CHAPTER - 6

BOUND STATES OF NON-ABELIAN DYONS WITH FERMIONS AND BOSONS

6.1 Introduction

The study of bound states of magnetic monopoles with fermions and bosons has a long history. Dirac, in his seminal paper on monopoles [14], showed that with the usual boundary condition of quantum mechanics, there exist no bound states of monopoles with electrons. The conclusion of Dirac, reinforced by other workers [74], had to be abandoned in the seventies following the theoretical observation of several unusual properties of the charge-pole system. It was shown that if the boundary conditions are chosen to ensure the self-adjointness of the Hamiltonian operator [75, 76], there can exist a spectrum of bound states with the lowest angular momentum value. There are several subtle problems in the charge-pole system which call for careful treatment and the question of the bound state formation between Dirac monopoles and charged particles is far from closed. It may be mentioned that the study of such bound states is also important in the context of experimental searches for monopoles [77].

The quantum mechanics of fermions and bosons in the background of non-abelian monopoles and dyons has been investigated by several workers. The bound state spectrum of a fermion in the background of a Wu-Yang monopole [37] and a dyon [73] of pure gauge theory was determined by Dereli, Swank and Swank [78]. They showed that while Wu-Yang monopoles have no bound states with fermions, dyons can have such bound states. For a 't Hooft-Polyakov monopole, Jackiw and Rebbi [47] demonstrated the existence
of non-degenerate zero energy bound states of monopoles with isospinor or isovector fermions. These solutions, incidentally, imply a doublet of solutions with fractional fermion number. A general analysis of the Dirac equation or Klein-Gordon equation in the background of the 't Hooft-Polyakov monopole is not possible at the moment because the regular monopole solution has not been cast in a closed form. In the PS limit where closed expression is available for the monopole solution, scattering solutions for the lowest partial wave were recently constructed by Marciano and Muzinich [79].

Bound states have not been obtained in Ref.79, probably because the Higgs-Fermi coupling is neglected in this work. Most of the studies have been done in the point limit of a 't Hooft-Polyakov monopole by allowing the size of the monopole core to tend to zero. In this limit the system is essentially abelian, and with special boundary conditions, there exist bound states in the lowest angular momentum channel [76]. Callias [76] has argued that for a regular monopole a finite number of bound states will exist.

In the asymptotic (point) limit of the PS monopole it has been shown by Cox and Yildiz [80] that the bound states can occur for all values of $\mathcal{J}$. It is the additional $-1/\kappa$ term present in the asymptotic Higgs field which is responsible for the bound states. Cox and Yildiz [80], however, have determined only the energy eigenvalues and did not construct the eigenfunctions. For PS dyon solutions in the point limit also, there exist an infinite number of bound states with all $\mathcal{J}$ values [81].

In addition to the regular monopole solution, there exist point singular monopoles and dyons [35] in the PS limit. In a recent work, Din and Roy [82] showed that an isospinor fermion in the background of a singular non-abelian
monopole has a well defined Hamiltonian with ordinary boundary conditions imposed on the wavefunctions at the origin. Monopole-fermion bound states were shown to occur for all $J$ values.

In this chapter we study the quantum mechanics of spin $1/2$ and spinless particles in the background of a point dyon potential. The form of the background dyon potential chosen is such that the asymptotic PS dyon solution arises as a special case. The background may be interpreted as due either to a point singular dyon or to a regular dyon solution with the size of the core neglected. In addition to the isospinor fermions and bosons which have already been discussed in the literature \[81, 83\] we have also studied isovector fermions and bosons. Exact bound state solutions to the Jackiw—Rebbi equations for all $J$, are obtained for isovector bosons and isospinor bosons and fermions. For isovector fermions a bound state has been obtained for the lowest angular momentum. Furthermore, we have shown that no bound state having $I_3 = 0$ exists for this system.

As mentioned above, part of our work concerning isospinor fermions and bosons overlaps that of Tang \[81, 83\] who considered the same problem with the asymptotic PS dyon as background. However, our method of solution is different. Tang introduced a singular string in the gauge potential by using a singular gauge transformation. The resulting abelianised equations are separated into radial and angular parts with the help of monopole harmonics \[84\]. We on the other hand, follow the method of Jackiw and Rebbi \[47\] who used spherical harmonics to separate radial and angular parts. The equivalence of the two procedures is demonstrated by utilizing a relationship between monopole harmonics and spherical harmonics which we have deduced.
In Section 6.2 we review briefly the classical SU(2) gauge theory and obtain the point singular background dyon potential. In Section 6.3 we obtain bound state solutions to the relevant Dirac and Klein-Gordon equations. We discuss various results in Section 6.4. The method of solving the radial equations of Section 6.3 is given in the Appendix 6.A. The proof of a relationship between spherical harmonics and monopole harmonics that we have deduced is given in the Appendix 6.B.

6.2 The background dyon potential

In Section 1.4 we have seen that the equations of motion following from the Lagrangian density,

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{2} \partial_{\mu} \phi^a \partial^{\mu} \phi^a - V(\phi), \]

(6.1)
can be reduced to

\[ r^2 J'' = 2 J K^2, \]

\[ r^2 H'' = H (2 K - m^2 r^2 + \frac{1}{g^2} H^2), \]

\[ r^2 K'' = K (K^2 - 1 + H^2 - J^2), \]

(6.2)

using the spherically symmetric 't Hooft-Polyakov-Julia-Zee ansatz [22, 23, 30]

\[ A_0^a = \frac{1}{g} r_a \frac{J(\omega)}{r^2}, \]

\[ A_i^a = \frac{1}{g} \epsilon_{a\lambda\mu} r_\mu A(\omega), \]

\[ \phi^a = \frac{1}{g} r_a \frac{H(\omega)}{r^2}, \]

(6.3)

where \( K(\omega) = 1 - r A(\omega). \)
We consider a particular solution of (6.2)

\[ K(n) = 0 \quad (\lambda(n) = \frac{4}{n}) \]
\[ H(n) = ar + b \]
\[ J(n) = cr + d, \]

where \( a, b, c, \) and \( d \) are arbitrary constants, as the background potential. Unlike the PS solution (1.81), this solution is singular at \( r = 0 \) and the classical energy is infinite. But at large distances this solution mimics the behaviour of a regular solution. In fact at large distances \( r \gg \frac{1}{(m_F^2)^{1/4}} \), this solution coincides with the PS dyon solution (1.93) with the identification

\[ a = \beta \cosh \eta \quad b = - \cosh \eta \]
\[ c = \beta \sinh \eta \quad d = - \sinh \eta. \]

So if the particles do not penetrate the dyon core, then (6.4) with (6.5) will be a good approximation to the regular solution. We can also consider (6.4) as a point dyon [35] solution. The relevance of such a solution is unclear at present, mainly because their classical energy is infinite. To our knowledge the quantum field theory of such objects has not been worked out so far. The fact that the singularity is at the origin seems to be a favourable point since one encounters a similar situation in the case of electron. The electric charge of this field configuration is given by

\[ Q = - \frac{4\pi d}{g}. \]
To include fermions we add to (6.1) the fermionic Lagrangian

\[ \mathcal{L}_\psi = \bar{\psi}_n (i \slashed{D} - M) \psi_n - g G \bar{\psi}_n T^a_{nm} \phi^a \psi_m \]  

(6.7)

where

\[ D_\mu \psi_n = \partial_\mu \psi_n - ig T^a_{nm} A^a_\mu \psi_m . \]

and the \( T^a \) are SU(2) generators satisfying

\[ [T^a, T^b] = i \epsilon_{abc} T^c. \]

\[ T^a_{nm} = T^a_{nm} \text{ for } I = \frac{1}{2} \text{ representation} \]

(6.8)

\[ = i \epsilon_{nam} \text{ for } I = 1 \text{ representation}. \]

We will consider fermions in the above two representations moving in the background potential (6.4).

