Chapter 2

Preliminaries

In this chapter, we present some known definitions, concepts, notations and results which will be used in rest of the thesis. Throughout the thesis, we denote by $\mathbb{N}$ (respectively, $\mathbb{R}$) the set of all natural numbers (respectively, real numbers). We use the notation $x_n \to x$ (respectively, $x_n \rightharpoonup x$) to indicate that the sequence $\{x_n\}$ converges strongly (respectively, weakly) to x. Let $T : X \to Y$ be a mapping. The range and domain of T are denoted by $R(T)$ and $D(T)$, respectively. If $T : X \to X$ is a mapping, then $\mathrm{Fix}(T)$ denotes the set of all fixed points of T, i.e., $\mathrm{Fix}(T) = \{x \in X : Tx = x\}$.

2.1 Basic Definitions, Properties and Results

We present some elementary results on the real sequences which will be used to study the convergence analysis of several algorithms.

Lemma 2.1.1. *[144, Lemma 1] Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be the sequences of nonnegative real numbers such that*

$$a_{n+1} \leq (1+\delta_n)a_n + b_n, \quad \text{for all } n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim\limits_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.1.2. *[149] Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be the sequences of nonnegative real numbers such that*

$$a_{n+1} \leq a_n + b_n, \quad \text{for all } n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then the limit $\lim\limits_{n\to\infty} a_n$ exists.

Lemma 2.1.3. *[186] Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \sigma_n, \quad \text{for all } n \in \mathbb{N},$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \sigma_n/\gamma_n \leq 0$, *or* $\sum_{n=0}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

The following lemma can be immediately obtained by using the properties of an inner product.

Lemma 2.1.4. *Let X be an inner product space. For all $x,\ y \in X$, we have*

(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$;

(ii) $\|x-y\|^2 \leq \|x\|^2 + \|y\|^2$;

(iii) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$, for all $\lambda \in [0,1]$;

(iv) $\|x-y\|^2 \leq \|x\|^2 + 2\langle y, y-x\rangle$.

Definition 2.1.1. Let K be a nonempty subset of a normed space X. A mapping $T : K \to X$ is said to be:

(i) L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x-y\|, \quad \text{for all } x,\ y \in K;$$

(ii) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x-y\|, \quad \text{for all } x,\ y \in K \text{ and all } n \in \mathbb{N};$$

(iii) k-contraction if there exists $k \in [0,1)$ such that

$$\|Tx - Ty\| \leq k\|x-y\|, \quad \text{for all } x, y \in K;$$

(iv) asymptotically k-strict psuedo-contractive [107] with sequence $\{\nu_n\}$ if there exist a constant $k \in [0,1)$ and a sequence $\{\nu_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} \nu_n = 0$ such that

$$\|T^n x - T^n y\|^2 \leq (1+\nu_n)\|x-y\|^2 + k\|x - T^n x - (y - T^n y)\|^2,$$

for all $x, y \in K$ and all $n \in \mathbb{N}$;

(v) asymptotically k-strict psuedo-contractive in the intermediate sense [160] with sequence $\{\nu_n\}$ if there exist a constant $k \in [0,1)$ and a sequence $\{\nu_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} \nu_n = 0$ such that

$$\|T^n x - T^n y\|^2 \leq (1+\nu_n)\|x-y\|^2 + k\|x - T^n x - (y - T^n y)\|^2 + c_n, \quad (2.1.1)$$

for all $x, y \in K$, $c_n \geq 0$, $n \in \mathbb{N}$ and $c_n \to 0$ as $n \to \infty$.

Definition 2.1.2. Let K be a nonempty subset of a normed space X. A mapping $T : K \to X$ is said to be:

(i) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$;

(ii) averaged [32] if it can be written as

$$T = (1-\alpha)I + \alpha S,$$

where $\alpha \in (0,1)$, I is the identity mapping on K and $S : K \to X$ is a nonexpansive mapping;

(iii) quasi-nonexpansive [150] if $\mathrm{Fix}(T)$ is nonempty and

$$\|Tx - p\| \leq \|x - p\|, \quad \text{for all } x \in K,\ p \in \mathrm{Fix}(T);$$

(iv) firmly quasi-nonexpansive [189] if $\mathrm{Fix}(T)$ is nonempty and

$$\|Tx - p\|^2 \leq \|x - p\|^2 - \|x - Tx\|^2, \quad \text{for all } x \in K,\ p \in \mathrm{Fix}(T);$$

(v) asymptotically quasi-nonexpansive [150] if $\mathrm{Fix}(T)$ is nonempty and there exists a sequence $\{\nu_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} \nu_n = 0$ such that

$$\|T^n x - p\| \leq (1+\nu_n)\|x - p\|, \quad \text{for all } x \in K,\ p \in \mathrm{Fix}(T) \text{ and all } n \in \mathbb{N};$$

(vi) strongly nonexpansive [29, 36] if T is nonexpansive and

$$\lim_{n\to\infty} \|(x_n - y_n) - (Tx_n - Ty_n)\| = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are bounded sequences in K and

$$\lim_{n\to\infty} (\|x_n - y_n\| - \|Tx_n - Ty_n\|) = 0.$$

(vi) strongly quasi-nonexpansive [112] if T is quasi-nonexpansive and

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

whenever $\{x_n\}$ is a bounded sequences in K and

$$\lim_{n\to\infty} (\|x_n - p\| - \|Tx_n - p\|) = 0, \quad \text{for some } p \in \text{Fix}(T).$$

Definition 2.1.3. Let K be a nonempty subset of an inner product space X. A mapping $T : K \to X$ is said to be:

(i) firmly nonexpansive if $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle$ for all $x, y \in K$;

(ii) cutter [36] if $\text{Fix}(T)$ is nonempty and

$$\langle x - Tx, p - Tx\rangle \leq 0, \quad \text{for all } x \in K \text{ and } p \in \text{Fix}(T).$$

Remark 2.1.1. The name cutter was proposed by Cegielski and Censor in [38]. Different names of the cutter are used in the literature. It was introduced and investigated by Bauschke and Combettes [20, Definition 2.2]. Combettes [65] called it $\mathfrak{T}$-class operators, Zaknoon [197], Segal [162] and Censor and Segal [48, 49] called it directed operator. However, Yamada and Ogura (see [189, 190]) and Măruster (see [121]) called it firmly quasi-nonexpansive. In [35], these operators were called separating operators.

