Chapter 3

Split Feasibility and Fixed Point Problems for Asymptotically k-strict Psuedo-contractive Mappings in Intermediate Sense

3.1 Introduction

During the last decade, the split feasibility problems (in short, SFP) are emerged as models of several problems, namely, signal processing, phase retrievals, image reconstruction, intensity-modulated radiation therapy, etc, see, for example, [7, 10, 30, 31, 32, 43]. Several iterative methods have appeared in the literature to compute the approximate solutions of such problems. For a comprehensive bibliography and survey on split feasibility problems, we refer to [10] and the references therein. Finding the common solution of a split feasibility problem and fixed point problem is one of the core interests of many researchers, see for example [10, 39, 40, 41, 73] and the references therein. Recently, Ceng et al. [40] introduced a relaxed extragradient method with regularization for finding a common element of the solution set of SFP and the set $\mathrm{Fix}(T)$ of the fixed points of a nonexpansive mapping T. Very recently, inspired by the work of Ceng et al. [40] and Xu [188], Deepho and Kumam [73] introduced and analyzed a relaxed extragradient method with regularization for finding a common element of the solution set Γ of the split feasibility problem and fixed points set $\mathrm{Fix}(T)$ of an uniformly Lipschitz continuous and asymptotically quasi-nonexpansive mappings in the setting of real Hilbert spaces. The purpose of this chapter is to consider and analyze the relaxed extragradient method with regularization proposed in [73] for finding a common element of Γ and $\mathrm{Fix}(T)$, where T is an asymptotically k-strict psuedo-contractive mapping in intermediate sense. We prove that the sequence generated by the considered algorithm converges weakly to an element of $\mathrm{Fix}(T) \cap \Gamma$.

3.2 An Algorithm and a Convergence Result

Very recently, Deepho and Kumam [73] proposed the following algorithm for finding the common element of the solution set Γ of the split feasibility problem and set $\mathrm{Fix}(T)$ of all fixed points of an asymptotically quasi-nonexpansive and Lipschitz continuous mapping in a real Hilbert space.

Algorithm 3.2.1. INITIALIZATION*: Take arbitrary* $x_1 \in C$*.*
ITERATIVE STEP*: For a given current* $x_n \in C$*, compute*

$$\begin{aligned} y_n &= P_C(I - \lambda_n \nabla f_{\alpha_n})(x_n), \\ x_{n+1} &= (1 - \beta_n)x_n + \beta_n T^n(y_n), \quad n \in \mathbb{N}, \end{aligned} \tag{3.2.1}$$

where $\nabla f_{\alpha_n} = \nabla f + \alpha_n I = A^*(I - P_Q)A + \alpha_n I$*, where* I *is an identity map and three sequences of parameters* $\{\alpha_n\}$*,* $\{\lambda_n\}$ *and* $\{\beta_n\}$ *satisfies the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n < \infty$*;*

(ii) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in \left(0, \frac{1}{\|A\|^2}\right)$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$*;*

(iii) $\{\beta_n\} \subset [c, d]$ *for some* $c, d \in (0, 1)$*.*

LAST STEP*: Update* $n := n + 1$*.*

We establish the following weak convergence result for Algorithm 3.2.1, where $T : C \to C$ is an uniformly continuous and asymptotically k-strict pseudo-contractive mapping in intermediate sense.

Theorem 3.2.1. *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H_1 *and* $T : C \to C$ *be an uniformly continuous and asymptotically k-strict pseudo-contractive mapping in intermediate sense with sequence* $\{\nu_n\}$ *such that* $\mathrm{Fix}(T) \cap \Gamma \neq \emptyset$ *and* $R(T) = C$*. Let* $\{x_n\}$ *and* $\{y_n\}$ *be the sequences in* C *generated by Algorithm 3.2.1. Assume that the sequences of parameters* $\{\alpha_n\}$*,* $\{\beta_n\}$*,* $\{\lambda_n\}$ *and* $\{\nu_n\}$ *satisfy the following conditions:*