For bosons instead of (6.7), we consider

\[ \mathcal{L}_u = D_\mu U^*_n D^\mu U_n - M^2 |U|^2 - g G U_n^* T^a_{nm} \phi^a U_n - \frac{1}{2} \phi^2 \phi^2 |U|^2. \]  

(6.9)

6.3 Solutions

i) Isospinor fermions

In this case the Dirac equation is given by

\[ i \gamma^\mu D_\mu \psi_n - \frac{i}{2} g G T^a_{nm} \phi^a \psi_m = M \psi_n. \]  

(6.10)
Putting \( \psi_n(x) = e^{-iEt} \psi_n(x') \) and substituting the gauge and Higgs field ansatz, we obtain the equation for \( \psi(x) \) as

\[
\left\{ \gamma^\mu \left[ D^\mu(x') - \frac{A(x)(x') \times \vec{x}}{2\mu} \right] + \frac{g}{2\mu} \frac{H(x')}{(\vec{r} \cdot \vec{x})} \right\} \psi(x') = (E - \beta M) \psi(x'),
\]

where \( \gamma^\mu \) and \( \beta \) are the Dirac matrices

\[
\gamma^\mu = \gamma^\nu \gamma^\mu, \quad \beta = \gamma^0
\]

We now proceed exactly as in Ref.47 to separate the radial and angular parts. But we use a different representation of Dirac matrices,

\[
\begin{align*}
\alpha^\mu &= \begin{pmatrix} 0 & i \sigma^\mu \\ -i \sigma^\mu & 0 \end{pmatrix}, \\
\beta &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{align*}
\]

It will be seen later that this choice is advantageous when we come to the solution of the radial equations.

Splitting the wavefunction into upper and lower components,

\[
\psi_n(x') = \begin{pmatrix} \chi^+(x') \\ \chi^-(x') \end{pmatrix},
\]

we find that the Dirac equation becomes

\[
\begin{align*}
\left\{ \gamma^\mu \partial_\mu - \frac{g}{2\mu} A(x)(x') \times \vec{r}_{nm} \right\} \chi^\pm_{jm} &= \pm \frac{g}{2\mu} \frac{H(x')}{(\vec{r} \cdot \vec{r}_{nm})} \chi^\mp_{jm} \\
- M \delta_{nm} \right\} \left\{ \gamma^\mu \partial_\mu - \frac{g}{2\mu} A(x)(x') \times \vec{r}_{nm} \right\} \chi^\mp_{jm} &= \pm \frac{g}{2\mu} \frac{H(x')}{(\vec{r} \cdot \vec{r}_{nm})} \chi^\pm_{jm}.
\end{align*}
\]

(6.14)
Here the first index on $\chi^\pm_{i_1}$ refers to the spin part and the second one to the isospin part. $\chi^\pm_{i_1n}$ is then expressed in terms of two scalar and vector functions,

$$\chi^\pm_{i_1n}(\vec{r}) = \left( g^\pm_{i_1}(\vec{r}) \delta_{i_1m} + \vec{g}^\pm_{i_1}(\vec{r}) \vec{a}_{i_1m} \right) \tau^a_{mn} \quad (6.15)$$

The scalar and vector functions are now expressible in terms of ordinary and vector spherical harmonics respectively,

$$g^\pm_{i_1}(\vec{r}) = g^\pm_{i_1}(\lambda) \ Y^M_J(\Omega)$$

$$g^\pm_{a}(\vec{r}) = \vec{P}^\pm_{a}(\lambda) \ \hat{a}_a \ Y^M_J(\Omega) + B^\pm_{a}(\lambda) \ \frac{j}{J} \ a_a \ Y^M_J(\Omega) + C^\pm_{a}(\lambda) \ \frac{j}{J} \ \epsilon_{abc} \ b_b \ d_c \ Y^M_J(\Omega) \quad (6.16)$$

Here $\frac{j}{J} = \sqrt{J(J+1)}$ and $J$ is the total angular momentum. Total angular momentum is obtained by combining orbital and spin angular momenta and isospin. In this case it takes values $0, 1, 2, \ldots \ldots \ B_o = C_o = 0$ by definition.

Substituting (6.15) and (6.16) in (6.14) we get eight radial equations:

$$\left( \frac{d}{dr} + \frac{j}{J} \frac{G_{J}(\lambda)}{2J} \right) P^\pm_J = - \frac{i}{\hbar} V^\pm_J = M C^\pm_J = \mp \left( \frac{J(\lambda)}{2J} \ P^\pm_J + E G^\pm_J \right) \quad (all \ J) \quad (6.17)$$

$$\left( \frac{d}{dr} + \frac{i}{\hbar} \frac{G_{J}(\lambda)}{2J} \right) G^\pm_J = - \frac{i}{\hbar} C^\pm_J = M P^\pm_J = \mp \left( \frac{J(\lambda)}{2J} \ G^\pm_J + E P^\pm_J \right) \quad (all \ J)$$

$$\left( \frac{d}{dr} + \frac{j}{J} \frac{G_{J}(\lambda)}{2J} \right) B^\pm_J = - \frac{i}{\hbar} P^\pm_J = \pm \left( \frac{J(\lambda)}{2J} \ B^\pm_J + E C^\pm_J \right) \quad (J > 0) \quad (6.17)$$

$$\left( \frac{d}{dr} + \frac{i}{\hbar} \frac{G_{J}(\lambda)}{2J} \right) C^\pm_J = - \frac{i}{\hbar} G^\pm_J = \pm \left( \frac{J(\lambda)}{2J} \ C^\pm_J + E B^\pm_J \right) \quad (J > 0).$$
where we have substituted \( A(\kappa) \) from (6.4). These equations, unlike the corresponding equations in Ref. 47 have, a \( \pm \) sign before \( E \) and \( J(\kappa) \). This is because of our choice of the representation for Dirac matrices.

The advantage of this choice is that the equations can now be transformed into a set of four independent coupled pairs of first order differential equations. Each of them can be decoupled and the resulting second order differential equations can be solved exactly. In contrast, the decoupling of equations in Ref. 47 gives a fourth order differential equation [82].

We now discuss \( J = 0 \) and \( J > 0 \) solutions separately.

**\( J = 0 \) solutions**

Setting

\[
P_0^\pm + G_0^\pm = R^\pm
\]

\[
P_0^\pm - G_0^\pm = S^\pm
\]

the \( J = 0 \) radial equations become

\[
\left[ \frac{d}{d \kappa} + \frac{1}{\hbar} \mp \left( \frac{B}{\hbar} + m_+ \right) \right] R^\pm = \mp \left( \frac{D}{\hbar} + \epsilon_+ \right) R^\mp \tag{6.19a}
\]

\[
\left[ \frac{d}{d \kappa} + \frac{1}{\hbar} \mp \left( \frac{B}{\hbar} + m_- \right) \right] S^\pm = \mp \left( \frac{D}{\hbar} + \epsilon_- \right) S^\mp , \tag{6.19b}
\]

where we have substituted \( J(\kappa) \) and \( H(\kappa) \) from (6.4). Also \( B = G b/\alpha \), \( D = \alpha/2 \), \( m_\pm = \left( a G b/\alpha \right) \pm M \), and \( \epsilon_\pm = \epsilon_\pm E \). Here we need solve only for the first set. The solution of the second set can be obtained by suitable replacements.
For solving (6.19a), consider one more dependent variable transformation,

\[ X^\pm = R^+ \pm R^- . \]  

(6.20)

(6.19a) now becomes

\[ \left( \frac{d}{d \tau} + \frac{4}{\hbar} \right) X^\pm = \left( \frac{B \pm D}{\hbar} + m \pm \epsilon \right) X^\mp , \]  

(6.21)

where we have suppressed the symbol over \( m \) and \( \epsilon \). This equation is exactly the same as that of the hydrogen atom problem in Dirac theory [85] if the \( B / \hbar \) term is absent, and it can be solved by a similar technique (see Appendix 6.A). For \( \epsilon < m \) we get discrete bound states and for \( \epsilon > m \) we get continuum states. Here we give only the bound state solutions. The continuous spectrum can be obtained by suitable replacements [85].

