Chapter 8

Iterative Methods for Split Hierarchical Monotone Variational Inclusion Problems

8.1 Introduction

Let H_1 and H_2 be real Hilbert spaces, $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty, closed and convex sets, $A : H_1 \to H_2$ be a bounded linear operator, and $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be two given mappings. Censor et al. [46] introduced the following split variational inequality problem (in short, SVIP):

$$\text{Find } x^* \in C \text{ such that } \langle f_1(x^*), x - x^* \rangle \geq 0, \quad \text{for all } x \in C, \tag{8.1.1}$$

and such that

$$y^* := Ax^* \in Q \text{ solves } \langle f_2(y^*), y - y^* \rangle \geq 0, \quad \text{for all } y \in Q. \tag{8.1.2}$$

Let Σ denote the solution set of SVIP, that is,

$$\Sigma = \{x \text{ solves } (8.1.1) : Ax \text{ solves } (8.1.2)\}.$$

If f_1 and f_2 are convex and differentiable, then SVIP is equivalent to the following split minimization problem:

$$\min f_1(x), \quad \text{subject to } x \in C, \tag{8.1.3}$$

and such that

$$y^* := Ax^* \in Q \text{ solves } \min f_2(y), \quad \text{subject to } y \in Q. \tag{8.1.4}$$

For the further detail on equivalence between a variational inequality and an optimization problem, we refer [8]. The SVIP also contains the split feasibility problem (SFP) as a special case. For further detail on SFP, we refer [10, 31] and the references therein.

If the sets C and Q are the set of fixed points of the mappings $T : H_1 \to H_1$ and $S : H_2 \to H_2$, respectively, then SVIP is called split hierarchical variational inequality problem (in short, SHVIP). It is introduced and studied by Ansari et al. [9]. Several special cases of SHVIP, namely, split convex minimization problem, split variational inequality problem defined over the solution set of a monotone variational inclusion problem, split variational inequality problem defined over the solution set of an equilibrium problem, are also considered in [9].

Let $M_1 : H_1 \rightrightarrows H_1$ and $M_2 : H_2 \rightrightarrows H_2$ be set-valued operators with nonempty values, and let $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be mappings. Then, inspired by the work in [46], Moudafi [130] introduced the following split monotone variational inclusion problem (in short, SMVIP):

$$\text{Find } x^* \in H_1 \text{ such that } 0 \in f_1(x^*) + M_1(x^*), \tag{8.1.5}$$

and such that

$$y^* := Ax^* \in H_2 \text{ solves } 0 \in f_2(y^*) + M_2(y^*). \tag{8.1.6}$$

Let Ξ denote the solution set of SMVIP, that is,

$$\Xi = \{x \text{ solves (8.1.5)} : Ax \text{ solves (8.1.6)}\}.$$

To solve the SMVIP, Moudafi [130] proposed the following iterative method: Let $\lambda > 0$ and x_0 be the initial guess. Compute

$$x_{n+1} = U(x_n + \gamma A^*(V - I)Ax_n), \quad \text{for all } n \in \mathbb{N}, \tag{8.1.7}$$

where $\gamma \in (0, 1/L)$ with L being the spectral radius of the operator A^*A, $U = J_\lambda^{M_1}(I - \lambda f_1)$, $V = J_\lambda^{M_2}(I - \lambda f_2)$, and $J_\lambda^{M_1}$ and $J_\lambda^{M_2}$ are the resolvents of M_1 and M_2, respectively, with parameter λ (see [198]). He obtained the following weak convergence result for iterative method (8.1.7).

Theorem 8.1.1. *[130, Theorem 2.1] Given a bounded linear operator $A : H_1 \to H_2$. Let $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be α_1- and α_2-inverse strongly monotone operators on H_1 and H_2, respectively, and M_1, M_2 be two maximal monotone operators, and set $\alpha := \min\{\alpha_1, \alpha_2\}$. Consider the operator $U := J_\lambda^{M_1}(I - \lambda f_1)$, $V := J_\lambda^{M_2}(I - \lambda f_2)$ with $\lambda \in (0, 2\alpha)$. Then, the sequence $\{x_n\}$ generated by (8.1.7) converges weakly to an element $x^* \in \Xi$, provided that $\Xi \neq \emptyset$ and $\gamma \in (0, 1/L)$.*

Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be mappings such that $\mathrm{Fix}(T) \neq \emptyset$ and $\mathrm{Fix}(S) \neq \emptyset$, where $\mathrm{Fix}(T)$ and $\mathrm{Fix}(S)$ denote the set of fixed points of T and S, respectively. Inspired by the work in [9] and [130], in this chapter, we introduce the following split hierarchical monotone variational inclusion problem (in short, SHMVIP):

$$\text{Find } x^* \in \mathrm{Fix}(T) \text{ such that } 0 \in f_1(x^*) + M_1(x^*), \tag{8.1.8}$$

and such that

$$y^* := Ax^* \in \mathrm{Fix}(S) \text{ solves } 0 \in f_2(y^*) + M_2(y^*). \tag{8.1.9}$$

We denote by Λ the set of solutions of SHMVIP, that is,

$$\Lambda = \{x \text{ solves } (8.1.8) : Ax \text{ solves } (8.1.9)\}.$$

We propose an iterative algorithm to compute the approximate solutions of SHMVIP. The weak convergence of the sequence generated by the proposed algorithm is studied. An example is presented to illustrate the proposed algorithm and result.

8.2 Algorithm and Convergence Result

Let $\phi : H \to H$ be a given single-valued mapping and $M : H \rightrightarrows H$ be a maximal monotone set-valued operator. Then,

$$0 \in \phi(x^*) + M(x^*) \quad \Leftrightarrow \quad x^* \in \mathrm{Fix}(J_\lambda^M(I - \lambda\phi)(x^*)). \tag{8.2.1}$$

Indeed, let $x^* \in \mathrm{Fix}(J_\lambda^M(I - \lambda\phi)(x^*))$. Then, $x^* = J_\lambda^M(I - \lambda\phi)(x^*)$. It follows that

$$x^* = (I+\lambda M)^{-1}(I-\lambda\phi)(x^*) \;\; \Leftrightarrow \;\; x^*-\lambda\phi(x^*) \in (I+\lambda M)(x^*) \;\; \Leftrightarrow \;\; 0 \in \phi(x^*)+M(x^*).$$

Let $M_1 : H_1 \rightrightarrows H_1$ and $M_2 : H_2 \rightrightarrows H_2$ be set-valued operators with nonempty values, and let $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be mappings. Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be mappings such that $\mathrm{Fix}(T) \neq \emptyset$ and $\mathrm{Fix}(S) \neq \emptyset$. Let $U := J_\lambda^{M_1}(I - \lambda f_1)$ and $V := J_\lambda^{M_2}(I - \lambda f_2)$. With the help of (8.2.1), (8.1.8) and (8.1.9) can be re-written as

$$\text{find } x^* \in \mathrm{Fix}(T) \quad \text{such that} \quad x^* \in \mathrm{Fix}(J_\lambda^{M_1}(I - \lambda f_1)), \tag{8.2.2}$$

and such that

$$y^* := Ax^* \in \mathrm{Fix}(S) \quad \text{solves} \quad y^* \in \mathrm{Fix}(J_\lambda^{M_2}(I - \lambda f_2)). \tag{8.2.3}$$

Now we propose the following algorithm to compute the approximate solutions of SHMVIP.

Algorithm 8.2.1. Initialization: *Take arbitrary* $x_1 \in H_1$.
Iterative Step: *For a given current* $x_n \in H_1$, *compute*

$$x_{n+1} = TU(x_n + \gamma A^*(SV - I)Ax_n), \quad n \in \mathbb{N}, \tag{8.2.4}$$

where $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$.
Last Step: *Update* $n := n + 1$.

Next we prove the weak convergence of the sequence generated by the Algorithm 8.2.1.

Theorem 8.2.1. *Let* $A : H_1 \to H_2$ *be a bounded linear operator,* $f_1 : H_1 \to H_1$ *be an* α_1*-inverse strongly monotone mapping,* $T : H_1 \to H_1$ *be a strongly nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$, $f_2 : H_2 \to H_2$ *be an* α_2*-inverse strongly monotone mapping,* $S : H_2 \to H_2$ *be a nonexpansive mapping such that* $\mathrm{Fix}(S) \neq \emptyset$, *and* $\alpha := \min\{\alpha_1, \alpha_2\}$. *Consider the operator* $U := J_\lambda^{M_1}(I - \lambda f_1)$ *and* $V := J_\lambda^{M_2}(I - \lambda f_2)$ *with* $\lambda \in (0, 2\alpha)$, *and* $M_1 : H_1 \rightrightarrows H_1$ *and* $M_2 : H_2 \rightrightarrows H_2$ *are two maximal monotone set-valued operators with nonempty values. Then the sequence* $\{x_n\}$ *generated by Algorithm 8.2.1 converges weakly to an element* $x^* \in \Lambda$.

