CHAPTER 1

Γ-SEMIGROUPS

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Γ-SEMIGROUPS

As a generalization of a semigroup, SEN [47], introduced the notion of Γ-semigroup in 1981 and developed some theory on Γ-semigroups [48], [49], [50] and [51]. JIROJKUL, SRIPAKORN, CHINRAM [26], extended many classical notions of semigroups to Γ-semigroups. DUTTA and CHATTERZEE [15] also studied the properties of green's relations in Γ-semigroups and they generalized the notions; idempotent elements, regular elements and semisimple elements in Γ-semigroups. DHEENA and ELAVARASAN[14] studied about right chain po-Γ-semigroup and obtained some characterizations of po-Γ-semigroup. MADHUSUDHANA RAO, ANJANEYULU and GANGADHARA RAO [32], [33], [34], [35], [36], and [37] developed the algebraic theory of Γ-subsemigroups and Γ-ideals in Γ-semigroups. In this thesis we made a study on the theory of Γ-semigroups and Γ-subsemigroups and also we exhibit some special elements in Γ-semigroups.

This chapter is divided into 3 sections. In section 1, the notion of a Γ-semigroup is introduced and some examples are given. Further the terms; commutative Γ-semigroup, quasi commutative Γ-semigroup and normal Γ-semigroup are introduced. It is proved that (1) if S is a commutative Γ-semigroup then S is a quasi commutative Γ-semigroup, (2) if S is a quasi commutative Γ-semigroup then S is a normal Γ-semigroup. Further the terms; left identity, right identity, identity, left zero, right zero, zero, left α-inverse, right α-inverse, α-inverse, left Γ-inverse, right Γ-inverse, Γ-inverse, unit elements of a Γ-semigroup are introduced. It is proved that if a is a left identity and b is a right identity of a Γ-semigroup, then $a = b$. It is also proved that any Γ-semigroup has at most one identity. It is proved that if a is a left zero and b is a right zero of a Γ-semigroup, then $a = b$ and it is also proved that any Γ-semigroup has at most one zero element. It is proved that if b is a left α-inverse and c is a right α-inverse of an element a of a Γ-semigroup S, then $b = c$ and it is also proved that the α-inverse of an element a in a Γ-semigroup S (if exists) is unique. Further it is proved that if b is a left Γ-inverse and c is a right Γ-inverse of an element a of a Γ-semigroup S, then $b = c$ and it is also proved that the Γ-inverse of an element a in a Γ-semigroup S (if exists) is unique.

In section 2, the terms; Γ-subsemigroup, Γ-subsemigroup generated by a subset, cyclic Γ-subsemigroup of a Γ-semigroup and cyclic Γ-semigroup are introduced. It is proved that (1) the nonempty intersection of any two **Γ**-subsemigroups of a **Γ**-semigroup S is a **Γ**-subsemigroup of S, (2) the nonempty intersection of any family of **Γ**-subsemigroups of a **Γ**-semigroup S is a **Γ**-subsemigroup of S. If S is a **Γ**-semigroup and A is a nonempty subset of S, then it is proved that $< A > = \{ a_1 \alpha_1 a_2 \alpha_2 \ldots\ldots a_{n-1} \alpha_{n-1} a_n : n \in N, a_1, a_2, \ldots, a_n \in A, \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \Gamma \}$. It is also proved that if A is a nonempty subset of a Γ-semigroup S, then the Γ-subsemigroup of S generated by A is the intersection of all Γ-subsemigroups of S containing A.