Lemma 2.1.5. *[166] Let $\{x_n\}$ and $\{z_n\}$ be the bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ such that $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n, \quad \text{for all } n \in \mathbb{N},$$

and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

A Banach space X is said to satisfy Opial's condition [143] if whenever $\{x_n\}$ is a sequence in X which converges weakly to x, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \text{for all}\, y \in H,\ y \neq x.$$

It is well-known that every Hilbert space H satisfies Opial's condition.

Lemma 2.1.6. *[143, Lemma 2, Demiclosedness Principle] Let K be a nonempty, closed and convex subset of a real Hilbert space H and $T : K \to K$ be a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. If the sequence $\{x_n\} \subset K$ converges weakly to x and the sequence $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$; In particular, if $y = 0$, then $x \in \text{Fix}(T)$.*

Next we have, Demiclosedness principle for asymptotically k-strict psuedo-contractive mapping in the intermediate sense.

Lemma 2.1.7. *[160, Demiclosedness Principle] Let K be a nonempty, closed and convex subset of a real Hilbert space H and $T : K \to K$ be a continuous asymptotically k-strict psuedo-contractive mapping in the intermediate sense. Then $I - T$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in K such that $x_n \rightharpoonup x \in K$ and $\lim_{m\to\infty} \limsup_{n\to\infty} \|x_n - T^m x_n\| = 0$, then $(I - T)x = 0$.*

Sahu et al. [160] extended Lemma 2.1.7 for uniformly continuous mappings and established the following result.

Lemma 2.1.8. *[160, Lemma 2.7] Let K be a nonempty, closed and convex subset of a real Hilbert space H and $T : K \to K$ be a uniformly continuous asymptotically k-strict pseudo-contractive mapping in the intermediate sense with sequence $\{\nu_n\}$. Let $\{x_n\}$ be a sequence in K such that $\|x_{n+1} - x_n\| \to 0$ and $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$. Then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.*

Definition 2.1.4. Let K be a nonempty subset of an inner product space X. A mapping $T : K \to X$ is said to be:

(i) monotone if $\langle x - y, Tx - Ty\rangle \geq 0$ for all $x, y \in K$;

(ii) α-inverse strongly monotone (α-ism) if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2, \quad \text{for all } x, y \in K;$$

(iii) β-strongly monotone if there exists a constant $\beta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \beta\|x - y\|^2, \quad \text{for all } x, y \in K;$$

(iv) strongly positive if there exists a constant $\alpha > 0$ such that

$$\langle Tx, x\rangle \geq \alpha\|x\|^2, \quad \text{for all } x \in K.$$

Definition 2.1.5. Let K be a nonempty convex subset of a normed space X. A mapping $T : K \to X$ is said to be hemicontinuous if it is continuous along the line segments in K.

Proposition 2.1.1. *[32, 36] Let K be a nonempty subset of an inner product space X and $T : K \to X$ be a mapping.*

(i) *If T is ν-ism, then it is $1/\nu$-Lipschitzian;*

(ii) *If T is ν-ism, then γT is $\frac{\nu}{\gamma}$-ism, for $\gamma > 0$;*

(iii) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > \frac{1}{2}$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism;*

(iv) *The composite of finitely many averaged mappings is averaged;*

(v) *If T_1, T_2,...,T_N are averaged mappings on K and have a common fixed point, then*

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 \dots T_N);$$

(vi) *Every firmly nonexpansive mapping is averaged;*

(vii) *Every firmly nonexpansive mapping having a fixed point is a cutter.*

Remark 2.1.2. [36] It can be seen that the class of averaged mappings and the class of firmly nonexpansive mappings are proper subclasses of the class of strongly nonexpansive mappings.

In the following lemma, we collect some important properties of the nonexpansive mappings, which will be used in the sequel.

Lemma 2.1.9. *Let H_1 and H_2 be real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^* such that $A \not\equiv 0$ and $S : H_2 \to H_2$ be a nonexpansive mapping. Then*

(i) $I - S$ *is* $\frac{1}{2}$-inverse strongly monotone;

(ii) $A^*(I - S)A$ *is* $\frac{1}{2\|A\|^2}$-inverse strongly monotone;

(iii) $U := I - \gamma A^*(I - S)A$ *is* $\gamma\|A\|^2$-averaged, for $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$;

(iiia) $\|Ux - Uy\|^2 \leq \|x - y\|^2 + \gamma(\gamma\|A\|^2 - 1)\|(S - I)Ax - (S - I)Ay\|^2$;

(iiib) $Ax \in \mathrm{Fix}(S) \Leftrightarrow x \in \mathrm{Fix}(U)$.