Proof. Let $p \in \Lambda$. Then $Tp = p$, $Up = p$, $S(Ap) = Ap$ and $V(Ap) = Ap$. Let $y_n := x_n + \gamma A^*(SV - I)Ax_n$ and consider

$$\begin{aligned}
\|y_n - p\|^2 &= \|x_n + \gamma A^*(SV - I)Ax_n - p\|^2 \\
&= \|x_n - p\|^2 + \gamma^2\|A^*(SV - I)Ax_n\|^2 \\
&\quad + 2\gamma\langle x_n - p, A^*(SV - I)Ax_n\rangle \\
&\leq \|x_n - p\|^2 + \gamma^2\|A\|^2\|(SV - I)Ax_n\|^2 \\
&\quad + 2\gamma\langle x_n - p, A^*(SV - I)Ax_n\rangle.
\end{aligned} \tag{8.2.5}$$

Consider the third term of inequality (8.2.5), we have

$$
\begin{aligned}
&\langle x_n - p, A^*(SV - I)Ax_n\rangle \\
&= \langle Ax_n - Ap, (SV - I)Ax_n\rangle \\
&= \langle (SV - I)Ax_n - Ap + Ax_n - (SV - I)Ax_n, (SV - I)Ax_n\rangle \\
&= \langle SVAx_n - Ap, SVAx_n - Ax_n\rangle - \|(SV - I)Ax_n\|^2 \\
&= \frac{1}{2}\|SVAx_n - Ap\|^2 + \frac{1}{2}\|SVAx_n - Ax_n\|^2 - \frac{1}{2}\|Ax_n - Ap\|^2 - \|(SV - I)Ax_n\|^2 \\
&= \frac{1}{2}\|SVAx_n - SVAp\|^2 + \frac{1}{2}\|SVAx_n - Ax_n\|^2 - \frac{1}{2}\|Ax_n - Ap\|^2 - \|(SV - I)Ax_n\|^2 \\
&\le \frac{1}{2}\|Ax_n - Ap\|^2 + \frac{1}{2}\|SVAx_n - Ax_n\|^2 - \frac{1}{2}\|Ax_n - Ap\|^2 - \|(SV - I)Ax_n\|^2 \\
&= -\frac{1}{2}\|(SV - I)Ax_n\|^2. \qquad (8.2.6)
\end{aligned}
$$

Combining (8.2.5) and (8.2.6), we obtain

$$
\begin{aligned}
\|y_n - p\|^2 &\le \|x_n - p\|^2 + \gamma^2\|A\|^2\|(I - SV)Ax_n\|^2 - \gamma\|(SV - I)Ax_n\|^2 \\
&= \|x_n - p\|^2 - \gamma(1 - \gamma\|A\|^2)\|(SV - I)Ax_n\|^2. \qquad (8.2.7)
\end{aligned}
$$

Since $\gamma \in \left(0, \frac{1}{\|A\|^2}\right)$, we have $\gamma(1 - \gamma\|A\|^2) > 0$, and thus

$$
\|y_n - p\| \le \|x_n - p\|. \qquad (8.2.8)
$$

From the above inequality (8.2.7), we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|TUy_n - TUp\|^2 \\
&\le \|Uy_n - Up\|^2 \\
&\le \|y_n - p\|^2 \\
&\le \|x_n - p\|^2 - \gamma(1 - \gamma\|A\|^2)\|(SV - I)Ax_n\|^2 \qquad (8.2.9) \\
&\le \|x_n - p\|^2.
\end{aligned}
$$

