J_\lambda^{M_1}[\alpha_n \sigma f(x_n) + (I - \alpha_n B)x_n], \quad n \in \mathbb{N}, \tag{4.3.3}$$

where $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$, $\sigma > 0$ and $\alpha_n \in [0, 1]$.

Since the resolvent operators are firmly nonexpansive, the strong convergence of the net $\{x_t\}$ (respectively, sequence $\{x_n\}$) generated by the Algorithm 4.3.1 (respectively, Algorithm 4.3.2) can be derived from Theorem 4.2.1 (respectively, Theorem 4.2.2).

Equilibrium problems via resolvent mappings: Let C be a nonempty closed convex subset of a Hilbert space H_1 and let $F : C \times C \to \mathbb{R}$ be a bifunction. Consider the following equilibrium problem: Find $z \in C$ such that

$$F(z, y) \geq 0, \quad \text{for all } y \in C. \tag{4.3.4}$$

The set of all $z \in C$ that satisfies (4.3.4) is denoted by $\text{EP}(C, F)$, i.e.,

$$\text{EP}(C, F) = \{z \in C \ : \ F(z, y) \geq 0 \text{ for all } y \in C\}.$$

It is well-known, (see [24, 67]) that the associated resolvent operator $T_\lambda^F : H \to C$ for $\lambda > 0$ defined by

$$T_\lambda^F x = \left\{ z \in C : F(z, y) + \frac{1}{\lambda}\langle y - z, z - x \rangle \geq 0, \text{ for all } y \in C \right\}, \tag{4.3.5}$$

is firmly nonexpansive and $\text{Fix}(T_\lambda^F) = EP(C,F)$. Let $T = T_\lambda^F$ and $S = S_\mu^G$, where $G : Q \times Q \to \mathbb{R}$ is another bifunction and S_μ^G is the resolvent operator of G as defined in (4.3.5). We consider the problem of finding $x^* \in \Omega_2$ such that

$$\langle \sigma f(x^*) - Bx^*, z - x^* \rangle \leq 0, \quad \text{for all} \quad z \in \Omega_2, \tag{4.3.6}$$

where $\Omega_2 = EP(C,F) \cap A^{-1}(EP(Q,G))$. Under these restrictions, Algorithms 4.2.1 and 4.2.2 reduces to the following Algorithms 4.3.3 and 4.3.4, respectively.

Algorithm 4.3.3. *For any $t \in \left(0, \frac{1}{\alpha - \sigma k}\right)$, define a net $\{x_t\} \subset H_1$ in the following way:*

$$x_t = T_\lambda^F[I - \gamma A^*(I - S_\mu^G)A]T_\lambda^F[t\sigma f(x_t) + (I - tB)x_t], \tag{4.3.7}$$

where $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$, $\alpha > 0$, $\sigma > 0$, $0 < k < 1$ and $\alpha - \sigma k > 0$.

Algorithm 4.3.4. *Choose arbitrary $x_1 \in H_1$, compute*

$$x_{n+1} = T_\lambda^F[I - \gamma A^*(I - S_\mu^G)A]T_\lambda^F[\alpha_n \sigma f(x_n) + (I - \alpha_n B)x_n], \quad n \in \mathbb{N}, \tag{4.3.8}$$

where $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$, $\sigma > 0$ and $\alpha_n \in [0,1]$.

Since the resolvent operators are firmly nonexpansive, the strong convergence of the net $\{x_t\}$ (respectively, sequence $\{x_n\}$) generated by the Algorithm 4.3.3 (respectively, Algorithm 4.3.4) can be derived from Theorem 4.2.1 (respectively, Theorem 4.2.2)