Chapter 7

Split Hierarchical Variational Inequality Problems and Fixed Point Problems for Nonexpansive Mappings

7.1 Introduction

A variational inequality problem in which the underlying set is a set of fixed points of a nonlinear operator is called hierarchical variational inequality problem. Recently, Ansari et al. [9] introduced the split hierarchical variational inequality problem (in short, SHVIP). More precisely, they considered the following split hierarchical Minty variational inequality problem (in short, SHMVIP) which requires to find a solution of a hierarchical Minty variational inequality problem (in short, HMVIP) such that its image under a nonlinear operator is a solution of another (HMVIP).

Let H_1 and H_2 be real Hilbert spaces, $f_1, T : H_1 \to H_1$ be mappings such that $\mathrm{Fix}(T) \neq \emptyset$, and $f_2, S : H_2 \to H_2$ be mappings with $\mathrm{Fix}(S) \neq \emptyset$. Let $A : H_1 \to H_2$ be an operator with $R(A) \cap \mathrm{Fix}(S) \neq \emptyset$, where $R(A)$ denotes the range of A. The split hierarchical variational inequality problem (SHVIP) is defined as follows:

$$\text{Find } x^* \in \mathrm{Fix}(T) \quad \text{such that} \quad \langle f_1(x^*), x - x^* \rangle \geq 0, \quad \text{for all } x \in \mathrm{Fix}(T), \tag{7.1.1}$$

and such that

$$Ax^* \in \mathrm{Fix}(S) \text{ solves } \langle f_2(Ax^*), y - Ax^* \rangle \geq 0, \quad \text{for all } y \in \mathrm{Fix}(S). \tag{7.1.2}$$

The solution set of SHVIP is denoted by Ψ.

Another problem which is closely related to SHVIP is the following split hierarchical Minty variational inequality problem (SHMVIP):

$$\text{Find } x^* \in \text{Fix}(T) \text{ such that } \langle f_1(x), x - x^* \rangle \geq 0, \quad \text{for all } x \in \text{Fix}(T), \tag{7.1.3}$$

and such that

$$Ax^* \in \text{Fix}(S) \text{ solves } \langle f_2(y), y - Ax^* \rangle \geq 0, \quad \text{for all } y \in \text{Fix}(S). \tag{7.1.4}$$

We denote by Δ the set of solutions of SHMVIP, that is,

$$\Delta = \{x \text{ solves (7.1.3)} : Ax \text{ solves (7.1.4)}\}.$$

It can be easily seen, by Minty lemma [125, Lemma 1] that if Fix(T) and Fix(S) are nonempty, closed and convex, and f_1 and f_2 are monotone and continuous, then SHVIP (7.1.1)-(7.1.2) and SHMVIP (7.1.3)-(7.1.4) are equivalent.

Ansari et al. [9] showed that several problems, namely, split convex minimization problem, split variational inequality problem over the solution set of a monotone variational inclusion problem, and split variational inequality problem over the solution set of an equilibrium problem, are particular cases of SHVIP. They proposed an iterative scheme for solving SHVIP and studied the weak convergence of the sequence generated by the proposed algorithm.

In this chapter, we give a common solution method for finding a fixed point of a nonexpansive mapping and a solution of a split hierarchical variational inequality problem. The weak convergence of such algorithm is studied. We also present an example to illustrate the proposed algorithm and the convergence result.

7.2 Algorithms and Convergence Results

Let $L : H_1 \to H_1$ be a nonexpansive mapping with Fix(L) $\cap \Delta \neq \emptyset$. We propose the following algorithm to compute a common element of the set of fixed points of L and set of solutions of SHMVIP.

Algorithm 7.2.1. INITIALIZATION: *Choose* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\lambda_n\}_{n=1}^{\infty} \subset (0,1)$. *Take arbitrary* $x_1 \in H_1$.
ITERATIVE STEP: *For a given current* $x_n \in H_1$, *compute*

$$\begin{aligned} z_n &= x_n - \gamma A^*(I - S(I - \beta_n f_2))Ax_n, \\ y_n &= T(I - \alpha_n f_1)z_n, \\ x_{n+1} &= \lambda_n x_n + (1 - \lambda_n)Ly_n, \quad n \in \mathbb{N}, \end{aligned} \tag{7.2.1}$$

where $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$.
LAST STEP: *Update* $n := n+1$.

When L is the identity mapping, Algorithm 7.2.1 reduces to the following algorithm.

Algorithm 7.2.2. INITIALIZATION: *Choose* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, $\{\lambda_n\}_{n=1}^{\infty} \subset (0,1)$. *Take arbitrary* $x_1 \in H_1$.
ITERATIVE STEP: *For a given current* $x_n \in H_1$, *compute*

$$\begin{aligned} z_n &= x_n - \gamma A^*(I - S(I - \beta_n f_2))Ax_n, \\ y_n &= T(I - \alpha_n f_1)z_n, \\ x_{n+1} &= \lambda_n x_n + (1-\lambda_n)y_n, \quad n \in \mathbb{N}, \end{aligned} \tag{7.2.2}$$

where $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$.
LAST STEP: *Update* $n := n+1$.