The solution for \( \epsilon < m \) is given by

\[ X^\pm = \sqrt{m \pm \epsilon} \quad e^{-\rho \gamma} \quad \rho \gamma - 1 \left( Q_2 \pm Q_1 \right) , \]  

(6.22)

where

\[ \rho = \pm \lambda \hbar , \quad \lambda = \sqrt{\epsilon^2 - m^2} , \quad \gamma = \sqrt{B^2 - D^2} , \]

\[ Q_1 = \frac{\gamma + (Bm - D\epsilon) / \lambda}{(Dm - B\epsilon) / \lambda} \quad {}_1F_1 \left( \frac{\gamma + (Bm - D\epsilon) / \lambda}{2\gamma + 1, \rho} \right) , \]  

\[ Q_2 = \frac{\gamma + (Bm - D\epsilon) / \lambda}{(Dm - B\epsilon) / \lambda} \quad {}_1F_1 \left( \frac{\gamma + (Bm - D\epsilon) / \lambda}{2\gamma + 1, \rho} \right) , \]  

and \( {}_1F_1 (a, b, \rho) \) are Kummer functions. For (6.22) to vanish at the origin,
we should set $B > D$. The corresponding solutions to (6.19) are

$$R^\pm = \alpha e^{-\beta/2} p, \quad \gamma^{-1} \left\{ \sqrt{m_+ + \varepsilon_+} \left( q_2^+ + q_2^- \right) \right\}$$

$$S^\pm = \beta e^{-\beta/2} \rho, \quad \gamma^{-1} \left\{ \sqrt{m_- - \varepsilon_-} \left( q_2^+ - q_2^- \right) \right\}$$

(6.24a)

(6.24b)

where

$$\varepsilon_\pm = 2 \lambda_\pm \eta, \quad \lambda_\pm = \sqrt{\varepsilon_\pm^2 - m_\pm^2}, \quad \gamma = \sqrt{B^2 - D^2}$$

$$Q_1^\pm = \frac{1}{F}(\gamma + \frac{B m_\pm - D \varepsilon_\pm}{\lambda_\pm}, \alpha \gamma + 1, \varepsilon_\pm)$$

(6.25)

$$Q_2^\pm = \frac{\gamma + (B m_\pm - D \varepsilon_\pm)/\lambda_\pm}{(D m_\pm - B \varepsilon_\pm)/\lambda_\pm} F(1 + \gamma + \frac{B m_\pm - D \varepsilon_\pm}{\lambda_\pm}, \alpha \gamma + 1, \varepsilon_\pm)$$

and $\alpha$ and $\beta$ are arbitrary constants (fixed by normalisation). The normalisation condition after the angular integration becomes

$$1 = \alpha \sum_{\pm} \int_0^\infty \rho^2 dr \left( R_0^\pm g_0^\pm + G_0^\pm g_0^\pm \right)$$

$$= \sum_{\pm} \int_0^\infty \rho^2 dr \left( R^\pm R^\pm + S^\pm S^\pm \right).$$

(6.26)

For the convergence of this integral the Kummer functions should reduce to polynomials. From this requirement we obtain the eigenvalue spectrum. The conditions are

$$\gamma + \frac{B m_+ - D \varepsilon_+}{\lambda_+} = - n_+$$

$$\gamma + \frac{B m_- - D \varepsilon_-}{\lambda_-} = - n_-$$

(6.27a)

(6.27b)
where $n_1$ and $n_2$ are positive integers. Zero is not possible because in this case $\gamma + (Bm - D\epsilon)/\lambda = (Dm - B\epsilon)/\lambda$ and $Q_4$ remains divergent*. Solving for $\epsilon_+$ and $\epsilon_-$ we get

$$
\frac{\epsilon_+}{m_+} = \frac{BD/(n_1 + \gamma)^2 \pm \sqrt{1 - (B^2 - D^2)/(n_1 + \gamma)^2}}{1 + D^2/(n_1 + \gamma)^2} \tag{6.28a}
$$

$$
\frac{\epsilon_-}{m_-} = \frac{BD/(n_1 + \gamma)^2 \pm \sqrt{1 - (B^2 - D^2)/(n_1 + \gamma)^2}}{1 + D^2/(n_1 + \gamma)^2} \tag{6.28b}
$$

Inspection of Eq. (6.28a,b) shows that it is not possible to satisfy these two equations simultaneously for the same value of energy. So we can take either (6.28a) and set $\beta = 0$ ($P_0^\pm = G_0^\pm$) or we take (6.28b) and set $\alpha = 0$ ($P_0^\pm = -G_0^\pm$). The corresponding energy values in terms of the original parameters are

$$
E_{n_1} = -\frac{c}{2} + \frac{M + aG/2}{1 + d^2/4(n_1 + \gamma)^2} \left[ \frac{bdG}{4(n_1 + \gamma)^2} \pm \sqrt{1 - \frac{b^2G^2 - d^2}{4(n_1 + \gamma)^2}} \right] \tag{6.29a}
$$

$$
E_{n_2} = \frac{c}{2} + \frac{M - aG/2}{1 + d^2/4(n_1 + \gamma)^2} \left[ \frac{bdG}{4(n_1 + \gamma)^2} \pm \sqrt{1 - \frac{b^2G^2 - d^2}{4(n_1 + \gamma)^2}} \right] \tag{6.29b}
$$

We get different solutions for each sign in (6.29) because $\epsilon_\pm$ depends on energy.

* If $n_1$ and $n_2$ are zero we get $Q_4^\pm = \beta \xi F_1(1, 2\gamma + 1, \lambda, \epsilon_\pm)$, which does not reduce to a polynomial.
As remarked earlier, in this case the eight coupled differential equations can be transformed to a system of four independent coupled pairs of first order differential equations. This is achieved by defining eight new functions in the following way:

\[
\begin{align*}
X_J^\pm &= P_J^\pm + G_J^\pm + B_J^\pm + C_J^\pm \\
Y_J^\pm &= P_J^\pm + G_J^\pm - B_J^\pm - C_J^\pm \\
Z_J^\pm &= P_J^\pm - G_J^\pm + B_J^\pm - C_J^\pm \\
W_J^\pm &= P_J^\pm - G_J^\pm - B_J^\pm - C_J^\pm.
\end{align*}
\]

The radial equations take the form

\[
\begin{align*}
D_{m^+}^\pm X_J^\pm &= \left[ \frac{j}{\kappa} \mp \left( \frac{p}{\kappa} + \epsilon_+ \right) \right] X_J^\pm \\
D_{m^-}^\pm Y_J^\pm &= \left[ -\frac{j}{\kappa} \mp \left( \frac{p}{\kappa} + \epsilon_- \right) \right] Y_J^\pm \\
D_{m^-}^\pm Z_J^\pm &= \left[ -\frac{j}{\kappa} \mp \left( \frac{p}{\kappa} + \epsilon_- \right) \right] Z_J^\pm \\
D_{m^+}^\pm W_J^\pm &= \left[ -\frac{j}{\kappa} \mp \left( \frac{p}{\kappa} + \epsilon_- \right) \right] W_J^\pm,
\end{align*}
\]

where \( D_{m^\pm} = \frac{d}{d\kappa} \pm \frac{1}{\kappa} \mp \left( \frac{p}{\kappa} + m \right) \) and \( \epsilon_\pm \) and \( m_\pm \) are the same as before.

These equations can be solved as in the previous case. (See also Appendix 6.A). We need solve only the first equation. Solutions to the remaining
three can be obtained by suitable replacements. We give only the final
results (for \( \varepsilon < m \)):

\[
\begin{align*}
\chi_\pm &= \alpha e^{-\kappa / 2} e^{\gamma - 1} \left[ \sqrt{m_+ \varepsilon_+ (Q_+^2(j) + Q_+^2)} \pm \sqrt{m_- \varepsilon_- (Q_+^2(j) - Q_+^2)} \right] \\
\gamma_\pm &= \beta e^{-\kappa / 2} e^{\gamma - 1} \left[ \sqrt{m_+ \varepsilon_+ (Q_-^2(j) + Q_-^2)} \pm \sqrt{m_- \varepsilon_- (Q_-^2(j) - Q_-^2)} \right] \\
Z_\pm &= \eta e^{-\kappa / 2} e^{\gamma - 1} \left[ \sqrt{m_+ \varepsilon_+ (Q_-^2(j) + Q_-^2)} \pm \sqrt{m_- \varepsilon_- (Q_-^2(j) - Q_-^2)} \right] \\
W_\pm &= \delta e^{-\kappa / 2} e^{\gamma - 1} \left[ \sqrt{m_+ \varepsilon_+ (Q_-^2(j) + Q_-^2)} \pm \sqrt{m_- \varepsilon_- (Q_-^2(j) - Q_-^2)} \right]
\end{align*}
\]

where

\[
\gamma = \sqrt{B^2 + j^2 - D^2}
\]

\[
Q_\pm = \frac{1}{2} \frac{B m_\pm - D e_\pm}{\lambda_\pm}
\]

