CHAPTER - 3

SYSTEMATIC DERIVATION OF MONOPOLE SOLUTIONS

3.1 Introduction

In this chapter we report a systematic study on the second order nonlinear differential equations of the 't Hooft-Polyakov theory. We have analysed these equations in the PS limit. The method we have used is the direct method of Hirota [59]. This method has been very useful in constructing solutions to (1+1) dimensional scalar field theories. A first application of the method to a coupled system of nonlinear differential equations was recently made by Hirota and Satsuma [60]. Here we apply this method to find solutions of the effectively one dimensional coupled field equations of the monopole theory.

All the exact monopole solutions, regular or point singular, reported in the literature were either obtained by guess work [29] or by integrating the first order Bogmolny equation [26, 34, 35]. However, all solutions of the second order field equations may not satisfy the Bogomolny equation*. All the solutions we have obtained using the Hirota method satisfied the Bogomolny equation and were reported earlier in the literature. Even though there are no new solutions, ours is a systematic method to generate all the monopole solutions from the second order field equations in the PS limit.

In Section 3.2 we introduce the Hirota method and Section 3.3 this method is applied to the 't Hooft-Polyakov equations in the PS limit and the solutions

* After the completion of this work we learned [40] that there are no such regular solutions for the case of the spherically symmetric ansatz (1.71) which is used in our work. In this connection see also papers by Frampton [38] and Kerner [39].
are obtained as a ratio of infinite series which depend upon a number of parameters. By adjusting the parameters the series and the solution are expressed in terms of elementary functions. This is discussed in Section 3.4.

3.2 Direct method of Hirota

Hirota developed a direct method [59] of finding exact solutions of a number of nonlinear differential equations. In this method the dependent variable is expressed as the ratio of two dependent variables \( g \) and \( f \). When this ratio is substituted in the original differential equation we get an equation with two dependent variables. The derivatives of functions can always be combined and expressed in terms of bilinear derivatives, the \( n^{th} \) order bilinear derivative being defined as

\[
D^n_{x} A(x,t) \cdot B(x,t) = \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^n A(x,t) B(x',t) \bigg|_{x = x'} .
\]

The nonlinear equation which contains bilinear derivatives of \( g \) and \( f \) is split into two coupled nonlinear equations. The functions \( g \) and \( f \) are then expanded as power series in a parameter \( \varepsilon \), as in perturbation theory. The individual functions in the power series are evaluated by successively integrating the differential equations that follow from equating the same powers of \( \varepsilon \) on both sides of the split nonlinear equations. Solutions can be obtained either by terminating the series by some technique or by summing the infinite power series.

3.3 Hirota's method for 't Hooft-Polyakov equations

In the PS limit the equations of motion of the SU(2) gauge theory (1.72) become

\[
\begin{align*}
\kappa^2 K'' &= K \left( K^2 - 1 + H^2 \right) , \\
\kappa^2 H'' &= 2K^2 H K^2 .
\end{align*}
\]
We make a dependent variable transformation,

\[ K(h) = \frac{A(h)}{B(h)} , \quad H(h) = \frac{C(h)}{B(h)} , \]

which modifies (3.2) into

\[ \kappa^2 \left( B D^2 A \cdot B - A D^2 B \cdot B \right) = A \left( A^2 - B^2 + C^2 \right) \] (3.4a)

\[ \kappa^2 \left( B D^2 C \cdot B - C D^2 B \cdot B \right) = 2C A^2 \] (3.4b)

where \( D^2 A \cdot B \) and \( D^2 B \cdot B \) are second order bilinear operators:

\[ D^2 A \cdot B = \left( \frac{d^2}{dh^2} - \frac{d}{dh} \right)^2 A(h) B(h') \bigg|_{h = h'} = A'' B - 2A'B' + A'B'' . \] (3.5)

We split (3.4b) using an arbitrary function \( \eta(h) \) to get

\[ \kappa^2 D^2 B \cdot B + \eta C B^2 = -2A^2 \] (3.6a)

\[ \kappa^2 D^2 C \cdot B + \eta C B = 0 \] (3.6b)

One readily verifies that solutions to (3.6) are solutions of (3.4b). One may consider alternative splitting patterns as well. However the present splitting procedure is advantageous because it reduces the degree of nonlinearity from three to two. (3.4a) now becomes

\[ \kappa^2 B D^2 A \cdot B = A \left( C^2 - (\eta + 1) B^2 - A^2 \right) . \] (3.6c)