This shows that $\|x_{n+1}-p\| \le \|x_n-p\|$ which implies that $\{\|x_n-p\|\}_{n=1}^{\infty}$ is a monotonic decreasing sequence, also it is bounded below by 0. Therefore, $\lim_{n\to\infty} \|x_n - p\|$ exists. Taking limit both the sides in (8.2.9), and notice that $\gamma(1 - \gamma\|A\|^2) > 0$, we have

$$
\lim_{n\to\infty} \|(SV - I)Ax_n\| = 0, \qquad (8.2.10)
$$

and since $y_n := x_n + \gamma A^*(SV - I)Ax_n$, we have $\|y_n - x_n\| = \gamma\|A\|\|(SV - I)Ax_n\|$. Thus, in view of (8.2.10), we have

$$
\lim_{n\to\infty} \|y_n - x_n\| = 0. \qquad (8.2.11)
$$

Since $\{x_n\}$ is a bounded sequence in a Hilbert space H_1, therefore it has a weakly convergent subsequence, say, $x_{n_i} \rightharpoonup x^*$. Further by Opial's condition [143], we can see that $x_n \rightharpoonup x^*$. Since A is bounded linear operator, $Ax_n \to Ax^*$. Since S is nonexpansive and V is averaged, therefore V is nonexpansive. Also, composition of nonexpansive mappings are nonexpansive, therefore SV is nonexpansive. Thus in view of $Ax_n \to Ax^*$, from (8.2.10) and closedness of $SV - I$ at 0, we obtain that $SV(Ax^*) = Ax^*$. Next we show that $V(Ax^*) = Ax^*$. Since

$$|\|SVAx_n - Ap\| - \|Ax_n - Ap\|| \leq \|SVAx_n - Ax_n\|.$$

Taking limit both the sides in the above inequality and by using (8.2.10), we obtain

$$\begin{aligned} \lim_{n\to\infty} |(\|SVAx_n - Ap\| - \|Ax_n - Ap\|)| &= 0 \\ \left|\lim_{n\to\infty} (\|SVAx_n - Ap\| - \|Ax_n - Ap\|)\right| &= 0 \\ \lim_{n\to\infty} (\|SVAx_n - Ap\| - \|Ax_n - Ap\|) &= 0. \end{aligned} \tag{8.2.12}$$

Since $S(Ap) = Ap$ and $V(Ap) = Ap$, by nonexpansiveness of S and V, we have

$$\|SV(Ax_n) - Ap\| \leq \|V(Ax_n) - Ap\| \leq \|Ax_n - Ap\|,$$

and therefore,

$$\|SV(Ax_n) - Ap\| - \|Ax_n - Ap\| \leq \|V(Ax_n) - Ap\| - \|Ax_n - Ap\| \leq 0.$$

Thus, we have

$$\begin{aligned} \lim_{n\to\infty} (\|SV(Ax_n) - Ap\| - \|Ax_n - Ap\|) &\leq \lim_{n\to\infty} (\|V(Ax_n) - Ap\| - \|Ax_n - Ap\|) \\ &\leq 0. \end{aligned} \tag{8.2.13}$$

From (8.2.12) and (8.2.13), we obtain

$$\lim_{n\to\infty} (\|V(Ax_n) - Ap\| - \|Ax_n - Ap\|) = 0. \tag{8.2.14}$$

Since V is averaged and every averaged map is strongly nonexpansive, we have V is strongly nonexpansive. Since $\{Ap\}$ and $\{Ax_n\}$ are bounded sequences, by the definition of strongly nonexpansiveness of V, we have

$$\lim_{n\to\infty} \|V(Ax_n) - Ax_n\| = 0.$$

Since V is nonexpansive, by demiclosedness principle, we have

$$V(Ax^*) = Ax^*.$$

Now, we are left to show that $Tx^* = x^*$ and $Ux^* = x^*$. By using nonexpansiveness of T and U, in view of (8.2.4) and (8.2.8), we have

$$0 \leq \|Uy_n - p\| - \|TUy_n - p\| \leq \|y_n - p\| - \|TUy_n - p\| \leq \|x_n - p\| - \|x_{n+1} - p\|.$$