\[
Q_\pm^2 = \frac{\gamma + (B m_\pm - D e_\pm)}{\lambda_\pm} - \frac{\gamma + (B m_\pm - D e_\pm)}{\lambda_\pm} \left( \gamma + 1, e_\pm \right)
\]

and \( e_\pm \) and \( \lambda_\pm \) are given by the same expressions as before. \( Q^\pm_\pm(j) \) is obtained by changing the sign of \( j \) in \( Q^\pm_\pm(j) \). As in the earlier case

the requirement of normalisability of wave functions leads to the conditions

\[
\begin{align*}
\gamma + \frac{B m_+ - D e_+}{\lambda_+} &= - \eta_+ \\
\gamma + \frac{B m_- - D e_-}{\lambda_-} &= - \eta_-
\end{align*}
\]
Here $\eta_1$ and $\eta_2$ can be zero contrary to the previous case. Depending on the sign of $Dm - Be$, one normalisable solution exists. Suppose $(Dm - B\epsilon) > 0$. In this case $Q_2(j) = 0$ and $Q_2(-j)$ is nonzero and divergent. If $(Dm - B\epsilon) < 0$, $Q_2(-j) = 0$ and $Q_2(j)$ is nonzero and divergent. Since $\epsilon$ is given by a quadratic equation it can have two roots. For $\eta_1 = 0$ and $\eta_2 = 0$, $\epsilon_\pm$ is given by

$$
\epsilon_+ = m_+ \frac{BD/j^2 + \sqrt{1 + (B^2 - D^2)/j^2}}{1 + B^2/j^2} \quad (6.35a)
$$

$$
\epsilon_- = m_- \frac{BD/j^2 + \sqrt{1 + (B^2 - D^2)/j^2}}{1 + B^2/j^2} \quad (6.35b)
$$

Hence for both $(m_+, \epsilon_+)$ and $(m_-, \epsilon_-)$ we obtain

$$
Dm - B\epsilon = m \frac{D + \sqrt{1 + (B^2 - D^2)/j^2}}{1 + B^2/j^2} \quad (6.36)
$$

Consider the case $D > 0$. Then if $|D| > (1 + (B^2 - D^2)/2)^{1/2}$ we find $(Dm - B\epsilon)$ always positive. So solutions containing $Q_2(-j)$ should be discarded (by setting $\beta = \delta = 0$). Similarly for $D < 0$ we should discard solutions containing $Q_2(j)$ ($\epsilon > 0$). If $|D| < (1 + (B^2 - D^2)/j^2)^{1/2}$, where $j_{\text{max}} = J_{\text{max}}(J_{\text{max}} + 1)$ is the highest value satisfying this condition (all lower angular momenta will evidently satisfy this), we should again consider two cases. For energy obtained by taking the upper sign in (6.35) we discard solutions containing $Q_2(j)$. For the lower sign the solution containing $Q_2(-j)$ should be discarded. This is the same for both positive and negative values of $D$ and for all angular momentum $J \leq J_{\text{max}}$.

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*Lowest nonzero value of $j^2 = J(J+1) < \ell^2$. If this value satisfies the above condition all higher angular momenta will automatically satisfy the same.
For integers \( n_1, n_2 > 0 \) the energy levels are the same as given in (6.29a) and (6.29b) except for the expression for \( \gamma \) when \( j = 0 \) we get exactly (6.29). Here again we should consider two separate cases since (6.29a) and (6.29b) cannot both be satisfied simultaneously for the same value of energy. Corresponding to each sign in (6.29a), we get two levels: \( \alpha = 0, \beta = \eta = \delta = 0 \) and \( \alpha = 0, \beta = 0, \eta = \delta = 0 \). In either case we get \( p^\pm_J = g_J^\pm \) and \( c_J^\pm = b_J^\pm \). Similarly for each sign in (6.29b) we get two solutions \( \alpha = \beta = 0, \eta = 0, \delta = 0 \) and \( \alpha = 0, \beta = 0, \eta = 0, \delta = 0 \). In this case we get solutions satisfying \( p^\pm_J = -g_J^\pm \) and \( c_J^\pm = -b_J^\pm \). Further discussion of these results will be given in Section 6.4.

**ii) Isovector fermions**

In order to facilitate solution we use a different set of Dirac matrices:

\[
\alpha = \begin{pmatrix} 0 & i \sigma \\ -i \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]  

(6.37)

With this representation the Dirac equation can be reduced to the form

\[
\left[ \alpha \cdot \gamma \delta_{nm} - A(\alpha) (\alpha \cdot \sigma_m - \sigma_m \alpha) \right] \chi^\pm_n \]

\[
= - \left[ \frac{H(\alpha)}{\hbar} \epsilon_{nm} \chi^\pm_n + M \delta_{nm} \pm \left( \frac{T(\alpha)}{\hbar} \epsilon_{nm} \chi^\pm_n + E \delta_{nm} \right) \right] \chi^\pm_m.
\]  

(6.38)

Separation of radial and angular parts is achieved by the use of vector spinor harmonics:

\[
\chi^\pm_n = F^\pm_{\lambda}(\alpha) \hat{\chi}_n \gamma^\pm_{JM} + F^\pm_{\lambda}(\alpha) \alpha \partial_m \gamma^\pm_{JM} + F^\pm_{\lambda}(\alpha) \partial_m \gamma^\pm_{JM} + F^\pm_{\lambda}(\alpha) \partial_m \gamma^\pm_{JM} + F^\pm_{\lambda}(\alpha) \partial_m \gamma^\pm_{JM} + F^\pm_{\lambda}(\alpha) \partial_m \gamma^\pm_{JM}.
\]  

(6.39)
where \( y_{JM} \) and \( y'_{JM} \) are spinor harmonics:

\[
\begin{align*}
\psi_{J M}(\vec{r}) &= \begin{pmatrix} \sqrt{\frac{J+M}{2\pi}} \ y_{J-M, M} \\ \sqrt{\frac{J-M}{2\pi}} \ y_{J+M, M} \end{pmatrix}, \\
\psi'_{J M}(\vec{r}) &= \begin{pmatrix} \sqrt{\frac{J+M+1}{2\pi}} \ y_{J-M, M+1} \\ \sqrt{\frac{J-M+1}{2\pi}} \ y_{J+M, M+1} \end{pmatrix}
\end{align*}
\]

(6.40)

and \( L_m = -i \epsilon_{mij} \partial_j \) is the angular momentum operator. \( F_{+}^\pm = F_{-}^\pm = 0 \) for \( J > \frac{1}{2} \) by definition. The radial equations are

\[
\begin{align*}
\left( \frac{d}{d \lambda} + \frac{j + \frac{1}{2}}{\lambda} \right) F_{3-}^{\pm} &= -(\mathbf{M} \pm \mathbf{E}) F_{3+}^{\mp} \quad \text{(all J)} \quad (6.41a) \\
\left( \frac{d}{d \lambda} - \frac{j - \frac{1}{2}}{\lambda} \right) F_{3+}^{\pm} &= -(\mathbf{M} \pm \mathbf{E}) F_{3-}^{\mp} \quad \text{(all J)} \quad (6.41b) \\
\end{align*}
\]

(6.42a)

(6.42b)

(6.43a)

(6.43b)

where \( j = J + \frac{1}{2} \). Here \( F_{+}^{\pm} \) and \( F_{-}^{\pm} \) do not depend on either the Higgs
field or the time component of gauge field. We will later see that $F_{1+}^\pm$ and $F_{2-}^\pm$ correspond to solutions having the third component of isospin $I_3 = 0$.

$J=\frac{1}{2}$ solutions

Solutions to (6.41) are readily obtained by directly decoupling them. There are no normalisable solutions corresponding to bound states ($E < M$); only a continuous spectrum exists.