Next we prove the weak convergence of the sequences generated by the Algorithm 7.2.1.

Theorem 7.2.1. *Let* $f_1 : H_1 \to H_1$ *be a monotone continuous mapping,* $T : H_1 \to H_1$ *be a cutter nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$, $f_2 : H_2 \to H_2$ *be a monotone continuous mapping and* $S : H_2 \to H_2$ *be a cutter strongly nonexpansive mapping such that* $\mathrm{Fix}(S) \neq \emptyset$. *Let* $A : H_1 \to H_2$ *be a bounded linear operator with* $R(A) \cap \mathrm{Fix}(S) \neq \emptyset$ *and let* $L : H_1 \to H_1$ *be a nonexpansive mapping with* $\mathrm{Fix}(L) \cap \Delta \neq \emptyset$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be the sequences generated by Algorithm 7.2.1 such that the following conditions hold:*

(i) *There exists a natural number* $n_\circ$ *such that*

$$\Delta \subset \bigcap_{n=n_\circ}^{\infty} \{z \in H_1 : \langle f_2(Ax_n), S(I - \beta_n f_2)Ax_n - Az \rangle \geq 0\};$$

(ii) $\{f_1(z_n)\}_{n=1}^{\infty}$ *is a bounded sequence;*

(iii) $\sum_{n=0}^{\infty} \alpha_n < \infty$;

(iv) $\lim_{n\to\infty} \beta_n = 0$;

(v) $0 < a \leq \lim_{n\to\infty} \lambda_n \leq b < 1$;

(vi) $\|x_{n+1} - x_n\| = o(\alpha_n)$ *and* $\alpha_n = o(\beta_n^2)$*;*

(vii) $\{f_2(Ax_n)\}_{n=1}^{\infty}$ *is a bounded sequence.*

Then the sequences $\{x_n\}$ *and* $\{y_n\}$ *converge weakly to an element* $x^* \in \mathrm{Fix}(L) \cap \Delta$*.*

Proof. Let $p \in \mathrm{Fix}(L) \cap \Delta$. Then $T(p) = p$, $L(p) = p$ and $S(Ap) = Ap$. Consider

$$\begin{aligned} \|z_n - p\|^2 &= \|x_n - \gamma A^*(I - S(I - \beta_n f_2))Ax_n - p\|^2 \\ &= \|x_n - p\|^2 + \gamma^2 \|A^*(I - S(I - \beta_n f_2))Ax_n\|^2 \\ &\quad - 2\gamma\langle x_n - p, A^*(I - S(I - \beta_n f_2))Ax_n\rangle \\ &\leq \|x_n - p\|^2 + \gamma^2\|A\|^2\|(I - S(I - \beta_n f_2))Ax_n\|^2 \\ &\quad - 2\gamma\langle x_n - p, A^*(I - S(I - \beta_n f_2))Ax_n\rangle, \quad \text{for all } n \geq 1. \end{aligned} \tag{7.2.3}$$

Since S is a cutter operator, we have

$$\begin{aligned} &\langle x_n - p, A^*(S(I - \beta_n f_2) - I)Ax_n\rangle \\ &= \langle Ax_n - Ap, (S(I - \beta_n f_2) - I)Ax_n\rangle \\ &= \langle S(I - \beta_n f_2)Ax_n - Ap + Ax_n - S(I - \beta_n f_2)Ax_n, (S(I - \beta_n f_2) - I)Ax_n\rangle \\ &= \langle S(I - \beta_n f_2)Ax_n - Ap, (S(I - \beta_n f_2) - I)Ax_n\rangle - \|(S(I - \beta_n f_2) - I)Ax_n\|^2 \\ &= \langle S(I - \beta_n f_2)Ax_n - Ap, (S(I - \beta_n f_2) - I)Ax_n + \beta_n f_2 Ax_n - \beta_n f_2 Ax_n\rangle \\ &\quad - \|(S(I - \beta_n f_2) - I)Ax_n\|^2 \\ &= \langle S(I - \beta_n f_2)Ax_n - Ap, S(I - \beta_n f_2)Ax_n - (I - \beta_n f_2)Ax_n\rangle \\ &\quad - \beta_n\langle S(I - \beta_n f_2)(Ax_n) - Ap, f_2(Ax_n)\rangle - \|(S(I - \beta_n f_2) - I)Ax_n\|^2 \\ &\leq -\|(S(I - \beta_n f_2) - I)Ax_n\|^2 - \beta_n\langle S(I - \beta_n f_2)(Ax_n) - Ap, f_2(Ax_n)\rangle. \end{aligned}$$

Since $p \in \Delta$, by condition (i), we have

$$\langle S(I - \beta_n f_2)(Ax_n) - Ap, f_2(Ax_n)\rangle \geq 0.$$

Since $\beta_n \in (0, 1)$ for all $n \in \mathbb{N}$, we further have

$$\beta_n\langle S(I - \beta_n f_2)(Ax_n) - Ap, f_2(Ax_n)\rangle \geq 0.$$

Therefore,

$$\langle x_n - p, A^*(S(I - \beta_n f_2) - I)Ax_n\rangle \leq -\|(S(I - \beta_n f_2) - I)Ax_n\|^2.$$