In Hirota's method the functions \( A, B \) and \( C \) are expanded as perturbation series. A consistent expansion of this kind is

\[ A(h) = \epsilon A_1(h) + \epsilon^2 A_2(h) + \cdots \]

\[ B(h) = 1 + \epsilon B_1(h) + \epsilon^2 B_2(h) + \cdots \] (3.7)

\[ C(h) = 1 + \epsilon C_1(h) + \epsilon^2 C_2(h) + \cdots \]
where $\varepsilon$ is a parameter. Substituting (3.7) in (3.6) and comparing the zeroth power of $\varepsilon$ on both sides we see that $\eta(x)$ should be zero for consistency. Hence (3.6) can be rewritten as

\begin{align}
\mathcal{\kappa}^2 D^2 B \cdot B &= -2A^2 \\
D^2 C \cdot B &= 0 \\
\mathcal{\kappa}^2 B D^2 A \cdot B &= A \left( c^2 - B^2 - A^2 \right) .
\end{align}

The functions $A_1, B_1, C_1, A_2, B_2, C_2, \ldots$ are obtained by integrating successively the linear equations which follow by substituting (3.7) in (3.8) and comparing coefficients of $\varepsilon, \varepsilon^2, \ldots$ respectively. For example, the first two sets of equations are

\begin{align}
\varepsilon &\Rightarrow \\
\frac{d^2 B_1}{d \lambda^2} &= 0 , \quad \frac{d^2 C_1}{d \lambda^2} = 0 , \quad \frac{d^2 A_1}{d \lambda^2} = 0 \\
\varepsilon^2 &\Rightarrow \\
\mathcal{\kappa}^2 \left( 2 D^2 B_2 \cdot 1 + D^2 B_1 \cdot B_1 \right) &= -2A_1^2 \\
D^2 C_2 \cdot 1 + D^2 C_1 \cdot B_1 + D^2 1 \cdot B_2 &= 0 \\
\mathcal{\kappa}^2 (D^2 A_2 \cdot 1 + D^2 A_1 \cdot B_1 + B_1 D^2 A_1 \cdot 1) &= 2A_1 (c_1 - B_1) .
\end{align}

We have choosen:

\begin{align}
A_1 &= a \lambda \\
B_1 &= b \lambda + d \\
C_1 &= c \lambda + d ,
\end{align}
as solutions of (3.9a) to insure the simplicity of the successive integrations. General solutions to (3.9a) lead to logarithmic functions in the second order, and hence the successive calculations become formidable.

With an initial set of solutions in the form (3.10) we obtain the following:

\[
\begin{align*}
A_1 &= a_1 \\
A_2 &= ac_1^2 \\
A_3 &= -d_1 ac_1^2 + ac_1^2 x_1^3/2! \\
A_4 &= d_1^2 ac_1 x_1^2 - 2d_1 ac_1^2 x_1^3/2! + ac_1^3 x_1^4/3! \\
A_5 &= -d_1^3 ac_1 x_1^3 + 3d_1^2 ac_1^2 x_1^3/2! - 3d_1 ac_1^3 x_1^4/3! + ac_1^4 x_1^5/5! \\
&\quad \vdots \\
B_1 &= b_1 x_1 + d_1 \\
B_2 &= E_1 x_1^2/2! \\
B_3 &= -d_1 E_1 x_1^2/2! + (E_1^2 b_1 - 2a_1^2 c_1) x_1^3/3! \\
B_4 &= d_1^2 E_1 x_1^2/2! - 2d_1 (E_1^2 b_1 - 2a_1^2 c_1) x_1^3/3! + (E_1^4 - 4 a_1^2 c_1^2) x_1^4/4! \\
B_5 &= -d_1^3 E_1 x_1^3/2! + 3d_1^2 (E_1^2 b_1 - 2a_1^2 c_1) x_1^3/3! - 3d_1 (E_1^4 - 4 a_1^2 c_1^2) x_1^4/4! \\
&\quad \vdots \\
C_1 &= c_1 x_1 + d_1 \\
C_2 &= (2bc - E_1^2) x_1^2/2! \\
C_3 &= -d_1 (2bc - E_1^2) x_1^3/2! + \left[ (c - 2b) E_1^2 + 2c \left[ b^2 + a_1^2 \right] \right] x_1^3/3! \\
C_4 &= d_1^2 (2bc - E_1^2) x_1^2/2! - 2d_1 \left[ (c - 2b) E_1^2 + 2c \left[ b^2 + a_1^2 \right] \right] x_1^3/3! \\
&\quad \vdots \\
&\quad + (4 a_1^2 c_1^2 + 4 bc E_1^2 - 3 E_1^4) x_1^4/4! \\
&\quad \vdots \\
&\quad \vdots
\end{align*}
\]

(3.11a)
where \( E = \sqrt{b^2 - a^2} \).