Equations (6.42), however, possess bound state solutions. Energy levels are similar to those of zero angular momentum isospinor fermions. To see this, define

$$X^\pm = F_{2-}^\pm + F_{3+}^\pm$$

$$Y^\pm = F_{2-}^\pm - F_{3+}^\pm$$  \hspace{1cm} (6.44)

The four mutually coupled equations now change to two independent coupled equations

$$(d/dn + \frac{4}{\hbar}) X^\pm = \left( \frac{B + D}{\hbar} + m_\pm \pm \epsilon_\pm \right) X^\pm$$  \hspace{1cm} (6.45a)

$$(d/dn + \frac{4}{\hbar}) Y^\pm = \left( \frac{B + D}{\hbar} + m_\pm \pm \epsilon_\pm \right) Y^\pm$$  \hspace{1cm} (6.45b)

where we have substituted for $H(n)$ and $J(n)$ and $B = Gb$, $D = c$, $m_\pm = G a \pm M$ and $\epsilon_\pm = c \pm E$. These equations are similar to (6.21) studied earlier. The solutions to (6.45a) and (6.45b) are not normalisable simultaneously. So the two solutions are $X^\pm \neq 0$, $Y^\pm = 0$ and $X^\pm = 0$, $Y^\pm \neq 0$. The corresponding energy levels are given by

$$E_{n_\pm} = c \pm \frac{M - \sqrt{c_G}}{1 + d^2/(\gamma^2 - r^2)} \left[ \frac{bdG}{c_G + r^2} \pm \sqrt{1 - \frac{b^4G^2 - d^2}{(c_G + r^2)^2}} \right]$$  \hspace{1cm} (6.46a)
respectively.

\( J > \frac{1}{2} \) solutions

As in the \( J = \frac{1}{2} \) case there are no normalisable state solutions for equations (6.41); only continuum solutions exist. We have not been able to solve the remaining equations for \( J > \frac{1}{2} \) since decoupling them leads to at least fourth order differential equations.

iii) Isospinor bosons

In this case the Klein-Gordon equation

\[
D_U D^A U(\vec{x}) = - \left( \gamma^2 + g^2 \gamma^3 \phi^2 + g G \gamma^3 \phi^4 \right) U(\vec{x})
\]

(6.47)

can be simplified to

\[
\left[ \nabla^2 - \frac{1}{\hbar^2} \right] U(\vec{x}) + \left( \frac{G H(\vec{x})}{\lambda} - \frac{E J(\vec{x})}{\lambda} \right) (\vec{E} \cdot \vec{x}) - \frac{1}{\hbar^2} \frac{E J(\vec{x})}{\lambda} + \frac{E^2 - M^2}{\hbar^2} \right] U(\vec{x}) = 0,
\]

(6.48)

where \( U(\vec{x}) = e^{i \vec{E} \cdot \vec{x}} U(\vec{x}) \). The angular part can be separated using spinor harmonics:

\[
U(\vec{r}) = F_+(\lambda) \Phi_+(\Omega) + F_-(\lambda) \Phi_-(\Omega).
\]

(6.49)

\( F_+(\lambda) \) and \( F_-(\lambda) \) obey the radial equations,

\[
\left[ \frac{d^2}{d\lambda^2} + \frac{\lambda}{\lambda} \frac{d}{d\lambda} + \frac{\lambda - \gamma^2}{\lambda^2} + \frac{\beta}{\lambda} + \epsilon^2 - m^2 \right] \begin{pmatrix} F_+ \\ F_- \end{pmatrix} = \begin{pmatrix} E \gamma - \frac{E d}{\lambda} - \frac{G b}{\lambda} \end{pmatrix} \begin{pmatrix} F_+ \\ F_- \end{pmatrix},
\]

(6.50)
where
\[ \gamma \approx \mathcal{J}(\mathcal{J}+1) + b^2 a^2 - \frac{d^2}{4} \]
\[ \beta = \frac{cd}{a^2} - 2ab \lambda \]
\[ \epsilon \approx E^2 + \frac{c^2}{4} \]
\[ m \approx M^2 + \frac{\lambda^2 a^2}{4}. \]

Here we need not consider lowest angular momentum (\( \mathcal{J} = 1/4 \)) and higher angular momenta (\( \mathcal{J} > 1/4 \)) separately, since both equations are valid for all \( \mathcal{J} \).

To solve (6.50) we define two new functions
\[ X^\pm = F_+ + F_- \quad \text{(6.52)} \]

The radial equations become
\[ \frac{d^2 X^\pm}{d \lambda^2} + \left[ \frac{4 - \gamma}{\lambda^2} \right] \frac{\beta}{\lambda} + \epsilon \approx m^2 X^\pm = 0 \quad \text{(6.53)} \]

where
\[ \epsilon \approx (E + c/2)^2 \]
\[ m \approx m^2 \pm Ga/2 \]
\[ \beta \approx \beta \pm \left( \frac{G a}{2} - E d \right). \]

The solution is given by
\[ X^\pm = e^{-P_{\pm}/2} P_{\pm}^{\gamma+1/2} \frac{\Gamma_{\gamma+1/2}}{\Gamma_{\gamma+1}} \left( \frac{\gamma+1}{2} - \frac{\beta_{\pm} a}{\lambda} , 2\gamma+1 , P_{\pm} \right), \quad \text{(6.55)} \]
\[ P_\pm = 2 \lambda_\pm \hat{r} \]
\[ \lambda_\pm = (\epsilon_\pm^2 - \varepsilon_\pm^2)^{3/4} \quad (6.56) \]

For (6.55) to vanish at the origin we require \( b \hat{\rho}^2 > d^2/4 \). Also for normalisability, the Kummer functions should reduce to polynomials. From this we arrive at the condition

\[ \gamma + \frac{d}{2} - \beta_\pm / 2 \lambda_\pm = - \eta \quad (6.57) \]

where \( \eta = 0, 1, 2 \ldots \). The energy levels are explicitly given by

\[ E_n^+ = \frac{\xi}{2} + \left\{ \frac{1}{2} \lambda (a b + 4 h^2ab) \pm \left[ (-G^2b^4 - 16a^2b^2d^2 + 8b^2ab^2) \right. \right. \]
\[ \left. \left. \times (4 \eta_f^{-1} + (\lambda^2 + \xi^2 - \frac{Ga}{\lambda^2}) \left( 1 + d^2/\eta_f^2 \right)^{3/2} \right] \right\} (1 + \frac{d^2}{\eta_f^3})^{-1} \quad (6.58a) \]

for the upper sign in (6.57) and

\[ E_n^- = - \frac{\xi}{2} + \left\{ \frac{1}{2} \lambda (a b + 4 h^2ab) \pm \left[ (-G^2b^4 - 16a^2b^2d^2 - 8b^2ab^2) \right. \right. \]
\[ \left. \left. \times (4 \eta_f^{-1} + (\lambda^2 + \xi^2 + \frac{Ga}{\lambda^2}) \left( 1 + d^2/\eta_f^2 \right)^{3/2} \right] \right\} (1 + \frac{d^2}{\eta_f^3})^{-1} \quad (6.58b) \]

for the lower sign in (6.57) and \( \eta_f = 3 \left( \eta + \gamma + 1/4 \right) \). As in the previous cases (6.58a) and (6.58b) cannot be satisfied simultaneously. So we take one solution as \( X^+ \neq 0 \), \( X^- = 0 \), the corresponding energy levels are given by (6.58a). Similarly eigenfunctions with \( X^+ = 0 \) and \( X^- \neq 0 \) will have energy values (6.58b). Since (6.57) can be satisfied only with a positive \( \beta_\pm \), the \( \pm \) sign appearing before the square root in (6.58) should be chosen to satisfy this.
iv) Isovector bosons

The static Klein-Gordon equation in this case can be written in the form

\[ \nabla^2 U_n + E^2 U_n + 2A(a) \left[ E_n \partial_i U_i - E_k \partial_n U_k \right] + 2iE \epsilon_{abc} \hat{e}_b \frac{J_a(a)}{\hbar} U_c \]

\[ - A_0(a) \left[ U_n + \hat{e}_n \hat{e}_k U_k \right] + \left( \frac{J(a)}{\hbar} \right)^2 \left[ U_n - \hat{e}_n \hat{e}_b U_b \right] \]

\[ = \left[ M + \left( \frac{\hbar H(a)}{\hbar} \right)^2 \right] U_n + \frac{G H(a)}{\hbar} \epsilon_{nab} \hat{e}_b \partial_c Y^m_j(a), \]