This implies that

$$\lim_{n\to\infty} (\|Uy_n - p\| - \|TUy_n - p\|) = 0. \tag{8.2.15}$$

From (8.2.8), we have that $\{y_n\}$ is a bounded sequence and since U is averaged, therefore it is nonexpansive. Thus, $\{Uy_n\}$ is also bounded. Since T is strongly nonexpansive, we have

$$\lim_{n\to\infty} \|Uy_n - TUy_n\| = 0. \tag{8.2.16}$$

In view of (8.2.4) and (8.2.8), by using the nonexpansiveness of TU, we have

$$0 \leq \|y_n - p\| - \|TUy_n - p\| \leq \|x_n - p\| - \|x_{n+1} - p\|.$$

It follows that

$$\lim_{n\to\infty} (\|y_n - p\| - \|TUy_n - p\|) = 0. \tag{8.2.17}$$

By using the nonexpansiveness of T and U, we have

$$\|TUy_n - p\| \leq \|Uy_n - p\| \leq \|y_n - p\|,$$

and therefore,

$$\|TUy_n - p\| - \|y_n - p\| \leq \|Uy_n - p\| - \|y_n - p\| \leq 0.$$

Thus, from (8.2.17), we have

$$\lim_{n\to\infty} (\|Uy_n - p\| - \|y_n - p\|) = 0. \tag{8.2.18}$$

From (8.2.8), we have that $\{y_n\}$ is a bounded sequence. Since $\{p\}$ a constant sequence, it is bounded. By strong nonexpansiveness of U, we have

$$\lim_{n\to\infty} \|Uy_n - y_n\| = 0. \tag{8.2.19}$$

Next consider for all $f \in H_1$,

$$\begin{aligned}\|f(y_n) - f(x^*)\| &= \|f(y_n) - f(x_n) + f(x_n) - f(x^*)\| \\ &\leq \|f(y_n) - f(x_n)\| + \|f(x_n) - f(x^*)\| \\ &\leq \|f\|\|y_n - x_n\| + \|f(x_n) - f(x^*)\|.\end{aligned}$$

Since $x_n \rightharpoonup x^*$ and from (8.2.11), we have $\lim_{n\to\infty} \|f(y_n) - f(x^*)\| = 0$, thus $y_n \rightharpoonup x^*$. Thus, in view of (8.2.19) and by applying demiclosedness principle, we have $Ux^* = x^*$. Again, since $y_n \rightharpoonup x^*$, in view of (8.2.19), we have $Uy_n \rightharpoonup x^*$. Thus, again in view of (8.2.16) and by applying demiclosedness principle, we have $Tx^* = x^*$. This completes the proof. □

8.3 Example

Now, we illustrate Algorithm 8.2.1 and Theorem 8.2.1 by the following example.

Example 8.3.1. *Let $H_1 = H_2 = H = \mathbb{R}$ and $M : H \rightrightarrows H$ be defined by*

$$M(x) = \begin{cases} \{1\}, & \text{if } x > 0, \\ [0,1], & \text{if } x = 0, \\ \{0\}, & \text{if } x < 0. \end{cases} \tag{8.3.1}$$

Then, as shown in [147], M is a set-valued maximal monotone operators. We define the mappings A, f_1, f_2, T, $S : H \to H$ by

$$Ax = \frac{x}{2}, \quad \text{for all } x \in H,$$

$$f_1 x = f_2 x = \frac{2x}{3}, \quad \text{for all } x \in H,$$

$$Tx = \frac{x}{3}, \quad \text{for all } x \in H,$$

and,

$$Sx = \frac{4x}{5} \quad \text{for all } x \in H,$$

respectively. It is easy to show that A is a bounded linear operator, f_1 and f_2 are $\frac{1}{3}$-ism, T is firmly nonexpansive, and thus T is strongly nonexpansive [36] and S is nonexpansive. Let $M_1(x) = M_2(x) = Mx$. Then, M_1 and M_2 are maximal monotone set-valued operators. Let $J_\lambda^{M_1}(x) = J_\lambda^{M_2}(x) = \frac{x}{2}$ be the resolvent operator. The values of $\{x_n\}$ with different values of n are reported in the Table 8.1. All codes are written in Matlab R2010.

The following table shows that the sequence $\{x_n\}$ with initial guess $x_1 = 10$, $x_1 = 15$ and $x_1 = 20$ converges to 0 which is the required solution.

Table 8.1 : Convergence Table

n	1	2	3	4	5	6	7	8	9	10
x_n	10	.7037	.0495	.0035	.0002	.0000	.0000	.0000	.0000	.0000
x_n	15	1.0556	.0743	.0052	.0004	.0000	.0000	.0000	.0000	.0000
x_n	20	1.4074	.0990	.0070	.0005	.0000	.0000	.0000	.0000	.0000