Thus, (7.2.3) becomes

$$\begin{aligned} &\|z_n - p\|^2 \\ &\leq \|x_n - p\|^2 + \gamma^2\|A\|^2\|(I - S(I - \beta_n f_2))Ax_n\|^2 \\ &\quad - 2\gamma\|(S(I - \beta_n f_2) - I)Ax_n\|^2 \\ &= \|x_n - p\|^2 - \gamma(2 - \gamma\|A\|^2)\|(S(I - \beta_n f_2) - I)Ax_n\|^2, \quad \text{for all } n \geq 1. \end{aligned} \tag{7.2.4}$$

Since $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$, we observe that $\gamma(2-\gamma\|A\|^2) > 0$, and hence,

$$\|z_n - p\| \le \|x_n - p\|, \quad \text{for all } n \ge 1. \tag{7.2.5}$$

Let $M := \sup\{\|f_1(z_n)\| : n \ge 1\}$. Then, for all $n \ge 1$, we have

$$\begin{aligned} \|y_n - p\| &= \|T(z_n - \alpha_n f_1(z_n)) - T(p)\| \\ &\le \|z_n - p\| + \alpha_n \|f_1(z_n)\| \\ &\le \|z_n - p\| + \alpha_n M \\ &\le \|x_n - p\| + \alpha_n M, \end{aligned} \tag{7.2.6}$$

$$\begin{aligned} \|x_{n+1} - p\| &= \|\lambda_n x_n + (1-\lambda_n) L y_n - p\| \\ &= \|\lambda_n (x_n - p) + (1-\lambda_n)(L y_n - p)\| \\ &\le \lambda_n \|x_n - p\| + (1-\lambda_n)\|y_n - p\| \\ &\le \lambda_n \|x_n - p\| + (1-\lambda_n)\|x_n - p\| + (1-\lambda_n)\alpha_n M \\ &\le \|x_n - p\| + (1-\lambda_n)\alpha_n M. \end{aligned}$$

Since $\sum \alpha_n < \infty$ and $0 < a \le \lim_{n\to\infty} \lambda_n \le b < 1$, we have $\Sigma_{n=1}^{\infty}(1-\lambda_n)\alpha_n < \infty$. Thus, by Lemma 2.1.2, the limit $\lim_{n\to\infty} \|x_n - p\|$ exists. Also, from (7.2.5)-(7.2.6), the limits $\lim_{n\to\infty} \|z_n - p\|$ and $\lim_{n\to\infty} \|y_n - p\|$ exist. This implies that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded sequences. Since $\{x_n\}$ is a bounded sequence, there exists a convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to some $x^* \in H_1$. Infact x_n converges weakly to x^*. It suffices to show that there is no subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup y^* \in H_1$ and $y^* \ne x^*$.

Indeed, if this is not true, then the well known Opial's theorem would imply

$$\begin{aligned} \lim_{n\to\infty} \|x_n - y^*\| &= \lim_{j\to\infty} \|x_{n_j} - x^*\| < \lim_{j\to\infty} \|x_{n_j} - y^*\| \\ &= \lim_{n\to\infty} \|x_n - y^*\| = \lim_{i\to\infty} \|x_{n_i} - y^*\| \\ &< \lim_{i\to\infty} \|x_{n_i} - x^*\| = \lim_{n\to\infty} \|x_n - y^*\|, \end{aligned}$$

which leads to a contradiction. Therefore, the sequence $\{x_n\}_{n=1}^{\infty}$ converges weakly to a solution x^*. Now, we will show that x^* is the required solution, that is $x^* \in \mathrm{Fix}(L) \cap \Delta$.

Now, consider

$$\begin{aligned} \|y_n - p\|^2 &= \|T(I - \alpha_n f_1)(z_n) - T(p)\|^2 \\ &\le \|(z_n - p) - \alpha_n f_1(z_n)\|^2 \\ &\le \|z_n - p\|^2 + \alpha_n^2 \|f_1(z_n)\|^2. \end{aligned} \tag{7.2.7}$$