(i) $\sum_{n=1}^{\infty} \alpha_n < \infty$;

(ii) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in \left(0, \frac{1}{\|A\|^2}\right)$ *and* $\sum_{n=1}^{\infty} \lambda_n < \infty$;

(iii) $\{\beta_n\} \subset [d, e]$ *for some* $d, e \in (0, 1)$, $0 < \beta_n < 1 - k < 1$ *and* $\sum_{n=1}^{\infty} \beta_n c_n < \infty$, *where* c_n *is defined by (2.1.1);*

(iv) $\sum_{n=1}^{\infty} \beta_n \nu_n < \infty$;

(v) $\{\nabla f_{\alpha_n}(x_n)\}_{n=1}^{\infty}$ *is a bounded sequence.*

Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge weakly to an element* $x^* \in \mathrm{Fix}(T) \cap \Gamma$.

Proof. Let $p \in \mathrm{Fix}(T) \cap \Gamma$ be arbitrarily chosen. Then, we have $T(p) = p \in C$ and $Ap \in Q$. Therefore, $P_C(p) = p$ and $P_Q(Ap) = Ap$. Since P_C is nonexpansive, we have

$$\begin{aligned} \|y_n - p\|^2 &= \|P_C(I - \lambda_n \nabla f_{\alpha_n})(x_n) - P_C(p)\|^2 \\ &\leq \|(x_n - p) - \lambda_n \nabla f_{\alpha_n}(x_n)\|^2 \\ &\leq \|x_n - p\|^2 + \lambda_n^2 \|\nabla f_{\alpha_n}(x_n)\|^2. \end{aligned} \tag{3.2.2}$$

Since $y_n \in C$ and $T^n y_n \in C$, we have

$$\begin{aligned} \|y_n - T^n y_n\|^2 &= \|P_C(I - \lambda_n \nabla f_{\alpha_n})(x_n) - P_C(T^n y_n)\|^2 \\ &\leq \|(x_n - T^n y_n) - \lambda_n \nabla f_{\alpha_n}(x_n)\|^2 \\ &\leq \|x_n - T^n y_n\|^2 + \lambda_n^2 \|\nabla f_{\alpha_n}(x_n)\|^2. \end{aligned} \tag{3.2.3}$$

By asymptotically k-strict pseudo-contractiveness in intermediate sense with sequence

$\{\nu_n\}$ of T, Lemma 2.1.4 (iii) and inequalities (3.2.2) and (3.2.3), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|(1-\beta_n)(x_n-p)+\beta_n(T^n y_n-p)\|^2 \\
&= (1-\beta_n)\|x_n-p\|^2+\beta_n\|T^n y_n-p\|^2-\beta_n(1-\beta_n)\|T^n y_n-x_n\|^2 \\
&= (1-\beta_n)\|x_n-p\|^2+\beta_n\{(1+\nu_n)\|y_n-p\|^2+k\|T^n y_n-y_n\|^2+c_n\} \\
&\quad -\beta_n(1-\beta_n)\|T^n y_n-x_n\|^2 \\
&\le (1-\beta_n)\|x_n-p\|^2+\beta_n(1+\nu_n)\{\|x_n-p\|^2+\lambda_n^2\|\nabla f_{\alpha_n}(x_n)\|^2\} \\
&\quad +\beta_n k\|x_n-T^n y_n\|^2+\beta_n k\lambda_n^2\|\nabla f_{\alpha_n}(x_n)\|^2 \\
&\quad +\beta_n c_n-\beta_n(1-\beta_n)\|T^n y_n-x_n\|^2 \\
&\le (1+\beta_n\nu_n)\|x_n-p\|^2+\beta_n c_n-\beta_n(1-\beta_n-k)\|x_n-T^n y\|^2 \\
&\quad +\beta_n\lambda_n^2\|\nabla f_{\alpha_n}(x_n)\|^2+\beta_n\nu_n\lambda_n^2\|\nabla f_{\alpha_n}(x_n)\|^2 \\
&\quad +k\beta_n\lambda_n^2\|\nabla f_{\alpha_n}(x_n)\|^2 \\
&\le (1+\beta_n\nu_n)\|x_n-p\|^2-\beta_n(1-\beta_n-k)\|x_n-T^n y\|^2 \\
&\quad +\beta_n\{c_n+(1+\nu_n)\lambda_n^2\|\nabla f_{\alpha_n}(x_n)\|^2+k\lambda_n^2\|\nabla f_{\alpha_n}(x_n)\|^2\} \\
&\le (1+\beta_n\nu_n)\|x_n-p\|^2-\beta_n(1-\beta_n-k)\|x_n-T^n y\|^2 \\
&\quad +\beta_n\{c_n+(1+\nu_n)\lambda_n^2 M+\lambda_n^2 M\} \\
&\le (1+\beta_n\nu_n)\|x_n-p\|^2+\beta_n\{c_n+(1+\nu_n)\lambda_n^2 M+\lambda_n^2 M\}. \qquad (3.2.4)
\end{aligned}$$