Proof. (i) Since S is nonexpansive, for all $x,\ y \in H_2$, we have

$$\begin{aligned}
\|x - y\|^2 &\geq \|Sx - Sy\|^2 \\
&= \|((I - S)x - (I - S)y) - (x - y)\|^2 \\
&= \|(I - S)x - (I - S)y\|^2 + \|x - y\|^2 - 2\langle x - y, (I - S)x - (I - S)y\rangle.
\end{aligned}$$

Therefore, we get

$$\langle x - y, (I - S)x - (I - S)y\rangle \geq \frac{1}{2}\|(I - S)x - (I - S)y\|^2.$$

This shows that $I - S$ is $\frac{1}{2}$-ism.

(ii) For all $x,\ y \in H_1$.

$$\begin{aligned}\langle x - y, A^*(I - S)Ax - A^*(I - S)Ay\rangle &= \langle Ax - Ay, (I - S)Ax - (I - S)Ay\rangle \\ &\geq \frac{1}{2}\|(I - S)Ax - (I - S)Ay\|^2 \\ &= \frac{1}{2\|A^*\|^2}\|A^*(I - S)Ax - A^*(I - S)Ay\|^2.\end{aligned}$$

Since $\|A\| = \|A^*\|$, we have $A^*(I - S)A$ is $\frac{1}{2\|A\|^2}$-ism.

(iii) From Proposition 2.1.1 (ii), we have $\gamma A^*(I - S)A$ is $\frac{1}{2\gamma\|A\|^2}$-ism. Since $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$, we have $\frac{1}{2\gamma\|A\|^2} > \frac{1}{2}$. So from Proposition 2.1.1 (iii), $U := I - \gamma A^*(I - S)A$ is averaged.

(iiia) For any $x,\ y \in H_1$, we have

$$\begin{aligned}&\|Ux - Uy\|^2 \\ &= \|(I - \gamma A^*(I - S)A)x - (I - \gamma A^*(I - S)A)y\|^2 \\ &= \|x - y\|^2 + \gamma^2\|A\|^2\|(I - S)Ax - (I - S)Ay\|^2 \\ &\quad - 2\gamma\langle x - y, A^*((I - S)Ax - (I - S)Ay)\rangle \\ &= \|x - y\|^2 + \gamma^2\|A\|^2\|(I - S)Ax - (I - S)Ay\|^2 \\ &\quad - 2\gamma\langle Ax - Ay, (I - S)Ax - (I - S)Ay\rangle \\ &\leq \|x - y\|^2 + \gamma^2\|A\|^2\|(I - S)Ax - (I - S)Ay\|^2 - \gamma\|(I - S)Ax - (I - S)Ay\|^2 \\ &\leq \|x - y\|^2 - \gamma\left(1 - \gamma\|A\|^2\right)\|(I - S)Ax - (I - S)Ay\|^2. \qquad (2.1.2)\end{aligned}$$

(iiib) Let there exist $x \in \mathrm{Fix}(U)$ and $z \in H_1$ such that $Az \in \mathrm{Fix}(S)$. It follows that $z \in \mathrm{Fix}(U)$ and in view of inequality (2.1.2), we have

$$\|x - z\|^2 = \|Ux - Uz\|^2 \leq \|x - z\|^2 - \gamma\left(1 - \gamma\|A\|^2\right)\|(I - S)Ax\|^2.$$

Since $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$, so we have $\|(I - S)Ax\|^2 = 0$, thus $Ax \in \mathrm{Fix}(S)$. It is obvious that $Ax \in \mathrm{Fix}(S)$ implies $x \in \mathrm{Fix}(U)$. □

2.2 Some Concepts from Set-valued Analysis

Let X be a real Banach space with topological dual space X^*, and $\langle ., . \rangle$ be the duality pairing between X and X^*.

Let $M : X \rightrightarrows X^*$ be a set-valued operator. The domain, range, graph and inverse of M are defined by

$$D(M) = \{x \in X : M(x) \neq \emptyset\}, \quad R(M) = \bigcup_{x \in D(M)} M(x),$$

$$G(M) = \{(x,\ x^*) : x^* \in Mx\} \quad \text{and} \quad M^{-1}(y) = \{x \in X : y \in Mx\},$$

respectively.

Definition 2.2.1. [110] A set-valued operator $M : X \rightrightarrows X^*$ is said to be

(i) monotone if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $(x, x^*),\ (y, y^*) \in G(M)$.

(ii) maximal monotone if its graph is not properly contained in the graph of any other monotone operator.

Let H be a real Hilbert space. It is well-known that when the set-valued operator $M : H \rightrightarrows H$ is maximal monotone, then for each $x \in H$ and $\lambda > 0$, there is a unique $z \in H$ such that $x \in (I + \lambda M)z$; See, for example [33, 167, 198] for further detail. In this case, the operator $J_\lambda^M := (I + \lambda M)^{-1}$ is called resolvent operator of M with parameter λ. It is known that J_λ^M is a single-valued and firmly nonexpansive mapping. Indeed, for any given $u \in H$, let $x, y \in J_\lambda^M(u)$. Then, $x, y \in (I + \lambda M)^{-1}(u)$ and thus $u - x \in \lambda Mx$ and $u - y \in \lambda My$. The monotonicity of λM implies that

$$\langle u - x - (u - y), x - y \rangle \geq 0.$$

This implies that $\|x - y\| \leq 0$ and thus $x = y$. Hence, J_λ^M is single-valued.