From (7.2.4), (7.2.7) and by Lemma 2.1.4 (iii), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\lambda_n x_n + (1-\lambda_n)Ly_n - p\|^2 \\ &= \|\lambda_n(x_n-p) + (1-\lambda_n)(Ly_n-p)\|^2 \\ &= \lambda_n\|x_n-p\|^2 + (1-\lambda_n)\|y_n-p\|^2 - \lambda_n(1-\lambda_n)\|Ly_n-x_n\|^2 \quad (7.2.8)\\ &\le \lambda_n\|x_n-p\|^2 + (1-\lambda_n)\{\|z_n-p\|^2 + \alpha_n^2\|f_1(z_n)\|^2\} \\ &\quad - \lambda_n(1-\lambda_n)\|Ly_n-x_n\|^2 \\ &\le \lambda_n\|x_n-p\|^2 + (1-\lambda_n)\|x_n-p\|^2 \\ &\quad - (1-\lambda_n)\gamma(2-\gamma\|A\|^2)\|(S(I-\beta_n f_2)-I)Ax_n\|^2 \\ &\quad + (1-\lambda_n)\alpha_n^2\|f_1(z_n)\|^2 - \lambda_n(1-\lambda_n)\|Ly_n-x_n\|^2 \\ &= \|x_n-p\| - (1-\lambda_n)\gamma(2-\gamma\|A\|^2)\|(S(I-\beta_n f_2)-I)Ax_n\|^2 \\ &\quad + (1-\lambda_n)\alpha_n^2\|f_1(z_n)\|^2 - \lambda_n(1-\lambda_n)\|Ly_n-x_n\|^2,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&(1-\lambda_n)\gamma(2-\gamma\|A\|^2)\|(S(I-\beta_n f_2)-I)Ax_n\|^2 + \lambda_n(1-\lambda_n)\|Ly_n-x_n\|^2 \\ &\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + (1-\lambda_n)\alpha_n^2\|f_1(z_n)\|^2.\end{aligned} \quad (7.2.9)$$

From the existence of the limit $\lim_{n\to\infty}\|x_n-p\|$ and the facts that $\alpha_n \to 0$, $\|f_1(z_n)\|$ is bounded, $0 < a \le \lim_{n\to\infty}\lambda_n \le b < 1$ and $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$, it follows that

$$\lim_{n\to\infty}\|(S(I-\beta_n f_2)-I)Ax_n\| = 0, \quad (7.2.10)$$

and

$$\lim_{n\to\infty}\|Ly_n-x_n\| = 0. \quad (7.2.11)$$

From (7.2.1), we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| \le \lim_{n\to\infty}\|Ly_n-x_n\| = 0. \quad (7.2.12)$$

Since T is a cutter operator, we have

$$\begin{aligned}&\langle p-y_n, z_n-y_n\rangle \\ &= \langle y_n-p, y_n-z_n\rangle \\ &= \langle T(I-\alpha_n f_1)z_n - p, T(I-\alpha_n f_1)z_n - (I-\alpha_n f_1)z_n + (I-\alpha_n f_1)z_n - z_n\rangle \\ &= \langle T(I-\alpha_n f_1)z_n - p, T(I-\alpha_n f_1)z_n - (I-\alpha_n f_1)z_n\rangle \\ &\quad + \langle T(I-\alpha_n f_1)z_n - p, -\alpha_n f_1 z_n\rangle \\ &\le \langle T(I-\alpha_n f_1)z_n - p, T(I-\alpha_n f_1)z_n - (I-\alpha_n f_1)z_n\rangle \\ &\quad + \alpha_n\|T(I-\alpha_n f_1)z_n - p\|\|f_1 z_n\|,\end{aligned}$$

and

$$\langle T(I-\alpha_n f_1)z_n - p, T(I-\alpha_n f_1)z_n - (I-\alpha_n f_1)z_n\rangle \leq 0.$$

This implies that

$$\langle p - y_n, z_n - y_n\rangle \leq \alpha_n \|T(I-\alpha_n f_1)z_n - p\|\|f_1 z_n\|. \tag{7.2.13}$$

From (7.2.4), (7.2.13) and by Lemma 2.1.4 (i), we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|z_n - p\|^2 - \|z_n - y_n\|^2 - 2\langle y_n - p, z_n - y_n\rangle \\
&\leq \|x_n - p\|^2 - \|x_n - \gamma A^*(I - S(I-\beta_n f_2))Ax_n - y_n\|^2 \\
&\quad - 2\langle y_n - p, z_n - y_n\rangle \\
&= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \gamma^2\|A\|^2\|(I - S(I-\beta_n f_2))Ax_n\|^2 \\
&\quad + 2\gamma\langle x_n - y_n, A^*(I - S(I-\beta_n f_2)Ax_n)\rangle + 2\langle p - y_n, z_n - y_n\rangle \\
&\leq \|x_n - p\|^2 - \|x_n - y_n\|^2 - \gamma^2\|A\|^2\|(I - S(I-\beta_n f_2))Ax_n\|^2 \qquad (7.2.14) \\
&\quad + 2\alpha_n\|T(I-\alpha_n f_1)z_n - p\|\|f_1(z_n)\| \\
&\quad + 2\gamma\|x_n - y_n\|\|A\|\|(S(I-\beta_n f_2) - I)Ax_n\|.
\end{aligned}$$