Thus we have,

$$\|x_{n+1}-p\|^2 \le (1+\beta_n\nu_n)\|x_n-p\|^2+b_n, \qquad (3.2.5)$$

where $b_n=\beta_n\{c_n+(1+\nu_n)\lambda_n^2 M+\lambda_n^2 M\}$. Since $\Sigma_{n=1}^{\infty}\beta_n\nu_n<\infty$, $0<\beta_n<1$, $\sum_{n=1}^{\infty}\beta_n c_n<\infty$, $\sum_{n=1}^{\infty}\lambda_n<\infty$, $0\le k<1$ and $\|\nabla f_{\alpha_n}(x_n)\|\le M$, where M is a constant, we conclude that $\Sigma_{n=1}^{\infty}b_n<\infty$. Therefore, by Lemma 2.1.1, we have

$$\lim_{n\to\infty}\|x_n-p\| \quad \text{exists.}$$

Also, from (3.2.2), we have

$$\lim_{n\to\infty}\|y_n-p\| \quad \text{exists.}$$

Thus, from (3.2.4), we obtain

$$\beta_n(1-\beta_n-k)\|T^n y_n-x_n\|^2<(1+\beta_n\nu_n)\|x_n-p\|^2-\|x_{n+1}-p\|^2+b_n. \qquad (3.2.6)$$

Since T is asymptotically k-strict pseudo-contractive mapping in the intermediate sense with sequence $\{\nu_n\}$, then $\lim_{n\to\infty}\nu_n=0$, and by the conditions (ii) and (iii), we have

$$\lim_{n\to\infty}\|T^n y_n-x_n\|=0. \qquad (3.2.7)$$

By using condition (ii) and equation (3.2.7) in inequality (3.2.3), we obtain

$$\lim_{n\to\infty} \|T^n y_n - y_n\| = 0. \tag{3.2.8}$$

From Algorithm 3.2.1 and (3.2.7), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \beta_n \|T^n y_n - x_n\| = 0. \tag{3.2.9}$$

Since $y_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n}(x_n))$ and by Proposition 2.3.1 (ii), as in [73], we have

$$\begin{aligned}\|y_n - p\|^2 &\le \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\lambda_n\alpha_n\|p\|\|p - x_n\| \\ &\quad + 2\lambda_n\|\nabla f_{\alpha_n}(x_n)\|\|y_n - p\| + 2\lambda_n\|\nabla f_{\alpha_n}(x_n)\|\|x_n - p\|. \end{aligned} \tag{3.2.10}$$