### 3.4 Solutions

Equation (3.11) together with (3.7) and (3.3) yield a solution to (3.2) labelled by five parameters \( a, b, c, d, \) and \( \epsilon \). But the series corresponding to (3.11) are too complicated to permit summation for arbitrary values of the parameters. However, by adjusting the parameters, one can make summable series out of (3.11) which can be expressed in terms of elementary functions. Before discussing this aspect, let us consider some simple cases of (3.11).

Setting \( a = b = c = d = 0 \) in (3.11), we find

\[
K = 0, \quad H = 1.
\]  

Likewise, the choice \( a = b, c = 0 \) yields

\[
K = \frac{\eta}{1 + \eta}, \quad H = \frac{1}{1 + \eta}
\]

where \( \eta = a \epsilon/(1 + d) \) is an arbitrary constant. Solutions (3.12) and (3.13) are precisely the point monopole solutions (1.84a) and (1.84b) discussed in Section 1.4.

For \( a \neq b, c = 0 \) and \( |d\epsilon| < 1 \), (3.11b) and (3.11c) lead to series for \( B(a) \) and \( C(a) \) which possess a representative term while the series (3.11a) terminates. Using binominal theorem, the series for \( B(a) \) becomes

\[
B(a) = 1 + d\epsilon + b(\epsilon a) + \frac{E^3}{1 + d\epsilon} (\epsilon a)^2 + \frac{bE^2}{(1 + d\epsilon)^2} (\epsilon a)^3 + \frac{E^4}{(1 + d\epsilon)^3} (\epsilon a)^4 + \frac{bE^3}{(1 + d\epsilon)^4} (\epsilon a)^5 + \ldots
\]

\[
= \frac{E}{\eta} \left( b \sinh \eta + E \cosh \eta \right)
\]
where

\[ \eta = \frac{\varepsilon}{1 + \delta \varepsilon} = \frac{\sqrt{b^2 - \alpha^2}}{1 + \delta \varepsilon} . \] (3.15)

Due to the large arbitrariness* of \( \delta \) and \( \varepsilon \) we can take \( \eta \) as an arbitrary constant independent of \( a \) and \( b \). After a straightforward calculation, the series for \( C(a) \) becomes

\[ C(a) = \frac{\eta}{a} \left[ (E - b \eta r) \cosh \eta r + (b - E \eta r) \sinh \eta r \right] . \] (3.16)

Transforming back to the dependent variables we find

\[ K(a) = \frac{p \eta r e^{\eta r}}{p^2 e^{2 \eta r} - 1} \] (3.17a)

\[ H(a) = -\eta r \frac{p^2 e^{2 \eta r} + 1}{p^2 e^{2 \eta r} - 1} + 1 \] (3.17b)

where \( p = a / (b - \sqrt{b^2 - \alpha^2}) \) is an arbitrary constant. This coincides** with the general point monopole solution (1.84d) obtained by Ju [35]. The Protogenov solution (1.84c) is a special case of (3.17) (with \( p = e^\alpha \)). By setting \( p = 1 \) one obtains the regular PS solution (1.81). We were not able to sum \( B(\alpha) \) and \( C(\alpha) \) series (3.11b) and (3.11c) for nonzero \( c \). However the \( A(\alpha) \) series (3.11a) can be summed in this case because it possesses a representative term. The function \( A(\alpha) \) obtained after summation can be substituted in (3.8a) to obtain an uncoupled nonlinear equation in \( B \). This can further be reduced to the one dimensional Liouville equation. From the known solutions of this equation \( B(\alpha) \) can be obtained. From a knowledge of \( A(\alpha) \) and \( B(\alpha) \), \( K(\alpha) \) can be evaluated. Then \( H(\alpha) \) can be constructed by direct substitution of \( K(\alpha) \) in (3.2a). However, this procedure does not give any new result. This we prove in Appendix 3.A.