Next we show that J_λ^M is firmly nonexpansive mapping. For any $x, y \in H$, let

$$J_\lambda^M(x) = (I + \lambda M)^{-1}(x) \text{ and } J_\lambda^M(y) = (I + \lambda M)^{-1}(y).$$

Then

$$x \in (I + \lambda M)(J_\lambda^M(x)) \text{ and } y \in (I + \lambda M)(J_\lambda^M(y)).$$

It follows that

$$\frac{1}{\lambda}\left(x - (J_\lambda^M(x))\right) \in M(J_\lambda^M(x)) \text{ and } \frac{1}{\lambda}\left(y - (J_\lambda^M(y))\right) \in M(J_\lambda^M(y)).$$

The monotonicity of M implies that

$$\left\langle J_\lambda^M(x) - J_\lambda^M(y), \frac{1}{\lambda}(x - J_\lambda^M(x)) - \frac{1}{\lambda}(y - J_\lambda^M(y)) \right\rangle \geq 0,$$

that is,

$$\left\langle J_\lambda^M(x) - J_\lambda^M(y), x - y \right\rangle \geq \left\| J_\lambda^M(x) - J_\lambda^M(y) \right\|^2.$$

Thus, J_λ^M is firmly nonexpansive.

Also, $0 \in M(x) \Leftrightarrow x \in \text{Fix}(J_\lambda^M)$. The set $M^{-1}(0) = \{z \in H : 0 \in Mz\}$ is called the set of zero points. Further, the set $\text{Fix}(J_\lambda^M)$ of fixed points of nonexpansive mapping J_λ^M is closed and convex. Since $M^{-1}(0) = \text{Fix}(J_\lambda^M)$, $M^{-1}(0)$ is closed and convex.

Lemma 2.2.1. *[33, 130] Let H be a real Hilbert space. Let $M : H \rightrightarrows H$ be a maximal monotone set-valued operator and $f : H \to H$ be an α-ism operator. Then the operator $J_\lambda^M(I - \lambda f)$ is averaged for $\lambda \in (0, 2\alpha)$.*

Lemma 2.2.2. *[168, Theorem 6.5.4] Let H be a real Hilbert space. Let $M : H \rightrightarrows H$ be a maximal monotone set-valued operator and $(x, u) \in H \times H$. If $\langle u - v, x - y \rangle \geq 0$ for all $(y, v) \in M$, then $(x, u) \in M$.*

2.3 Metric Projection

Let K be a nonempty, closed and convex subset of a real Hilbert space H. For an arbitrary point $x \in H$, consider the set

$$\left\{ z \in K : \|x - z\| = \min_{y \in C} \|x - y\| \right\}.$$

It is known that this set is always a singleton. Let P_K be a mapping from H onto K satisfying

$$\|x - P_K x\| = \min_{y \in C} \|x - y\|.$$

Such a mapping P_K is called the metric projection. It is also known as the nearest point projection, proximity mapping or best approximation operator.

Proposition 2.3.1. *[7, 84] Let K be a nonempty, closed and convex subset of a Hilbert space H. Then*

(i) *$z = P_K x$ if and only if $\langle x - z, y - z \rangle \geq 0$* for all *$y \in K$;*

(ii) *$z = P_K x$ if and only if $\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$ for all $y \in K$;*

(iii) *$\langle P_K x - P_K y, x - y \rangle \geq \|P_K x - P_K y\|^2$ for all $x, y \in H$;*

(iv) *P_K is nonexpansive, that is, $\|P_K(x) - P_K(y)\| \leq \|x - y\|$ for all $x, y \in H$;*

(v) *P_K is monotone, that is, $\langle P_K(x) - P_K(y), x - y \rangle \geq 0$ for all $x, y \in H$.*

2.4 Basic Concepts and Results from Geometry of Banach Spaces

Let X be a real Banach space with topological dual space X^*, and $\langle ., . \rangle$ be the duality pairing between X and X^*. The normalized duality mapping $J : X \rightrightarrows X^*$ is defined by

$$J(x) := \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \text{ for all } x \in X\}.$$

For further details on normalized duality, we refer to [7, 61, 167]. Let $S(X)$ be the unit sphere centered at the origin of X. The space X is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \text{ exists} \tag{2.4.1}$$

for all $x, y \in S(X)$. The space X is said to have a uniformly Gâteaux differentiable norm if for each $y \in S(X)$, the limit (2.4.1) is attained uniformly for $x \in S(X)$. X is said to be uniformly smooth if the limit (2.4.1) converges uniformly in $x, y \in S(X)$. A Banach space X is said to be strictly convex if $\|(x+y)/2\| < 1$ whenever $x, y \in S(X)$ and $x = y$. The space X is said to be uniformly convex if for all $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $x, y \in S(X)$ and $\|x - y\| \geq \epsilon$ imply $\|(x + y)/2\| \leq 1 - \delta$. The duality mapping J from a smooth Banach space X into X^* is said to be weakly sequentially continuous if $Jx_n \overset{*}{\rightharpoonup} Jx$ whenever $x_n \rightharpoonup x$, where $\overset{*}{\rightharpoonup}$ means weak* convergence in the dual space. We also know the following properties (see [167] for detail):

(i) $J(x) \neq \emptyset$ for each $x \in X$;

(ii) If X is strictly convex, then J is one to one, that is,

$$x \neq y \Rightarrow J(x) \cap J(y) = \emptyset;$$

(iii) If X is reflexive, then J is a mapping of X onto X^*;

(iv) If X is smooth, then the duality mapping J is single valued;

(v) X is uniformly convex if and only if X^* is uniformly smooth.

Lemma 2.4.1. *[17, 27, 157] Let X be a strictly convex and reflexive Banach space and let $M : X \rightrightarrows X^*$ be a monotone operator. Then M is maximal monotone if and only if $R(J + \lambda M) = X^*$ for all $\lambda > 0$.*

Lemma 2.4.2. *[28] Let K be a nonempty closed convex subset of a uniformly convex Banach space X and $T : K \to X$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence of K such that $x_n \rightharpoonup x$ and $\|(I - T)x_n\| \to 0$, then $(I - T)x = 0$, that is, x is a fixed point of T, where I is the identity mapping on K.*

Lemma 2.4.3. *[110, 185] Let X be a uniformly convex Banach space. Then for any given number $r > 0$, there exists a continuous strictly increasing function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|tx + (1 - t)y\|^2 \leq t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)g(\|x - y\|),$$

for all $x,\ y \in X$ with $\|x\| \leq r$ and $\|y\| \leq r,\ \ t \in [0, 1]$.