The angular part can be separated using vector spherical harmonics,

\[ U_n = X_J(a) \hat{e}_n Y^m_J(\Omega) + Y_J(a) \frac{\hbar}{j} \hat{e}_n Y^m_J(\Omega) + \frac{Z_J}{ij} \epsilon_{nab} \hat{e}_b \partial_c Y^m_j(a), \]

where \( Y^m_J(a) \) are ordinary spherical harmonics and \( j = \sqrt{J(J+1)} \). \( J \) is the total angular momentum and it takes values 0,1,2, ... . \( Y_0 \) and \( Z_0 \) are zero by definition. The radial equations are

\[ X_J'' + \frac{\hbar}{n} X_J' - \frac{i^2}{n^2} X_J + \left[ E^2 - M^2 - \left( \hbar H(a)/\hbar \right)^2 \right] X_J = 0 \quad (\text{all } J) \quad (6.60a) \]

\[ Y_J'' + \frac{\hbar}{n} Y_J' + \frac{1}{n^2} Y_J + \left[ E^2 + \left( \frac{J(a)}{\hbar} \right)^2 - M^2 - \left( \hbar H(a)/\hbar \right)^2 \right] Y_J \]

\[ = \left[ \frac{2E J(a) - G H(a)}{\hbar} \right] Z_J \quad (J > 0) \quad (6.60b) \]

\[ Z_J'' + \frac{\hbar}{n} Z_J' + \frac{1}{n^2} Z_J + \left[ E^2 + \left( \frac{J(a)}{\hbar} \right)^2 - M^2 - \left( \hbar H(a)/\hbar \right)^2 \right] Z_J \]

\[ = \left[ \frac{2E J(a) - G H(a)}{\hbar} \right] Y_J \quad (J > 0), \quad (6.60c) \]

where we have substituted for \( A(a) \).
We note that (6.60a) is independent of the electric degree of freedom of the monopole. We will later prove that \( X_{J}(a) \) corresponds to the third component of isospin zero. Contrary to the case of isospinor fermions there is an interaction with the Higgs field. This is due to the addition of a fourth power boson term (last term in (6.9)) in the Lagrangian. Note that \( X_{J} \) does not couple to the linear term in the Higgs field. Since (6.60a) is the only equation for the lowest partial wave \( (J = 0) \) we conclude that the electric charge of the dyon has no effect on the lowest angular momentum boson. The lowest angular momentum bound state corresponds to a singlet with \( I_{3} = 0 \).

Solutions to (6.60a) are easily obtained in terms of Kummer functions. Substituting for \( H(Q) \) we get

\[
X_{J} = \alpha e^{-\rho/\lambda} \rho^{\gamma + 3/2} {}_{1}F_{1}\left( \frac{\alpha b k_{2}}{\lambda}, 2\gamma + 1, \rho \right),
\]

where

\[
\begin{align*}
\rho &= 2\lambda \lambda \\
\lambda &= \left( M^{2} + k^{2}a^{2} - E^{2} \right)^{3/2} \\
\gamma &= \left( \frac{a}{4} + \frac{a^{2}b^{2} + j^{2}}{4} \right)^{3/2}
\end{align*}
\]

and \( \alpha \) is an arbitrary constant. The Kummer function should reduce to polynomials for normalisability. From this we get the condition

\[
\gamma + \frac{a}{2} + \frac{abk_{2}^{2}}{\lambda} = -n,
\]

where \( n = 0, 1, 2, \ldots \). To satisfy this condition, either \( a \) or \( b \) should
be negative. In such a case only we get bound states. The expression for the bound state energy is

\[ E_n = \pm \left[ \frac{\hbar^2 a^2}{2m} - (a b^2)/(n^2 + \gamma)^2 \right]. \] (6.64)

There are other types of higher angular momentum states obtainable from (6.60b) and (6.60c). To deduce this we set

\[ \gamma \pm \zeta_j = \mathcal{W}_j^\pm. \] (6.65)

(6.60c) and (6.60b) now become

\[
\frac{d^2 \mathcal{W}_j^\pm}{d\gamma^2} + \frac{2}{\lambda_j} \frac{d \mathcal{W}_j^\pm}{d\lambda_j} + \frac{1-\delta_j}{\lambda_j^2} \mathcal{W}_j^\pm + \left[ E^2 - \left( \frac{J(\alpha)}{\lambda_j} \right)^2 - M^2 - \left( \frac{b H(\alpha)}{\lambda_j} \right)^2 \right] \mathcal{W}_j^\pm
\]

\[ = \pm \left[ \frac{\alpha \epsilon(\gamma - G H(\alpha))}{\gamma} \right] \mathcal{W}_j^\pm. \] (6.66)

Substituting for \( J(\alpha) \) and \( H(\alpha) \),

\[
\left[ \frac{d^2}{d\gamma^2} + \frac{1}{\lambda_j^2} \mathcal{W}_j^\pm + \frac{\beta_j}{\lambda_j} + \epsilon_j^\pm - m_j^\pm \right] \mathcal{W}_j^\pm = 0, \] (6.67)

where

\[
\gamma^2 = j^2 + k^2 b^2 - d^2 - \frac{3}{4}
\]

\[ \beta_j = acd - 2k^2 b + 2E d \pm G b \]

\[ \epsilon_j^\pm = E \mp c \] (6.68)

\[ m_j^\pm = M^2 + k^2 a^2 \pm G a. \]

The solutions are given by

\[ \mathcal{W}_j^\pm = \alpha_j^\pm e^{-\frac{\beta_j}{\lambda_j}} \frac{1}{\Gamma_1 \left( \gamma + \frac{1}{2} - \frac{\beta_j}{\lambda_j^2} \right)} \left[ 2 \gamma + 1, p_j^\pm \right]. \] (6.69)
For normalisability

$$\gamma + \frac{1}{2\alpha} - \frac{p_z}{2\lambda} = -n.$$  \hfill (6.70)

For a bound state $p_z$ should be positive. This fact will be used to calculate the energy levels which follow from (6.70). The energy corresponding to the upper sign solution in (6.70) is

$$E_n^+ = c + \left\{ d \left( g_{zb} - h_{ab} \right) \pm \left[ \left( - \frac{g_{zb}^2}{4} - h_{ab} h_{ab} + G h_{ab} h_{ab} \right) \times (4\eta_1)^{-1} \right] \right\} \left[ \frac{1}{4} + \frac{p_z^2}{\eta_1^2} \right]^{-\frac{1}{2}}$$

and that corresponding to the lower sign solution is

$$E_n^- = -c + \left\{ d \left( -g_{zb} + h_{ab} \right) \pm \left[ \left( \frac{g_{zb}^2}{4} - h_{ab} h_{ab} - G h_{ab} h_{ab} \right) \times (4\eta_1)^{-1} \right] \right\} \left[ \frac{1}{4} + \frac{p_z^2}{\eta_1^2} \right]^{-\frac{1}{2}},$$

where $\eta_1 = 4\left( n + \frac{1}{2} + \gamma \right)$. The sign before the square root is chosen in such a way that $p_z$ given in (6.68) is positive. As in the isospinor case, it is not possible to satisfy the two conditions (6.70) simultaneously. So we should set $\alpha^+ \neq 0, \alpha^- = 0$ for solutions having energy given in (6.71a) and the solutions corresponding to the energy level (6.71b) have $\alpha^+ = 0, \alpha^- \neq 0$. 
6.4 Results and discussion

In order to obtain more information about the solutions we gauge-transform them to the string gauge. For the isospinor fermions the transformation is given by [79]

\[ \psi' = \psi u^T \]  

(6.72)

where

\[ u = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) e^{-i\phi} \\ -\sin(\theta/2) e^{i\phi} & \cos(\theta/2) \end{pmatrix}. \]  

(6.73)

Substituting for \( \psi \) and \( u^T \) we find, explicitly,

\[
\chi^z = i \begin{pmatrix} \left( \frac{p^z}{j} \right) \sin(\theta/2) + \frac{B^z - C^z}{j} & \left( \frac{p^z}{j} \right) \sin(\theta/2) + \frac{B^z + C^z}{j} \\ \left( \frac{p^z}{j} \right) \cos(\theta/2) + \frac{B^z + C^z}{j} & \left( \frac{p^z}{j} \right) \cos(\theta/2) - \frac{B^z - C^z}{j} \end{pmatrix} \psi^M(\theta, \phi).
\]

(6.74)

As already remarked the wave functions corresponding to the energy levels given by (6.34a) satisfy \( p^z_j = G^z_j \) and \( B^z_j = C^z_j \). Hence from the explicit form of the wavefunction given above, it is clear that it describes an isospin-up bound state. Similarly, the energy levels given by (6.34b) are \( I^z = -1/2 \) levels. If we consider the scattering behaviour of the solutions the above property will lead to charge-conserved scattering. This is the same as in the case of fermions in the point monopole background.
having lowest angular momentum [82]. But in a regular dyon field the situation is different. Charge exchange scattering occurs in this case [79].

