Chapter 1

Introduction

1.1 Split Feasibility Problem

The split feasibility problem (in short, SFP) is formulated as:

$$\text{Find } x^* \in C \quad \text{such that} \qquad Ax^* \in Q, \tag{1.1.1}$$

where C and Q are nonempty, closed and convex subsets of finite dimensional Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and A is an $m \times n$ real matrix. It was introduced by Censor and Elfving [44] in 1994.

The theory of split feasibility problems is a rich source of inspiration in mathematical, engineering and biological sciences. It is an effective tool to study the existence of solutions of constrained and inverse problems arising in optimization, control theory, operational research, engineering sciences, medical sciences, etc. Due to the extraordinary utility and broad applicability in many areas of applied mathematics, most notably in image denoising, signal processing, image reconstruction, approximation theory, control theory, biomedical engineering, communications and geophysics with particular progress in intensity modulated radiotherapy, the split feasibility problems are continuously receiving great attentions, see for instance [6, 19, 32, 46, 57, 122, 200] and the references therein.

Many iterative methods have been proposed in the literature to solve the SFP (1.1.1), see for example [31, 71, 151, 161, 182, 188, 191, 192, 201, 202] and the references therein. Censor and Elfving [44] proposed a method to calculate the solutions of the SFP (1.1.1) which is mainly dependent on the inverse of the matrix A, where it was assumed that A^{-1} exists. Since involvement of the computation of the inverse of A is itself a big task, this method could not attract many people. Byrne [31] proposed the so called CQ algorithm for solving SFP (1.1.1). Compared with the Censor and Elfving's algorithm [44], Byrne's algorithm [31] can be easily executed, since it deals only with the orthogonal projections and there is no need to compute the matrix inverse. However, to implement the CQ algorithm [31], one has to compute the largest eigenvalue of the matrix $A^\top A$, which is again not an easy task, here $A^\top$ is the transpose of the matrix A. Not only this, there are several questions. Namely, what will happen when C and Q are such whose projection could not be calculated? Does there exist a way to select the step size in CQ algorithm [31] which does not depend on the largest eigenvalue of the matrix $A^\top A$? To overcome with all these difficulties, various authors proposed and studied several modifications of the CQ algorithm, see for example [31, 113, 117, 151, 191, 192, 200] and references therein.

Xu [188] considered the split feasibility problem in the setting of infinite-dimensional Hilbert spaces, which is defined as:

$$\text{Find } x^* \in C \quad \text{such that} \quad Ax^* \in Q, \tag{1.1.2}$$

where C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively, and A is a bounded linear operator from H_1 to H_2. We use Γ to denote the solution set of the SFP (1.1.2), i.e.,

$$\Gamma = \{x \in C : Ax \in Q\} = C \cap A^{-1}Q.$$

He extended the CQ algorithm for SFP (1.1.2). One can find a large number of articles on CQ algorithm and its varient forms in the literature, see, for instance, [39, 40, 41, 71, 182, 184, 193, 195]. Among all the iterative methods proposed in the literature, the simplest iterative procedure is the gradient projection method. Infact, Xu [188] has shown that the CQ algorithm [188] is indeed a special case of the gradient projection method for solving convex minimization problems. The CQ algorithm can be viewed as a fixed point algorithm for averaged mappings, and hence fixed point algorithms can be used to solve SFP (1.1.2). There are large number of fixed point methods in the literature, see, e.g., [18, 54, 55, 61, 90, 101, 116, 119, 127, 146, 148, 163] which can be used to solve SFP (1.1.2). The weak convergence of the CQ algorithm is studied in [188]. Xu [188] also applied Mann's algorithm to solve SFP (1.1.2) and proposed the averaged CQ algorithm. The weak convergence of the sequences generated by averaged CQ algorithm is also studied under certain assumptions. In most of the cases, strong convergence is more desirable than the weak convergence. Therefore, Xu [188] considered Tikhonovs regularization for the SFP (1.1.2) and established the strong convergence results. It is also shown that the minimum-norm solutions of SFP (1.1.2) can be obtained. Further, Xu [187] presented a modified KM algorithm and established its strong convergence. Wang and Xu [182] proposed a modified CQ algorithm and proved its strong convergence by introducing an approximating curve. Dang and Gao [71] presented a KM-CQ-like algorithm for solving SFP (1.1.2) and proved its strong convergence.

During the last decade, the split feasibility problems have been extended, generalized and studied in the finite / infinite dimensional spaces. Several iterative methods have been proposed and analyzed. Weak and strong convergence results for the algorithms have been studied. For further details on split feasibility problems, their generalizations and applications, we refer to [7, 30, 31, 32, 39, 40, 41, 44, 45, 47, 66, 68, 72, 73, 113, 114, 126, 132, 140, 151, 152, 181, 183, 187, 188, 191, 203, 204] and the references therein. A comprehensive theory on SFPs can be found in the recent survey article in the book [10, Chapter 9].

Very recently, using the methods of [135, 141, 164], Takahashi [169] first attempted to consider the split feasibility problem (1.1.2) in the setting of Banach spaces. He proposed so called hybrid method and proved the strong and weak convergence of the sequences generated by the proposed method. The results presented in [169] seems to be the first outside Hilbert spaces.

1.2 Split Equality Problem

Let H_1, H_2 and H_3 be real Hilbert spaces, and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Moudafi [131] introduced the following split equality problem (in short, SEP):

$$\text{Find } x \in C \text{ and } y \in Q \quad \text{such that} \quad Ax = By. \tag{1.2.1}$$

This kind of problem allows asymmetric and partial relations between the variables x and y. The decomposition methods in partial differential equations (in short, PDEs) and intensity modulated radiation therapy (in short, IMRT) can be modelled as SEP. In decision sciences, this problem allows us to consider agents who interplay only via some components of their control variables; for further details, see Attouch et al. [15] and the references therein. In IMRT this amounts to envisage a weak coupling between the vector of doses absorbed in all voxels and that of the radiation intensity

[43]. So this is one of the most important and useful generalization of split feasibility problem.

In order to solve problem (1.2.1), Moudafi [131] used a fixed point formulation and proposed alternating CQ algorithm. Further, Byrne and Moudafi [34] proposed simultaneous split equality algorithm for finding the approximate solutions of SEP (1.2.1). Later on Moudafi [132] proposed relaxed alternating CQ algorithm for solving SEP (1.2.1). Several different and innovative iterative scheme for solving the SEP (1.2.1) can be found in the literature, see [34, 56, 58, 59, 60, 75, 118, 131, 132, 180] and the references therein.