Lemma 2.4.4. *[167] Let X be a smooth Banach space. Then $\langle x - y, Jx - Jy\rangle \geq 0$ for all $x,\ y \in X$. Furthermore, if X is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

Next we give the characterization of a metric projection in the setting of a strictly convex and reflexive Banach space X.

Lemma 2.4.5. *[167] Let X be a strictly convex and reflexive Banach space, K be a nonempty, closed and convex subset of X, $x \in X$ and $z \in K$. Then the following conditions are equivalent*

(i) *$z = P_K x$;*

(ii) $\langle z - y, J(x - z)\rangle \geq 0, \quad$ for all $y \in K$.

Let X be a reflexive, strictly convex and smooth Banach space. Let $M : X \rightrightarrows X^*$ be a maximal monotone operator. Then for $\lambda > 0$ and $x \in X$, Takahashi and Yao [174] consider the following sets:

$$Q_\lambda x = \{z \in X : 0 \in J(z - x) + \lambda M(z)\},$$

and

$$J_\lambda x = \{z \in X : Jx \in Jz + \lambda M(z)\}.$$

They also proved that these sets are singleton. In other words,

$$Q_\lambda = (I + \lambda J^{-1}M)^{-1} \text{ and } J_\lambda = (J + \lambda M)^{-1}J.$$

Such Q_λ and J_λ are known as metric resolvent and relative resolvent of M for $\lambda > 0$.

Remark 2.4.1. In the setting of Hilbert spaces, the duality mapping J reduces to the identity mapping I, therefore $Q_\lambda = J_\lambda$.

Now we gather, some well-known properties of relative resolvent [110, 123]:

(i) $J_\lambda : X \to D(M)$ is a single-valued mapping;

(ii) $M^{-1}(0) = \text{Fix}(J_\lambda)$ for each $\lambda > 0$.

The modulus of smoothness of a Banach space X is the function $\rho : [0,\ \infty) \to [0,\ \infty)$ defined by

$$\rho(\tau) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x,\ y \in X, \|x\| = 1,\ \|y\| = \tau\right\}.$$

It is known that X is uniformly smooth [2] if and only if $\lim_{\tau\to 0}\rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be q-uniformly smooth [2] if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$, where c is q-uniformly smooth constant. For more information about geometry of Banach spaces, we refer [7, 61, 64, 153, 167].

Lemma 2.4.6. *[185] Let X be a 2-uniformly smooth Banach space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|\kappa y\|^2, \quad \text{for all } x, y \in X.$$

where $\kappa > 0$ is the 2-uniformly smooth constant of X.

Let X be a smooth Banach space. Following Alber [5] and Kamimura and Takahashi [104], let $\phi : X \times X \to \mathbb{R}$ be the mapping defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \text{for all } x,\ y \in X. \tag{2.4.2}$$

If $X = H$ is a Hilbert space, then we have $\phi(x, y) = \|x - y\|^2$ for all $x,\ y \in X$. We know that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \text{for all } x,\ y \in X. \tag{2.4.3}$$

If X is strictly convex, then

$$\phi(x, y) = 0 \Leftrightarrow x = y. \tag{2.4.4}$$

Let K be a nonempty, closed and convex subset of a smooth, reflexive and strictly convex Banach space X. For an arbitrary point $x \in X$, consider the set

$$\left\{ x_0 \in K : \phi(x, x_0) = \min_{y \in K} \phi(x, y) \right\}.$$

It is known that this set is always a singleton, see [5]. Let Π_K be a mapping from X onto K satisfying

$$\phi(x, \Pi_K) = \min_{y \in K} \phi(x, y). \tag{2.4.5}$$

Such a mapping Π_K is called the generalized projection from X onto K.

Lemma 2.4.7. *[5, 104] Let X be a reflexive, strictly convex and smooth Banach space, K be a nonempty closed convex subset of X and $x \in X$. Then*

(i) $x_0 = \Pi_K x \Leftrightarrow \langle x_0 - y, Jx - Jx_0 \rangle \geq 0$ for each $y \in K$;

(ii) $\phi(y, \Pi_K x) + \phi(\Pi_K x, x) \leq \phi(y, x)$ for each $y \in K$.

Since the normalized duality mapping J on a Hilbert space is the identity mapping, we have $P_K = \Pi_K$ in the setting of Hilbert spaces.

Lemma 2.4.8. *[104] Let X be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\phi(x_n, y_n) = 0$, then $\|x_n - y_n\| = 0$.*

Lemma 2.4.9. *[104] Let $r > 0$ and let X be a uniformly convex and smooth Banach space. Then*

$$g(\|y - z\|) \leq \phi(y, z),$$

for all $y, z \in X_r = \{w \in X : \|w\| \leq r\}$, where $g : [0, \infty) \to [0, \infty)$ is a continuous, strictly increasing and convex function with $g(0) = 0$.

Definition 2.4.1. Let K be a nonempty, closed and convex subset of a smooth Banach space X and $T : K \to K$ be a mapping. A point $a \in K$ is called an asymptotic fixed point [154] of T if there exists a sequence $\{x_n\}$ such that $x_n \rightharpoonup a$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points is denoted by $\widehat{\text{Fix}}(T)$.