In the isovector case, the gauge transformation is given by

\[ \Psi' = \Psi u^T, \]  

(6.75)

where

\[ u = \begin{pmatrix} \cos \theta \cos \phi + \sin \phi \sin \theta \sin \phi & \cos \phi \sin \phi (\cos \theta - 1) & -\sin \theta \cos \phi \\ \cos \phi \sin \phi (\cos \theta - 1) & \cos \theta \sin^2 \phi + \cos^2 \phi & -\sin \theta \sin \phi \\ \sin \theta \cos \phi & \sin \phi \sin \theta & \cos \theta \end{pmatrix} \]  

(6.76)

where the isospin components in \( \Psi \) are written as a 1x3 row matrix:

\[ \chi^z = \begin{pmatrix} \rho_1, \rho_2, \rho_3 \end{pmatrix} \begin{pmatrix} F_{1+}^z \hat{\alpha}_3 + F_{2+}^z \hat{\alpha}_2 + F_{3+}^z L_3 \hat{\gamma}_{JM} \\ + F_{3-}^z L_3 \hat{\gamma}_{JM} \end{pmatrix} + \begin{pmatrix} \text{similar matrix with } F_{i+}^z \leftrightarrow F_{i-}^z \end{pmatrix} \]  

(6.77)

After the transformation we get

\[ \chi^z = \begin{pmatrix} \left( F_{1+}^z \cos \phi - i F_{3-}^z \sin \phi \right) \frac{\partial}{\partial \theta} \\ \left( F_{1-}^z \sin \phi + i F_{3+}^z \cos \phi \right) \frac{\partial}{\partial \phi} \end{pmatrix} \begin{pmatrix} F_{1+}^z \hat{\gamma}_{JM} \\ + F_{3+}^z L_3 \hat{\gamma}_{JM} \end{pmatrix} + \begin{pmatrix} \text{similar matrix with } F_{i+}^z \leftrightarrow F_{i-}^z \end{pmatrix} \]  

(6.78)

From this it is obvious that \( F_{1+}^z \) and \( F_{1-}^z \) correspond to solution with \( I_3 = 0 \).
In the isovector boson case we have the row matrix

\[
U' = \begin{pmatrix}
\frac{1}{3} \left[ (\gamma \cos \phi - i \gamma \sin \phi) \frac{2}{\beta} \right] Y^m - \frac{2}{3} \left[ (\gamma \sin \phi + i \gamma \cos \phi) \frac{2}{\beta} \right] Y^m \\
- \frac{1}{\sin \theta} \left[ (\gamma \sin \phi + i \gamma \cos \phi) \frac{2}{\beta} \right] Y^m - \frac{1}{\sin \theta} \left[ (\gamma \cos \phi - i \gamma \sin \phi) \frac{2}{\beta} \right] Y^m 
\end{pmatrix}
\] (6.79)

Here also the solution with \( X \neq 0 \), \( Y = Z = 0 \) corresponds to a state with \( I_3 = 0 \).

We shall now compare our solutions with those given by Tang [81,83] and prove the equivalence of the two methods used to achieve the angular separation. For this it is enough to consider the isospinor bosons. Tang [83] studied this problem by applying a singular gauge transformation which creates a singular string in the gauge potential. Then using monopole harmonics the separation of radial angular parts was achieved. Our model reduces to that considered in Ref. 83 if we set \( \epsilon = 0 \) and take the parameters \( a, b, c, \) and \( d \) as in (6.5). So we can compare the eigenvalues and eigenfunctions in these cases. Since the eigenfunctions are gauge-dependent this comparison must be made in the same gauge. For this we gauge-transform our solutions to the gauge used in Ref. 83.

\[
U' = u U, \tag{6.80}
\]

where \( u \) is the unitary matrix given in (6.73). This transformation yields,

\[
U' = \begin{pmatrix}
(F_+ - F_-) \left\{ \sqrt{\frac{J+M}{2J}} \cos (\Theta/2) Y^{M-N_L}_J - \sqrt{\frac{J-M}{2J}} e^{-i \phi} \sin (\Theta/2) Y^{M+N_L}_J \right\} \\
(F_+ + F_-) \left\{ \sqrt{\frac{J+M}{2J}} e^{i \phi} \sin (\Theta/2) Y^{M+N_L}_J + \sqrt{\frac{J-M}{2J}} \cos (\Theta/2) Y^{M-N_L}_J \right\}
\end{pmatrix}
\] (6.81)

where use has been made of the recurrence relations of associated Legendre
polynomials [86]. This can be further shown to be equal to (see Appendix 6.B)

\[ U' = \left( \begin{array}{c} \frac{F_+ + F_-}{\sqrt{2}} \gamma_{l\nu_4, j, m} \\ \frac{F_+ - F_-}{\sqrt{2}} \gamma_{l\nu_2, j, m} \end{array} \right), \quad (6.82) \]

where \( \gamma_{l, j, m} \) are monopole harmonics [84]. From the study of radial equations in Section 6.3 we have seen that either \((F_+ + F_-)\) or \((F_+ - F_-)\) is nonzero, but not both. The radial equations obeyed by \((F_+ + F_-)\) and \((F_+ - F_-)\) are the same as studied in Ref. 83. The angular parts are also the same. So we conclude that both the methods are equivalent. From (6.82) we also see that the states having energy values \((6.58a)\) are isospin-up states and those having energy values \((6.58b)\) are isospin-down states.

Finally we comment on the degeneracy of the system. For massless particles in a pure monopole \((M = c = 0)\) field that the pair of quantisation conditions for both fermions and bosons, \((6.27a, b), (6.34a, b), (6.57)\) and \((6.70)\), can be satisfied simultaneously. The degeneracy of the energy levels in this case for fixed \(n\) and \(J\) is as given in the following table.

<table>
<thead>
<tr>
<th>Description of particles</th>
<th>Lowest total angular momentum</th>
<th>Degeneracy of bond states for fixed (n) and (J) ((n \neq 0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isospinor fermions</td>
<td>0 ((&lt; 1))</td>
<td>2 ((2J+1))</td>
</tr>
<tr>
<td>Isospinor bosons</td>
<td>(\frac{3}{2} ((= 1))</td>
<td>2 ((2J+1))</td>
</tr>
<tr>
<td>Isovector fermions</td>
<td>(\frac{1}{2} ((&lt; 1))</td>
<td>2 ((2J+1))</td>
</tr>
<tr>
<td>Isovector bosons</td>
<td>0 ((&lt; 1))</td>
<td>(2(2J+1))</td>
</tr>
</tbody>
</table>

Two types of bound states
1) \((2J+1)\) \((I_3 = 0)\)
The inclusion of either a nonzero mass (M40) or dyon degree freedom (c ≠ 0) or both, changes the degeneracy of the spectrum. The pair of the above-mentioned conditions are not satisfied simultaneously. The degeneracy is half of that given in the above table except for the I3 = 0 state of the isovector fermions. Unlike the c = 0, M = 0 case the energy levels are not symmetrically distributed on both sides of zero. We also note that the total number of states for the case of monopoles with M ≠ 0, agree with the counting as given in Ref. 80. For isovector fermions there are no bound states with I3 = 0 and agreement with Ref. 80 is obtained only if we count the unbound solutions along with the bound states.