Thus, from (7.2.8) and (7.2.14), we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \lambda_n\|x_n - p\|^2 + (1-\lambda_n)\|y_n - p\|^2 - \lambda_n(1-\lambda_n)\|Ly_n - x_n\|^2 \\
&\leq \lambda_n\|x_n - p\|^2 + (1-\lambda_n)\|x_n - p\|^2 - (1-\lambda_n)\gamma^2\|A\|^2\|(I - S(I-\beta_n f_2))Ax_n\|^2 \\
&\quad - (1-\lambda_n)\|x_n - y_n\|^2 - \lambda_n(1-\lambda_n)\|Ly_n - x_n\|^2 \\
&\quad + 2(1-\lambda_n)\alpha_n\|T(I-\alpha_n f_1)z_n - p\|\|f_1(z_n)\| \\
&\quad + 2\gamma(1-\lambda_n)\|x_n - y_n\|\|A\|\|(S(I-\beta_n f_2) - I)Ax_n\| \\
&\leq \|x_n - p\|^2 - (1-\lambda_n)\|x_n - y_n\|^2 - (1-\lambda_n)\gamma^2\|A\|^2\|(I - S(I-\beta_n f_2))Ax_n\|^2 \\
&\quad + 2(1-\lambda_n)\alpha_n\|T(I-\alpha_n f_1)z_n - p\|\|f_1(z_n)\| - \lambda_n(1-\lambda_n)\|Ly_n - x_n\|^2 \\
&\quad + 2\gamma(1-\lambda_n)\|x_n - y_n\|\|A\|\|(S(I-\beta_n f_2) - I)Ax_n\|
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
(1-\lambda_n)\|x_n - y_n\|^2 \leq \|x_n - p\|^2 &- \|x_{n+1} - p\|^2 - \lambda_n(1-\lambda_n)\|Ly_n - x_n\|^2 \\
&+ 2(1-\lambda_n)\alpha_n\|T(I-\alpha_n f_1)z_n - p\|\|f_1(z_n)\| \\
&+ 2\gamma(1-\lambda_n)\|x_n - y_n\|\|A\|\|(S(I-\beta_n f_2) - I)Ax_n\|.
\end{aligned}$$

Taking limit as $n \to \infty$, and taking into account $\alpha_n \to 0$, $0 < a \leq \lim_{n\to\infty} \lambda_n \leq b < 1$, $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$ and from the equations (7.2.10), (7.2.11) we have

$$\|x_n - y_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{7.2.15}$$

$$\|y_n - Ly_n\| = \|y_n - x_n + x_n - Ly_n\|$$
$$\leq \|y_n - x_n\| + \|x_n - Ly_n\|.$$

From (7.2.11) and (7.2.15), we obtain

$$\|y_n - Ly_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{7.2.16}$$

Since $\|x_n - y_n\| \to 0$ and $x_n \rightharpoonup x^* \in H_1$, we have, $y_n \rightharpoonup x^* \in H_1$. Thus by demiclosed principle, $y_n \rightharpoonup x^*$ and $\|y_n - Ly_n\| \to 0$, we have

$$Lx^* = x^*.$$

Thus we have shown that $x^* \in \text{Fix}(L)$.

Now we are left to show that $x^* \in \Delta$. From (7.2.1), we obtain

$$\|z_n - x_n\| = \gamma\|A\|\|(I - S(I - \beta_n f_2))Ax_n\|.$$

By (7.2.10), we have

$$\|z_n - x_n\| \to 0 \quad \text{as } n \to \infty, \tag{7.2.17}$$

and

$$\|z_n - y_n\| \leq \|x_n - y_n\| + \gamma\|A\|\|(I - S(I - \beta_n f_2))Ax_n\|.$$

Equations (7.2.10) and (7.2.15) yield that

$$\|z_n - y_n\| \to 0 \quad \text{as } n \to \infty. \tag{7.2.18}$$

From the definition of y_n, we have

$$\|y_n - Tz_n\| = \|T(z_n - \alpha_n f_1(z_n)) - Tz_n\|$$
$$\leq \|z_n - \alpha_n f_1(z_n) - z_n\|$$
$$\leq \alpha_n\|f_1(z_n)\|.$$

This implies that

$$\lim_{n\to\infty} \|y_n - Tz_n\| = 0, \tag{7.2.19}$$

and

$$\|y_n - Ty_n\| = \|y_n - Tz_n + Tz_n - Ty_n\|$$
$$\leq \|y_n - Tz_n\| + \|z_n - y_n\|.$$

From (7.2.18) and (7.2.19), we get

$$\lim_{n\to\infty} \|y_n - Ty_n\| \to 0.$$

Since $y_n \rightharpoonup x^*$ and $\|y_n - Ty_n\| \to 0$, by demiclosed principle, we obtain

$$Tx^* = x^*.$$

Let $v_n := Ax_n - \beta_n f_2(Ax_n)$ for all $n \geq 1$. We observe that

$$\begin{aligned}
0 &\leq \|v_n - Ap\| - \|Sv_n - SAp\| \\
&= \|Ax_n - \beta_n f_2(Ax_n) - Ap\| - \|Sv_n - SAp\| \\
&= \|Ax_n - Sv_n + Sv_n - \beta_n f_2(Ax_n) - Ap\| - \|Sv_n - SAp\| \\
&\leq \|Ax_n - Sv_n\| + \|Sv_n - SAp\| + \beta_n\|f_2(Ax_n)\| - \|Sv_n - SAp\| \\
&= \|Ax_n - S(Ax_n - \beta_n(f_2(Ax_n)))\| + \beta_n\|f_2(Ax_n)\| \\
&= \|(S(I - \beta_n f_2) - I)Ax_n\| + \beta_n\|f_2(Ax_n)\|.
\end{aligned}$$