* The only condition on \( \alpha \) and \( \varepsilon \) is \( |\delta \varepsilon| < 1 \).

** Comparing (3.17) with (1.84d) one notices a sign change. This does not matter because (3.2) is symmetric under the operations \( K \rightarrow -K \) and \( H \rightarrow -H \) (either together or separately).
3.A Appendix

From (3.11a) the \((n+1)^{th}\) term of \(A(a)\) series is given by

\[
A_{n+1} = a_n (-1)^n d^n \sum_{k=0}^{n-1} (-1)^k (\frac{cr}{d})^{k+1} \frac{(n-1)!}{(k+1)!} \binom{n-1}{k}.
\]  

(3.18)

From this

\[
A_{n+2} = a_n (-1)^n \frac{d^{n+1}}{n+1} \sum_{k=0}^{n} (-1)^k \frac{(cr/d)^k}{(k+1)!} \frac{n!}{k!(n-k)!} \sum_{k=0}^{n} (-1)^k \frac{(cr/d)^k}{(k+1)(n-k)}
\]

\[
= a_n (-1)^n \frac{d^{n+1}}{n+1} \left(\frac{cr}{d}\right) \sum_{k=0}^{n} (-1)^k \left(\frac{cr/d}{k+1}\right) \frac{(n+1)!}{k!(n-k)!}
\]

\[
= a_n (-1)^n \frac{d^{n+1}}{n+1} \left(\frac{cr}{d}\right) L_n^1 (cr/d),
\]

(3.19)

where \(L_n^1 (x)\) is the associated Laguerre Polynomial [61]. \(A(a)\) now becomes

\[
A(a) = e \alpha n \left[ 1 + (\frac{cr}{d}) \sum_{n=0}^{\infty} (-1)^n \frac{(ed)^{n+1}}{n+1} L_n^1 (cr/d) \right].
\]  

(3.20)

Using the relation [62]

\[
\sum_{n=0}^{\infty} a_n d^n = \int_0^\infty e^{-t} \sum_{n=0}^{\infty} \frac{a_n t^n d^n}{n!} dt,
\]

(3.21)

\(A(a)\) can be rewritten as

\[
A(a) = e \alpha n \left[ 1 + e cr \int_0^\infty dt e^{-t} \sum_{n=0}^{\infty} (-1)^n \frac{(td)^n}{(n+1)!} L_n^1 (cr/d) \right].
\]  

(3.22)

After performing the summation [61] we get

\[
A(a) = e \alpha n \left[ 1 + e cr \int_0^\infty dt (-cr\alpha e)^{-t/2} e^{-t(b+d\alpha)} J_{1/2} (2 \sqrt{cr\alpha e}) \right].
\]  

(3.23)
which upon integration [61] yields, for \( c < 0 \),

\[
A(\alpha) = e \alpha r e^{\frac{\epsilon c r}{(1 + \alpha e)}}
\]  

(3.24)

Substituting this in (3.8a), an uncoupled nonlinear equation for \( B \) is obtained:

\[
D^2 B \cdot B = -2(ea)^2 \exp \left[ 2\frac{\epsilon cr}{(\alpha + \alpha e)} \right].
\]  

(3.25)

By putting

\[
B(\alpha) = ea \exp \left[ \frac{\epsilon c r}{1 + \alpha e} - f(\alpha) \right]
\]  

(3.26)

we get the one dimensional Liouville equation,

\[
f'' = e^{2f}.
\]  

(3.27)

Three distinct solutions of this equation are [35]

\[
f = -\ln (r + \beta)
\]  

(3.28a)

\[
f = -\ln \left( \frac{\sin(\alpha (r + \beta))}{\alpha} \right)
\]  

(3.28b)

\[
f = \alpha r + \ln |2\alpha \eta^{\frac{1}{2}}| - \ln \left( 1 - \eta e^{2\alpha r} \right).
\]  

(3.28c)

Using (3.26), (3.24) and (3.3), \( K(\alpha) \) can be calculated for each solution (3.28). In each case \( H(\alpha) \) can be constructed by direct substitution of \( K(\alpha) \) in (3.2a). The solutions which follow from (3.28a), (3.28b) and (3.28c) can be reduced to the form (1.84b), (1.84c) and (1.84d) respectively.