Consequently, by asymptotically k-strict pseudo-contractiveness in intermediate sense with sequence $\{\nu_n\}$ of T, utilizing Lemma 2.1.4 (iii) and inequality (3.2.10), we conclude that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\beta_n)(x_n - p) + \beta_n(T^n y_n - p)\|^2 \\ &= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|T^n y_n - p\|^2 - \beta_n(1-\beta_n)\|T^n y_n - x_n\|^2 \\ &\le (1-\beta_n)\|x_n - p\|^2 + \beta_n\{(1+\nu_n)\|y_n - p\|^2 + k\|y_n - T^n y_n\| + c_n\} \\ &\quad - \beta_n(1-\beta_n)\|T^n y_n - x_n\|^2. \\ &\le (1-\beta_n)\|x_n - p\|^2 + \beta_n(1+\nu_n)\{\|x_n - p\|^2 - \|x_n - y_n\|^2 \\ &\quad + 2\lambda_n\alpha_n\|p\|\|p - x_n\| + 2\lambda_n\|\nabla f_{\alpha_n}(x_n)\|\|y_n - p\| \\ &\quad + 2\lambda_n\|\nabla f_{\alpha_n}(x_n)\|\|x_n - p\|\} + \beta_n k\|y_n - T^n y_n\|^2 + \beta_n c_n \\ &\quad - \beta_n(1-\beta_n)\|T^n y_n - x_n\|^2. \end{aligned}$$

Taking limits both the sides and using the conditions (i)-(iv), equations (3.2.7) and (3.2.8), we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.2.11}$$

From Algorithm 3.2.1, we have

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|P_C(I - \lambda_{n+1}\nabla f_{\alpha_{n+1}})(x_{n+1}) - P_C(I - \lambda_n\nabla f_{\alpha_n})(x_n)\| \\ &\le \|x_{n+1} - x_n\| + \lambda_{n+1}\|\nabla f_{\alpha_{n+1}}(x_{n+1})\|^2 + \lambda_n\|\nabla f_{\alpha_n}(x_n)\|^2. \end{aligned}$$

Taking limits both the sides and using condition (ii) and (3.2.9), we have

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{3.2.12}$$

Since $\|y_{n+1} - y_n\| \to 0$, $\|T^n y_n - y_n\| \to 0$ as $n \to \infty$ and T is uniformly continuous, we obtain from Lemma 2.1.8 that $\|Ty_n - y_n\| \to 0$ as $n \to \infty$. Since $\{x_n\}$ is a bounded sequence in Hilbert space H_1, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to some $x^* \in H_1$. In fact, $x_n \rightharpoonup x^*$.

Indeed, let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \bar{x}$. Assume $x^* \neq \bar{x}$. From Opial condition [143], we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - x^*\| &= \liminf_{i\to\infty} \|x_{n_i} - x^*\| < \liminf_{i\to\infty} \|x_{n_i} - \bar{x}\| \\
&= \lim_{n\to\infty} \|x_n - \bar{x}\| = \liminf_{j\to\infty} \|x_{n_j} - \bar{x}\| \\
&< \liminf_{j\to\infty} \|x_{n_j} - x^*\| = \lim_{n\to\infty} \|x_n - x^*\|.
\end{aligned}$$

This contradict our assumption $x^* \neq \bar{x}$. Hence, $x_{n_j} \rightharpoonup x^*$. This shows that every subsequence of $\{x_n\}$ converges weakly to x^*. This implies that $x_n \rightharpoonup x^*$, and for all $f \in H_1$, we have $f(x_n) \to f(x^*)$. Next we show that $y_n \rightharpoonup x^*$. For all $f \in H_1$, we consider

$$\begin{aligned}
\|f(y_n) - f(x^*)\| &= \|f(y_n) - f(x_n) + f(x_n) - f(x^*)\| \\
&\leq \|f(y_n) - f(x_n)\| + \|f(x_n) - f(x^*)\| \\
&\leq \|f\|\|y_n - x_n\| + \|f(x_n) - f(x^*)\|.
\end{aligned}$$