From the condition (iv) and (7.2.10), we have

$$\lim_{n\to\infty}(\|v_n - Ap\| - \|Sv_n - SAp\|) = 0.$$

The boundedness of v_n and strongly nonexpansiveness of S imply that

$$\lim_{n\to\infty}\|Sv_n - v_n\| = 0. \tag{7.2.20}$$

From the definition of v_n and the condition (iv), we get

$$\lim_{n\to\infty}\|v_n - Ax_n\| = 0, \tag{7.2.21}$$

and

$$\begin{aligned}
\|v_n - SAx_n\| &\leq \|v_n - Sv_n\| + \|Sv_n - S(Ax_n)\| \\
&\leq \|v_n - Sv_n\| + \|v_n - Ax_n\|.
\end{aligned}$$

From (7.2.20) and (7.2.21), we have

$$\lim_{n\to\infty}\|v_n - S(Ax_n)\| = 0,$$

and thus

$$\lim_{n\to\infty}\|Ax_n - S(Ax_n)\| = 0. \tag{7.2.22}$$

Since $x_n \rightharpoonup x^* \in H_1$, we have $Ax_n \to Ax^* \in H_2$. From (7.2.22) and by demiclosed principle, we obtain

$$S(Ax^*) = Ax^*.$$

Let $q_n := z_n - \alpha_n f_1(z_n)$. By Lemma 2.1.4 (iv) and inequality (7.2.5), we have

$$
\begin{aligned}
\|y_n - p\|^2 &= \|T(I - \alpha_n f_1)z_n - Tp\|^2 \\
&\le \|z_n - p - \alpha_n f_1(z_n)\|^2 \\
&\le \|z_n - p\|^2 + 2\langle \alpha_n f_1(z_n), \alpha_n f_1(z_n) - z_n + p\rangle \\
&= \|z_n - p\|^2 + 2\langle \alpha_n f_1(z_n), p - z_n\rangle + 2\alpha_n^2\|f_1(z_n)\|^2 \\
&\le \|x_n - p\|^2 + 2\alpha_n\langle f_1(z_n), p - z_n\rangle + 2\alpha_n^2\|f_1(z_n)\|^2. \qquad (7.2.23)
\end{aligned}
$$

From the definition of x_{n+1}, (7.2.23) and using monotonocity of f_1, we have

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &\le \lambda_n\|x_n - p\|^2 + (1 - \lambda_n)\|y_n - p\|^2 \\
&\le \lambda_n\|x_n - p\|^2 + (1 - \lambda_n)\{\|x_n - p\|^2 + 2\alpha_n\langle f_1(z_n), p - z_n\rangle \\
&\quad + 2\alpha_n^2\|f_1(z_n)\|^2\} \\
&\le \|x_n - p\|^2 + 2(1 - \lambda_n)\alpha_n\langle f_1(z_n), p - z_n\rangle \\
&\quad + 2(1 - \lambda_n)\alpha_n^2\|f_1(z_n)\|^2 \qquad (7.2.24) \\
&= \|x_n - p\|^2 + 2(1 - \lambda_n)\alpha_n\langle f_1(z_n) - f_1(p) + f_1(p), p - z_n\rangle \\
&\quad + 2(1 - \lambda_n)\alpha_n^2\|f_1(z_n)\|^2 \\
&= \|x_n - p\|^2 + 2(1 - \lambda_n)\alpha_n\langle f_1(z_n) - f_1(p), p - z_n\rangle \\
&\quad + 2(1 - \lambda_n)\alpha_n\langle f_1(p), p - z_n\rangle + 2(1 - \lambda_n)\alpha_n^2\|f_1(z_n)\|^2 \\
&= \|x_n - p\|^2 - 2(1 - \lambda_n)\alpha_n\langle f_1(p) - f_1(z_n), p - z_n\rangle \\
&\quad + 2(1 - \lambda_n)\alpha_n\langle f_1(p), p - z_n\rangle + 2(1 - \lambda_n)\alpha_n^2\|f_1(z_n)\|^2 \\
&= \|x_n - p\|^2 + 2(1 - \lambda_n)\alpha_n\langle f_1(p), p - z_n\rangle + 2(1 - \lambda_n)\alpha_n^2\|f_1(z_n)\|^2,
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&2(1 - \lambda_n)\langle f_1(p), z_n - p\rangle \\
&\le \left(\frac{\|x_n - p\|^2 - \|x_{n+1} - p\|^2}{\alpha_n}\right) + 2(1 - \lambda_n)\alpha_n\|f_1(z_n)\|^2 \\
&\le \left\{\frac{(\|x_n - p\| + \|x_{n+1} - p\|)(\|x_n - p\| - \|x_{n+1} - p\|)}{\alpha_n}\right\} + 2(1 - \lambda_n)\alpha_n\|f_1(z_n)\|^2 \\
&\le M_1\left(\frac{\|x_n - p\| - \|x_{n+1} - p\|}{\alpha_n}\right) + 2(1 - \lambda_n)\alpha_n\|f_1(z_n)\|^2 \\
&\le M_1\left(\frac{\|x_n - x_{n+1}\|}{\alpha_n}\right) + 2(1 - \lambda_n)\alpha_n\|f_1(z_n)\|^2,
\end{aligned}
$$