The mapping $T : K \to K$ is said to be:

(i) firmly nonexpansive type [110] if

$$\phi(Tx,Ty)+\phi(Ty,Tx)+\phi(Tx,x)+\phi(Ty,y) \\ \leq \phi(Tx,y)+\phi(Ty,x), \text{ for all } x,\ y\in K; \quad (2.4.6)$$

(ii) relatively nonexpansive (see [110, 123, 124]) if the following properties are satisfied:

(iia) $\mathrm{Fix}(T)\neq\emptyset$;

(iib) $\phi(p,Tx)\leq\phi(p,x)$ for all $p\in\mathrm{Fix}(T), x\in K$;

(iic) $\widehat{\mathrm{Fix}}(T)=\mathrm{Fix}(T)$.

(iii) strongly relatively nonexpansive (see [110, 154]) if the following properties are satisfied:

(iiia) T is relative nonexpansive;

(iiib) $\lim_{n\to\infty}\phi(Tx_n,x_n)=0$ whenever $\{x_n\}$ is bounded sequence in K and $\lim_{n\to\infty}(\phi(p,x_n)-\phi(p,Tx_n))=0$ for some $p\in\mathrm{Fix}(T)$.

(iv) nonspreading [111] if

$$\phi(Tx,Ty)+\phi(Ty,Tx)\leq\phi(Tx,y)+\phi(Ty,x), \quad \text{for all } x,\ y\in K;$$

(v) generalized nonspreading [96, 109] if there are $\alpha,\ \beta,\ \gamma,\ \delta\in\mathbb{R}$ such that

$$\alpha\phi(Tx,Ty)+(1-\alpha)\phi(x,Ty)+\gamma\{\phi(Ty,Tx)-\phi(Ty,x)\} \\ \leq\beta\phi(Tx,y)+(1-\beta)\phi(x,y)+\delta\{\phi(y,Tx)-\phi(y,x)\}, \text{ for all } x,\ y\in K. \quad (2.4.7)$$

Remark 2.4.2. [110, 123] Let X be a reflexive, strictly convex and smooth Banach space with its dual X^* and $M:X\rightrightarrows X^*$ be a maximal monotone operator. Then the relative resolvent $J_\lambda:X\to D(M)$ of M is firmly nonexpansive type mapping.

Lemma 2.4.10. *[110, Theorem 5.2] Let X be a strictly convex Banach space whose norm is uniformly Gâteaux differentiable. Let K be a nonempty closed convex subset of X and $T:K\to K$ be a firmly nonexpansive type mapping such that* Fix(T) *is nonempty. Then T is strongly relatively nonexpansive.*

Lemma 2.4.11. *[124, Proposition 2.4] Let X be a strictly convex and smooth Banach space, K be a closed convex subset of X, and T be a relatively nonexpansive mapping from K into itself. Then* Fix(T) *is closed and convex.*

Remark 2.4.3. A generalized nonspreading mapping is nonspreading if $\alpha = 1$, $\gamma = 1$, $\beta = 1$ and $\delta = 0$.

Remark 2.4.4. If X is a Hilbert space, then $\phi(x, y) = \|x - y\|$. Therefore, we obtain

$$\begin{aligned}&\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 + \gamma\{\|Tx - Ty\|^2 - \|x - Ty\|^2\}\\ &\leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 + \delta\{\|Tx - y\|^2 - \|x - y\|^2\}, \text{ for all } x,\ y \in K.\end{aligned} \tag{2.4.8}$$

This implies that

$$\begin{aligned}&(\alpha+\gamma)\|Tx - Ty\|^2 + \{1 - (\alpha+\gamma)\}\|x - Ty\|^2\\ &\qquad \leq (\beta+\delta)\|Tx - y\|^2 + \{1 - (\beta+\delta)\}\|x - y\|^2, \text{for all } x,\ y \in K.\end{aligned} \tag{2.4.9}$$

In this case, T is called generalized hybrid mapping (see, [96, 109]). Observe that if $\mathrm{Fix}(T) \neq \emptyset$, then $\phi(p, Ty) \leq \phi(p, y)$, for all $p \in \mathrm{Fix}(T)$ and $y \in K$. Indeed, putting $x = p \in \mathrm{Fix}(T)$ in (2.4.7), we obtain

$$\begin{aligned}&\alpha\phi(p, Ty) + (1-\alpha)\phi(p, Ty) + \gamma\{\phi(Ty, p) - \phi(Ty, p)\}\\ &\qquad \leq \beta\phi(p, y) + (1-\beta)\phi(p, y) + \delta\{\phi(y, p) - \phi(y, p)\}.\end{aligned} \tag{2.4.10}$$

So, we have

$$\phi(p, Ty) \leq \phi(p, y). \tag{2.4.11}$$

Lemma 2.4.12. *[109] Let X be a uniformly smooth and strictly convex Banach space, K be a nonempty closed convex subset of X and T be a generalized nonspreading mapping of K into itself such that* $\mathrm{Fix}(T) \neq \emptyset$. *Then* $\widehat{\mathrm{Fix}}(T) = \mathrm{Fix}(T)$ *and* $\mathrm{Fix}(T)$ *is closed and convex.*

2.5 Variational Inequality Problem

The theory of variational inequalities is well known and well developed branch of nonlinear analysis and optimization. It has many applications in different areas of science, social science, engineering, and management. There are several monographs on variational inequalities, but we mention here few [8, 76, 108]. Let K be a nonempty, closed and convex subset of a real Hilbert space H and $f : K \to H$ be a mapping. The variational inequality problem $\mathrm{VIP}(K, f)$ is to:

$$\text{Find } x^* \in K \quad \text{such that} \quad \langle f(x^*), x - x^*\rangle \geq 0, \quad \text{for all } x \in K. \tag{2.5.1}$$

Another problem closely related to VIP(K, f) is known as the Minty variational inequality problem MVIP(K, f). The MVIP(K, f) is to:

$$\text{Find } x^* \in K \quad \text{such that} \quad \langle f(x), x - x^* \rangle \geq 0, \quad \text{for all } x \in K. \tag{2.5.2}$$

The trivial unlikeness of two proposed problems is the linearity of variational inequalities. In fact, the Minty variational inequality MVIP(K, f) is linear but the variational inequality VIP(K, f) is not. However, under the (hemi)continuity and monotonicity of f, the solution sets of these problems are same.