From (3.2.11), we conclude that $\lim_{n\to\infty} \|f(y_n) - f(x^*)\| = 0$, for all $f \in H_1$. Hence $y_n \rightharpoonup x^*$. Thus we have shown that $\{x_n\}$ and $\{y_n\}$ converges weakly to x^*. Now we will show that x^* is the required solution, that is $x^* \in \text{Fix}(T) \cap \Gamma$. Note that T is uniformly continuous and $\|Ty_n - y_n\| \to 0$, we see that $\|y_n - T^m y_n\| \to 0$, for all $m \in \mathbb{N}$. Thus, by Lemma 2.1.7, we obtain that $x^* \in \text{Fix}(T)$. Now we show that $x^* \in \Gamma$. Let

$$Sw_1 = \begin{cases} \lambda_n \nabla f w_1 + N_C w_1, & \text{if } w_1 \in C, \\ \emptyset, & \text{if } w_1 \notin C, \end{cases} \tag{3.2.13}$$

where $N_C w_1 = \{z \in H_1 : \langle w_1 - u, z\rangle \geq 0 \text{ for all } u \in C\}$. To show that $x^* \in \Gamma$, it is sufficient to show that $0 \in Sx^*$. Let $(w_1, z) \in G(S)$, we have $z \in Sw_1 = \lambda_n \nabla f w_1 + N_C w_1$, and hence, $z - \lambda_n \nabla f w_1 \in N_C w_1$. So, we have $\langle w_1 - u, z - \lambda_n \nabla f w_1\rangle \geq 0$, for all $u \in C$. Since $w_1 \in C$, from Algorithm 3.2.1, we have $y_n = P_C(I - \lambda_n \nabla f_{\alpha_n} x_n)$, so from Proposition 2.3.1 (i), we have

$$\langle x_n - \lambda_n \nabla f_{\alpha_n} x_n - y_n, y_n - w_1\rangle \geq 0,$$

and

$$\langle w_1 - y_n, y_n - x_n + \lambda_n \nabla f_{\alpha_n} x_n\rangle \geq 0.$$

Since, $z - \lambda_n \nabla f w_1 \in N_C w_1$ and $y_n \in C$, it follows that

$$\begin{aligned}
\langle w_1 - y_n, z\rangle &\geq \langle w_1 - y_n, \lambda_n \nabla f w_1\rangle \\
&\geq \langle w_1 - y_n, \lambda_n \nabla f w_1\rangle - \langle w_1 - y_n, y_n - x_n + \lambda_n \nabla f \alpha_n x_n\rangle \\
&\geq \langle w_1 - y_n, \lambda_n \nabla f w_1\rangle - \langle w_1 - y_n, y_n - x_n + \lambda_n \nabla f x_n\rangle \\
&\quad - \lambda_n \alpha_n \langle w_1 - y_n, x_n\rangle \\
&= \langle w_1 - y_n, \lambda_n \nabla f w_1 - \lambda_n \nabla f y_n\rangle + \langle w_1 - y_n, \lambda_n \nabla f y_n - \lambda_n \nabla f x_n\rangle \\
&\quad - \langle w_1 - y_n, y_n - x_n\rangle - \lambda_n \alpha_n \langle w_1 - y_n, x_n\rangle \\
&\geq \langle w_1 - y_n, \lambda_n \nabla f y_n - \lambda_n \nabla f x_n\rangle - \langle w_1 - y_n, y_n - x_n\rangle \\
&\quad - \lambda_n \alpha_n \langle w_1 - y_n, x_n\rangle.
\end{aligned}$$

Taking limit as $n \to \infty$, and by the fact that $y_n \rightharpoonup x^*$, we obtain

$$\langle w_1 - x^*, z\rangle \geq 0, \quad \text{as} \quad n \to \infty.$$

Since $\langle w_1 - x^*, z - 0\rangle \geq 0$, for every $(w_1, z) \in G(S)$. Therefore by maximality of S we have $0 \in Sx^*$. This further implies that $x^* \in \mathrm{VI}(C, \nabla f)$ (see [157, Theorem 3]). Finally Proposition 2.5.3 implies that $x^* \in \Gamma$, therefore $x^* \in \mathrm{Fix}(T) \cap \Gamma$. Thus we have shown that $\{x_n\}$ and $\{y_n\}$ converges weakly to x^* and $x^* \in \mathrm{Fix}(T) \cap \Gamma$. This completes the proof. $\square$