where $M_1 = \sup\{\|x_n - p\| + \|x_{n+1} - p\|, \quad n \ge 1\} < \infty$. Taking limit both the sides and taking into account that $0 < a \le \lim_{n\to\infty} \lambda_n \le b < 1$, $\alpha_n \to 0$, $\|x_n - x_{n+1}\| = o(\alpha_n)$ and $z_n \rightharpoonup x^*$, we have

$$\langle f_1(p), x^* - p\rangle \le 0,$$

and thus

$$\langle f_1(p), x^* - p\rangle \le 0, \quad \text{for all} \quad p \in \mathrm{Fix}(T),$$

that is, $x^* \in \mathrm{Fix}(T)$ solves (7.1.3).

In order to complete the proof, we have to show that $Ax^* \in \mathrm{Fix}(S)$ solves (7.1.4). Since $\alpha_n = o(\beta_n^2)$, we may assume that $\alpha_n \le \beta_n^2$ for all $n \ge 1$. From (7.2.9), for all $n \ge 1$, we have

$$\begin{aligned}
&(1-\lambda_n)\gamma(2-\gamma\|A\|^2)\|(S(I-\beta_n f_2) - I)Ax_n\|^2 \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n^2\|f_1(z_n)\|^2 \\
&\le \{(\|x_n - p\| + \|x_{n+1} - p\|)(\|x_n - p\| - \|x_{n+1} - p\|)\} + \alpha_n^2\|f_1(z_n)\|^2 \\
&\le M_2\|x_n - x_{n+1}\| + \alpha_n^2\|f_1(z_n)\|^2,
\end{aligned}$$

where $M_2 = \sup\{\|x_n - p\| + \|x_{n+1} - p\| \,:\, n \ge 1\} < \infty$. Therefore, for all $n \ge 1$, we have

$$\begin{aligned}
(1-\lambda_n)\gamma(2-\gamma\|A\|^2)\frac{\|Ax_n - Sv_n\|^2}{\beta_n^2} &\le \frac{\|x_n - x_{n+1}\|}{\beta_n^2}M_2 + \frac{\alpha_n^2}{\beta_n^2}\|f_1(z_n)\|^2 \\
&\le \frac{\|x_n - x_{n+1}\|}{\alpha_n}M_2 + \alpha_n\|f_1(z_n)\|^2.
\end{aligned}$$

Subsequently, since $\alpha_n \to 0$, $\|x_{n+1} - x_n\| = o(\alpha_n)$, $\gamma(2-\gamma\|A\|^2) > 0$ and $0 < a \le \lim_{n\to\infty} \lambda_n \le b < 1$, we have

$$\lim_{n\to\infty} \frac{\|Ax_n - Sv_n\|}{\beta_n} = 0. \tag{7.2.25}$$

For all $n \ge 1$, by Lemma 2.1.4 (iv) and the monotonicity of f_2, we compute

$$\begin{aligned}
&\|Sv_n - SAp\|^2 \\
&\le \|v_n - Ap\| \\
&\le \|Ax_n - \beta_n f_2(Ax_n) - Ap\|^2 \\
&\le \|Ax_n - Ap\|^2 + 2\langle \beta_n f_2(Ax_n), \beta_n f_2(Ax_n) - Ax_n + Ap\rangle \\
&\le \|Ax_n - Ap\|^2 + 2\beta_n\langle f_2(Ax_n), Ap - Ax_n\rangle + 2\beta_n^2\|h(Ax_n)\|^2 \qquad (7.2.26)\\
&\le \|Ax_n - Ap\|^2 - 2\beta_n\langle f_2(Ap) - h(Ax_n), Ap - Ax_n\rangle \\
&\quad + 2\beta_n^2\|f_2(Ax_n)\|^2 + 2\beta_n\langle f_2(Ap), Ap - Ax_n\rangle \\
&\le \|Ax_n - Ap\|^2 + 2\beta_n^2\|f_2(Ax_n)\|^2 + 2\beta_n\langle f_2(Ap), Ap - Ax_n\rangle.
\end{aligned}$$