The following lemma establishes the relation between variational inequality problem (2.5.1) and Minty variational inequality problem (2.5.2).

Lemma 2.5.1 (Minty Lemma)**.** *[108] Let $F : K \to H$ be a monotone and continuous along the segments (i.e., $F(x + ty) \to F(x)$ as $t \to 0$). Then the solution sets of VIP(K, F) and MVIP(K, F) are same. Moreover, if F is strongly monotone, then the solution of VIP(K, F) is unique.*

Let $f : K \to \mathbb{R}$ be a given function. Consider the constrained minimization problem:

$$\min_{x \in K} f(x). \tag{2.5.3}$$

If f is differentiable, then the VIP$(K, \nabla f)$ provides the necessary and sufficient conditions for a solution of the problem (2.5.3). For further details and applications of variational inequalities, we refer to [8, 108] and the references therein.

The following result provides the equivalence between a variational inequality problem and a fixed point problem.

Proposition 2.5.1. *Let K be a nonempty closed convex subset of a real Hilbert space H and $F : K \to H$ be a mapping. Then $x^* \in K$ is a solution of* VIP(K, F) *if and only if for any $\gamma > 0$, x^* is a fixed point of the mapping $P_K(I - \gamma F) : K \to K$, that is,*

$$x^* = P_K(x^* - \gamma F(x^*)), \tag{2.5.4}$$

where $P_K(x^* - \gamma F(x^*))$ denotes the projection of $(x^* - \gamma F(x^*))$ onto K, and I is the identity mapping.

In view of the above proposition and discussion, we have the following proposition.

Proposition 2.5.2. *Let K be a nonempty closed convex subset of a real Hilbert space H and $F : K \to H$ be a convex and Fréchet differential function. Then the following statement are equivalent:*

(i) $x^* \in K$ *is a solution of (2.5.3);*

(ii) $x^* \in K$ *solves* VIP(K, F) *(2.5.1);*

(iii) $x^* \in K$ *is a solution of (2.5.4).*

From the above equivalence, we have the following gradient projection method.

Theorem 2.5.2. [Projection Gradient Method] *Let $F : K \to H$ be a L-Lipschitzian and β-strongly monotone mapping. Let $\gamma > 0$ be a constant such that $\gamma < \frac{2\beta}{L^2}$. Then,*

(i) *$P_K(I-\gamma F) : K \to K$ is a contraction mapping and there exist a solution $x^* \in K$ of the* VIP(K, F)*;*

(ii) *The sequence $\{x_n\}$ generated by the following iterative process:*

$$x_{n+1} = P_K(I - \gamma F)(x_n), \quad \text{for all } n \in \mathbb{N},$$

converges strongly to a solution x^* of the VIP(K, F).

In view of Proposition 2.5.2 and Theorem 2.5.2, we have the following method for finding an approximate solution of a convex and differentiable minimization problem.

Theorem 2.5.3. *Let $f : K \to \mathbb{R}$ be a convex and differentiable function such that the gradient ∇f is L-Lipschitzian and β-strongly monotone mapping. Let $\{\gamma_n\}$ be a sequence of strictly positive real numbers such that*

$$0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < \frac{2\beta}{L^2}. \tag{2.5.5}$$

Then the sequence $\{x_n\}$ generated by the following gradient projection method

$$x_{n+1} = P_K(I - \gamma\nabla f)(x_n), \quad \text{for all } n \in \mathbb{N}, \tag{2.5.6}$$

converges strongly to a unique solution of the minimization problem (2.5.3).

The sequence $\{x_n\}$ generated by the method (2.5.6) converges weakly to a unique solution of the minimization problem (2.5.3) even when ∇f is not necessary strongly monotone.

The following result provide an equivalence between SFP (1.1.2) and variational inequality problem (2.5.1).

Proposition 2.5.3. *[39] Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint operator A^*. For given $x^* \in H_1$, the following statement are equivalent.*

(i) x^* *solves* SFP *(1.1.2);*

(ii) x^* *solves fixed point equation* $P_C(I - \lambda A^*(I - P_Q)A)x^* = x^*$*;*

(iii) x^* *solves variational inequality problem* (VIP) *of finding* $x^* \in C$ *such that*

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad \text{for all } x \in C,$$

where $\nabla f = A^*(I - P_Q)A$.

2.6 Variational Inclusion

In 2003, Fang and Huang [78] considered the following variational inclusion problem:

$$\text{Find } x^* \in H \ \text{ such that } \ 0 \in f(x^*) + M(x^*), \tag{2.6.1}$$

where 0 is the zero vector in the Hilbert space H, $f : H \to H$ is a single-valued (nonlinear) mapping and $M : H \rightrightarrows H$ be a set-valued operator.