6.4 Appendix

In this section we solve the first order coupled equations

\[
\frac{d}{d\lambda} + \left( \frac{1 \pm i}{\rho} \right) X^\pm = \left( \frac{B \pm D}{\rho} + m \pm \epsilon \right) X^\pm ,
\] (6.83)

which is exactly similar to that of the hydrogen atom problem in Dirac theory [85] if the \( \frac{3}{r} \) term is absent. This can be solved by a similar procedure. Dividing (6.83) throughout by \( 2\lambda = 2\sqrt{m^2 - \epsilon^2} \) we get

\[
\begin{align*}
\frac{d}{d\epsilon} + \left( \frac{1 \pm i}{\rho} \right) X^+ &= \left( \frac{B + D}{\rho} + \frac{1}{2} \sqrt{\frac{m + \epsilon}{m - \epsilon}} \right) X^-
\end{align*}
\] (6.84)

and

\[
\begin{align*}
\frac{d}{d\rho} + \left( \frac{1 \pm i}{\rho} \right) X^- &= \left( \frac{B - D}{\rho} + \frac{1}{2} \sqrt{\frac{m - \epsilon}{m + \epsilon}} \right) X^+
\end{align*}
\]

where \( \rho = 2\lambda \). Using the substitution

\[
X^\pm = \sqrt{m \pm \epsilon} e^{-i\rho/2} \rho^{-1} (Q_4 \pm Q_4),
\] (6.85)

with

\[
\gamma = \sqrt{B^2 + j^2 - D^2},
\]
(6.84) can be reduced to

\[
\left( \rho \frac{d}{dp} + \gamma + j \right) (Q_1 + Q_2) - \rho Q_2 = (B + D) \sqrt{\frac{m - \epsilon}{m + \epsilon}} (Q_1 - Q_2)
\]

(6.86)

\[
\left( \rho \frac{d}{dp} + \gamma - j \right) (Q_1 - Q_2) + \rho Q_2 = -(B - D) \sqrt{\frac{m + \epsilon}{m - \epsilon}} (Q_1 + Q_2).
\]

The sum and difference of these gives

\[
\left( \rho \frac{d}{dp} + \gamma + \frac{Bm - De}{\lambda} \right) Q_1 = (-j + \frac{Dm - Be}{\lambda}) Q_2 \quad (6.87)
\]

\[
\left( \rho \frac{d}{dp} + \gamma - \frac{Bm - De}{\lambda} - \rho \right) Q_2 = (-j - \frac{Dm - Be}{\lambda}) Q_1.
\]

Now elimination of either \( Q_2 \) or \( Q_1 \) gives

\[
\rho \frac{d^2 Q_1}{dp^2} + (2\gamma + 1 - \rho) \frac{dQ_1}{dp} - \left( \gamma + \frac{Bm - De}{\lambda} \right) Q_1 = 0 \quad (6.88a)
\]

and

\[
\rho \frac{d^2 Q_2}{dp^2} + (2\gamma + 1 - \rho) \frac{dQ_2}{dp} - \left( \gamma + 1 + \frac{Bm - De}{\lambda} \right) Q_2 = 0 \quad (6.88b)
\]

respectively. In deriving this we have used the relation

\[
j^2 - \left( \frac{Dm - Be}{\lambda} \right)^2 = \gamma^2 - \left( \frac{Bm - De}{\lambda} \right)^2. \quad (6.89)
\]

The solutions of (6.88) are

\[
Q_1 = \alpha \frac{F_1}{\lambda} \left( \gamma + \frac{Bm - De}{\lambda}, 2\gamma + 1, \rho \right)
\]

\[
Q_2 = \beta \frac{F_1}{\lambda} \left( 1 + \gamma + \frac{Bm - De}{\lambda}, 2\gamma + 1, \rho \right), \quad (6.90)
\]

where the constants \( \alpha \) and \( \beta \) are related:

\[
\beta = \frac{\gamma + (Bm - De)/\lambda}{-j + (Dm - Be)/\lambda} \alpha. \quad (6.91)
\]
which is obtained by substituting (6.90) in either of the equations in (6.87) and putting $P = 0$.

6.B. Appendix

In this section we will prove the following results:

\[
\sqrt{2} \left[ \frac{\sqrt{J+M}}{\sqrt{J-M}} \cos(\theta/2) \, Y_{J-M}^{n} + \frac{\sqrt{J-M}}{\sqrt{J+M}} \, e^{-i\phi} \sin(\theta/2) \, Y_{J+M}^{n} \right] = Y_{J}^{n}, \quad J, M
\]  

(6.92a)

\[
\sqrt{2} \left[ \frac{\sqrt{J+M}}{\sqrt{J-M}} \sin(\theta/2) \, Y_{J-M}^{n} + \frac{\sqrt{J-M}}{\sqrt{J+M}} \, \cos(\theta/2) \, Y_{J+M}^{n} \right] = Y_{J}^{n}, \quad J, M
\]  

(6.92b)

Putting $x = \cos \theta$, $j = J - M$, $m = M - k$, the left hand side of (6.92a) can be written as

\[
\sqrt{\frac{j+m+1}{j+1} \sqrt{1+x}} \, Y_{j}^{m} (x, \phi) + \sqrt{\frac{j-m}{j+1} \sqrt{1-x}} \, e^{-i\phi} \, Y_{j}^{m+1} (x, \phi)
\]

\[
= (-1)^{m} \sqrt{\frac{(j-m)!}{4\pi (j+m+1)!}} \left\{ \frac{(1-x)^{m/2} \sqrt{1+x}}{\sqrt{2}} \right\} \text{F2} \left( \frac{m-j}{2}, m-j+1 ; m+1 ; \frac{1-x}{2} \right) \text{e}^{im\phi}
\]

\[
= (-1)^{m} \sqrt{\frac{(j+m+1)!}{4\pi (j-m)!}} \left\{ \frac{(1-x)^{m/2} \sqrt{1+x}}{\sqrt{2}} \right\} \text{F2} \left( \frac{m-j}{2}, m-j+1 ; m+1 ; \frac{1-x}{2} \right) \text{e}^{im\phi}
\]

\[
= (-1)^{m} \sqrt{\frac{(j+m+1)!}{4\pi (j-m)!}} \left\{ (1-x)^{m/2} \sqrt{1+x} \right\} \text{P}_{j+m}^{m} (x)
\]

\[
= (-1)^{m} \sqrt{\frac{(j+M+1)!}{4\pi (j-M)!}} \left\{ (1-x)^{m/2} \sqrt{1+x} \right\} \text{P}_{j-M}^{m} (x)
\]

\[
\times \text{P}_{j-M, m+1/2}^{m-1/2} (x) \text{e}^{i (M-1/2) \phi}
\]
\[ \begin{align*}
&= \sqrt{\frac{(2J+1)(J-M)!}{4\pi (J+\frac{1}{2})!(J-\frac{1}{2})!}} \left(1-x\right)^{-\frac{(M-\frac{1}{2})}{2}} (1+x)^{-\frac{(M+\frac{1}{2})}{2}} \\
&\quad \times P_{J+M}^{-\left(M-\frac{1}{2}\right), \left(-M+\frac{1}{2}\right)}(x) e^{i(M-\frac{1}{2})\phi} \\
&= Y_{J,M}(\theta, \phi),
\end{align*} \]

(6.93)

where \( \mathcal{F}_1(a, \beta; \gamma; x) \) are hypergeometric functions and \( P_{\alpha}^\beta(x) \) are Jacobi polynomials [61, 87]. For deriving (6.93) we have used the relations [61, 87]

\[ P_{n}^\alpha(x) = \frac{(n+m)!}{(n-m)!} \left(1-x\right)^{\frac{n}{2}} a \mathcal{F}_1(m-n, m+n+1; m+1; \frac{x-a}{m}) \]  \hspace{1cm} (6.94)

\[ P_{n}^\alpha(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} a \mathcal{F}_1(-n, n+2+\beta+1; \gamma+1; \frac{x-a}{\alpha}) \]  \hspace{1cm} (6.95)

\[ a \mathcal{F}_1(a, \beta; \gamma; x + \frac{\alpha}{\gamma} \mathcal{F}_1(a+1, \beta+1; \gamma+1; y) = a \mathcal{F}_1(a, \beta+1; \gamma; y) \]  \hspace{1cm} (6.96)

and [84]

\[ P_{n+\alpha+\beta}^{-\alpha, -\beta}(x) = a^{-\alpha-\beta} (x-1)^{\alpha} (x+1)^{\beta} P_{n}^\alpha(x) \]  \hspace{1cm} (6.97)

\[ Y_{n, J, m}(x, \phi) = \sqrt{\frac{(2J+1)(J-M)!}{4\pi (J-q)! (J+q)!}} \left(1-x\right)^{-\frac{(q+M)}{2}} (1+x)^{-\frac{(q-M)}{2}} \\
\quad \times P_{J+M}^{-\left(q-M\right), \left(q+M\right)}(x) e^{i(q-M)\phi}. \]  \hspace{1cm} (6.98)

(6.92b) can be proved similarly. In this case instead of (6.96) we should use the relation

\[ a \mathcal{F}_1(a, \beta; \gamma; y) - \frac{\alpha}{\gamma} (1-y) a \mathcal{F}_1(a+1, \beta+1; \gamma+1; y) = \frac{Y-x}{Y} a \mathcal{F}_1(a, \beta+1; \gamma+1; y) \]  \hspace{1cm} (6.99)