This gives

$$\begin{aligned}
&2\langle f_2(Ap), Ax_n - Ap\rangle \\
&\le \left\{\frac{\|Ax_n - Ap\|^2 - \|Sv_n - SAp\|^2}{\beta_n}\right\} + 2\beta_n\|f_2(Ax_n)\|^2 \\
&\le \left\{\frac{(\|Ax_n - Ap\| - \|Sv_n - SAp\|)(\|Ax_n - Ap\| + \|Sv_n - SAp\|)}{\beta_n}\right\} \\
&\quad + 2\beta_n\|f_2(Ax_n)\|^2 \\
&\le \left(\frac{\|Ax_n - Sv_n\|}{\beta_n}\right) M_4 + 2\beta_n\|f_2(Ax_n)\|^2,
\end{aligned}$$

where $M_4 := \sup\{\|Ax_n - Ap\| + \|Sv_n - SAp\| : n \geq 1\} < \infty$. From (7.2.25), condition (iv) and $Ax_n \rightharpoonup Ap$, we obtain

$$\langle f_2(Ap), Ax^* - Ap\rangle \le 0, \quad \text{for all } Ap \in \mathrm{Fix}(S),$$

that is, Ax^* solves (7.1.4). This completes the proof. □

7.3 Example

Now, we illustrate Algorithm 7.2.1 and Theorem 7.2.1 by the following example.

Example 7.3.1. *Let $H_1 = H_2 = \mathbb{R}^2$ with inner product and norm are given by $\langle x, y\rangle = x_1y_1 + x_2y_2$ and $\|x\| = |x_1| + |x_2|$, respectively, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Let $C = \{x \in \mathbb{R}^2 : \|x\| \le \sqrt{2}\}$ be a nonempty closed subspace of $\mathbb{R}^2$. Let $T : C \to C$ be defined by*

$$T(x, y) = \left(\frac{2}{3}x + \frac{1}{3}y, \frac{1}{3}x + \frac{2}{3}y\right), \quad \textit{for all } (x, y) \in C.$$

Then T is a cutter nonexpansive mapping.

Let $S : C \to C$ be defined as

$$S(x, y) = \left(\frac{1}{10}x + \frac{1}{10}y, \frac{1}{10}x + \frac{1}{10}y\right), \quad \textit{for all } (x, y) \in C.$$

Then S is firmly nonexpansive, and it has a fixed point $(0, 0)$. Thus being firmly nonexpansive, S is strongly nonexpansive. Also, every firmly nonexpansive mapping with a fixed point is cutter (see [9, 36]). Thus, S is cutter strongly nonexpansive mapping.

Let $f_1, f_2 : C \to C$ *be operators defined by*

$$f_1(x,y) = \left(\frac{1}{3}x - \frac{1}{3}y, -\frac{1}{3}x + \frac{1}{3}y\right), \quad \textit{for all } (x,y) \in C,$$

and

$$f_2(x,y) = \left(\frac{1}{2}x - \frac{1}{2}y, -\frac{1}{2}x + \frac{1}{2}y\right), \quad \textit{for all } (x,y) \in C.$$

Then f_1 *and* f_2 *are monotone.*

Let $L : C \to C$ *be defined by*

$$L(x,y) = \left(\frac{1}{5}x + \frac{1}{5}y, \frac{1}{5}x + \frac{1}{5}y\right), \quad \textit{for all } (x,y) \in C.$$

Then L *is nonexpansive.*

Let $A : C \to C$ *be defined by*

$$A(x,y) = \left(\frac{1}{2}x, \frac{1}{2}y\right), \quad \textit{for all } (x,y) \in C.$$

Then A *is a bounded linear operator and* $\|A\|^2 = \frac{1}{4}$.

Let $\alpha_n = (1/2n^2, 1/2n^2)$, $\beta_n = (1/n, 1/n)$, $\gamma \in (0,8)$ *and* $\lambda_n \in (0,1)$. *Then the sequences* x_n *and* y_n *generated by Algorithm 7.2.1 with initial guess* $x^1 = (1,-1)$ *converges to* $(0,0)$ *which is a fixed point of* T *and* L, *whereas* $A(0,0) = (0,0)$ *which is the fixed point of* S, *where* $x^1 = (x_1^1, x_2^1)$. *Thus,* $(0,0)$ *is the required solution.*

Table 7.1 : Convergence Table

No. of Iterations (n)	y^n	x^n
1	(-.2222, .2222)	(1, -1)
2	(1.0e-004) (-.1528, .1528)	(1.0e-004) (.5000, -.5000)
3	(1.0e-008) (.-1605, .1605)	(1.0e-008) (.5000, -.5000)
4	(1.0e-012) (-.2447, .2447)	(1.0e-012) (.7500, -.7500)
5	(1.0e-016) (-.4931, .4931)	(1.0e-015) (.1500, -.1500)
6	(1.0e-019) (-.1238, .1238)	(1.0e-019) (.3750, -.3750)
7	(1.0e-023) (-.3722, .3722)	(1.0e-022) (.1125, -.1125)
8	(1.0e-026) (-.1305, .1305)	(1.0e-026) (.3938, -.3938)
9	(1.0e-030) (-.5223, .5223)	(1.0e-029) (.1575, -.1575)
10	(1.0e-033) (-.2352, .2352)	(1.0e-033) (.7088, -.7088)
11	(1.0e-036) (-.1177, .1177)	(1.0e-036) (.5544, -.5544)
12	(0, 0)	(0, 0)
13	(0, 0)	(0, 0)
14	(0, 0)	(0, 0)
15	(0, 0)	(0, 0)
16	(0, 0)	(0, 0)