When M is maximal monotone and f is strongly monotone and L-Lipschitzian, problem (2.6.1) has been studied by Huang [98]. Let $\phi : H \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. It is known [157] that the subdifferential $\partial\phi$ of ϕ defined by

$$\partial\phi(x) = \{z \in H : \phi(x) + \langle z, y - x \rangle \leq \phi(y), \text{ for all } y \in H\}, \quad x \in H,$$

is a maximal monotone operator. If $M = \partial\phi$, then problem (2.6.1) reduces to the following problem:

$$\text{Find } \ x^* \in H \text{ such that } \langle fx^*, y - x^* \rangle + \phi(x^*) - \phi(y) \geq 0, \text{ for all } y \in H,$$

which is called a nonlinear variational inequality and has been studied by many authors, see, for example, [16, 85, 86, 87, 138, 165]. If in addition $f \equiv 0$ then problem (2.6.1) reduces to the following convex minimization problem:

$$\text{Find } \ x^* \in H \text{ such that } \phi(x^*) \leq \phi(y), \text{ for all } y \in H.$$

Let K be a nonempty, closed and convex subset of H. If we define

$$M_h v = \begin{cases} hv + N_K v, & \text{if } v \in K, \\ \emptyset, & \text{if } v \notin K, \end{cases} \tag{2.6.2}$$

where $N_K(v) = \{z \in H : \langle y - v, z\rangle \leq 0, \text{ for all } y \in K\}$ is the normal cone of K at a point $v \in K$. $h : K \to H$ is a given mapping, then under certain continuity assumption on h, Rockafellar [157, Theorem 3] showed that M_h is a maximal monotone operator and

$$M_h^{-1}(0) = \mathrm{VIP}(K, h). \tag{2.6.3}$$

So, if $M \equiv N_K$, then problem (2.6.1) reduces to the following problem:

$$\text{Find } x^* \in K \text{ such that } \langle f(x^*), y - x^*\rangle \geq 0, \text{ for all } y \in K,$$

which is the classical variational inequality, see [91, 165].

If $M \equiv 0$ and $f \equiv I - T$ where I is an identity mapping and $T : H \to H$ is a nonlinear mapping, then problem (2.6.1) is equivalent to the fixed point problem of T:

$$\text{Find } x^* \in H \text{ such that } x^* = Tx^*. \tag{2.6.4}$$

Further, if $H = \mathbb{R}^n$ the problem (2.6.1) becomes the generalized equation introduced by Robinson [155]. If $f \equiv 0$ and M is maximal monotone operator, then the problem (2.6.1) becomes the inclusion problem:

$$\text{Find } x^* \in H \text{ such that } 0 \in M(x^*). \tag{2.6.5}$$

It is introduced by Martinet [120] and generalized by Rockafellar [158]. Problem (2.6.5) is very important in the area of optimization and related fields. One of the most popular method for solving (2.6.5) is the proximal point algorithm. Later many authors studied problem (2.6.5), see Brézis and Lions [25], Lions [115], Passty [145], Güler [88], Kamimura and Takahashi [103, 105] and Solodov and Svaite [164].

Thus notion of variational inclusion problems provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complimentarity problems, optimal control, mathematical economics, equilibria, game theory and so forth. In the last decade, it has been extended and generalized; See for example [1, 3, 4, 52, 53, 55, 63, 69, 74, 78, 79, 80, 92, 97, 99, 100, 106, 139, 159, 178, 179] and the references therein.

2.7 Equilibrium Problems

Let K be a nonempty, closed and convex subset of a uniformly smooth, strictly convex and reflexive Banach space X and $F : K \times K \to \mathbb{R}$ be a bifunction. The equilibrium problem (in short, EP) is to find $x \in K$ such that

$$F(x, y) \geq 0, \quad \text{for all } y \in K. \tag{2.7.1}$$

The set of solutions of (2.7.1) is denoted by $\mathrm{EP}(K, F)$ i.e.,

$$\mathrm{EP}(K, F) = \{x \in K : F(x, y) \geq 0, \quad \text{for all } y \in K\}.$$

The EP (2.7.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequality problems, complementarity problems, saddle point problems, Nash equilibrium problems and others, see, for instance, [24, 42, 81, 171] and the references therein. It was first considered by Nikaido and Isoda [136] to prove the existence of a solution of a game problem. The key role in the theory of EP's was made by Ky Fan [77]. After the work of Blum and Otteli [24] many mathematician have started to study the EP again. For further details on equilibrium problems, we refer to [11, 21, 22, 23, 24, 26, 50, 51, 82, 83, 89] and the references therein.

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in K$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in K$;

(A3) $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$ for all $x, y, z \in K$;

(A4) for each $x \in X$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Takahashi and Zembayashi [176] obtained the following result.

Lemma 2.7.1. *Let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). For $r > 0$, define a resolvent operator of F by $T_r^F : X \to K$, for all $x \in X$, by*

$$T_r^F x = \left\{ z \in K : F(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \text{ for all } y \in K \right\}. \tag{2.7.2}$$

Then the following assertions hold:

(i) *T_r^F is single-valued;*

(ii) *T_r^F is a firmly nonexpansive type mapping;*

(iii) $\mathrm{Fix}(T_r^F) = \mathrm{EP}(K, F)$;

(iv) $\mathrm{EP}(K, F)$ *is closed and convex.*

If we take $X = H$ a Hilbert space, then J reduces to the identity operator. Now we have the following well known Lemma.

Lemma 2.7.2. *[67] Let $F : K \times K \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). For $r > 0$, define a resolvent operator of F by $T_r^F : H \to K$, for all $x \in H$, by*

$$T_r^F x = \left\{ z \in K : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \text{ for all } y \in K \right\}. \tag{2.7.3}$$

Then the following assertions hold:

(i) T_r^F *is single-valued;*

(ii) T_r^F *is a firmly nonexpansive mapping;*

(iii) $\mathrm{Fix}(T_r^F) = \mathrm{EP}(K, F)$;

(iv) $\mathrm{EP}(K, F)$ *is closed and convex.*