In section 3, the terms; α-idempotent, Γ-idempotent, strongly Γ-idempotent, midunit, *r*-element, regular element, left regular element, right regular element, completely regular element, (α, β)-inverse of an element, semisimple element, intra regular element, left α-cancellative element, right α-cancellative element, α-cancellative element, left Γ-cancellative element, right Γ-cancellative element, Γ-cancellative element, strongly left Γ-cancellative element, strongly right Γ-cancellative element and strongly Γ-cancellative element in a Γ-semigroup are introduced. Also the terms; idempotent Γ-semigroup and generalized commutative Γ-semigroup are introduced. It is proved that every *α*-idempotent element of a Γ-semigroup is regular. It is proved that every **Γ**-ideal of a regular **Γ**-semigroup S is a regular **Γ**-subsemigroup of S. It is proved that if a Γ-semigroup S is a regular Γ-semigroup then every principal Γ-ideal is generated by an idempotent. Further it is also proved that, in a Γ-semigroup, *a* is a regular element if and only if *a* has an (*α, β*)-inverse. It is proved that, (1) if '*a*' is a completely regular element of a Γ-semigroup S, then *a* is regular and semisimple, (2) if *a* is a completely regular element of Γ-semigroup then *a* is both left regular and right regular, (3) if '*a*' is a left regular element of a Γ- semigroup S, then *a* is semisimple, (4) if '*a*' is a right regular element of a Γ- semigroup S, then *a* is semisimple, (5) if '*a*' is a regular element of a Γ- semigroup S, then *a* is semisimple and (6) if '*a*' is a intra regular element of a Γ-semigroup S, then *a* is semisimple.

The contents of section 1 and section 3 in chapter 1 are published in "International Journal of Mathematical Sciences, Technology and Humanities" under the title 'Duo chained Γ-semigroups' [19].

1.1. Γ-SEMIGROUP

In this section, the notion of a Γ-semigroup is introduced and some examples are given. Further the terms; commutative Γ-semigroup, quasi commutative Γ-semigroup and normal Γ-semigroup are introduced. It is proved that (1) if S is a commutative Γ-semigroup then S is a quasi commutative Γ-semigroup, (2) if S is a quasi commutative

Γ-semigroup then S is a normal Γ-semigroup. Further the terms; left identity, right identity, identity, left zero, right zero, zero, left α-inverse, right α-inverse, α-inverse, left Γ-inverse, right Γ-inverse, Γ-inverse, unit elements of a Γ-semigroup are introduced. It is proved that if a is a left identity and b is a right identity of a Γ-semigroup, then $a = b$. It is also proved that any Γ-semigroup has at most one identity. It is proved that if a is a left zero and b is a right zero of a Γ-semigroup, then $a = b$ and it is also proved that any Γ-semigroup has at most one zero element. It is proved that if b is a left α-inverse and c is a right α-inverse of an element a of a Γ-semigroup S, then $b = c$ and it is also proved that the **α**-inverse of an element a in a **Γ**-semigroup S (if exists) is unique. Further it is proved that if b is a left Γ-inverse and c is a right Γ-inverse of an element a of a Γ-semigroup S, then $b = c$ and it is also proved that the Γ-inverse of an element a in a **Γ**-semigroup S (if exists) is unique.

We now introduce the notion of a Γ-semigroup which is due to SEN [47].

DEFINITION 1.1.1 : Let S and Γ be two non-empty sets. Then S is called a ***Γ-semigroup*** if there exist a mapping from $S \times \Gamma \times S$ to S which maps $(a, \alpha, b) \rightarrow a\alpha b$ satisfying the condition : $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in$ S and $\gamma, \mu \in \Gamma$.

NOTE 1.1.2 : Let S be a Γ-semigroup. If A and B are two subsets of S, we shall denote the set $\{ a\gamma b : a \in A , b \in B \text{ and } \gamma \in \Gamma \}$ by AΓB.

In the following some examples of Γ-semigroups are given

EXAMPLE 1.1.3 : Let S be the set of all non-positive integers and Γ be the set of all non-positive even integers. If $a\alpha b$ denote as usual multiplication of integers for $a, b \in$ S and $\alpha \in \Gamma$, then S is a Γ-semigroup.

EXAMPLE 1.1.4 : Let Q be the set of rational numbers and Γ = N be the set of natural numbers. Define a mapping from $Q \times \Gamma \times Q$ to Q by $a\alpha b$ = usual product of a, α, b; for $a, b \in Q$, $\alpha \in \Gamma$. Then Q is a Γ-semigroup.

EXAMPLE 1.1.5 : Let $S = \{ 5n + 4 : n$ is a positive integer$\}$ and $\Gamma = \{ 5n + 1 :$ n is a positive integer$\}$. Then S is a Γ-semigroup with the operation defined by $a\alpha b = a + \alpha + b$ where $a, b \in S$, $\alpha \in \Gamma$ and + is the usual addition of integers.

EXAMPLE 1.1.6 : Let S be the set of all integers of the form $4n+1$ where n is an integer and Γ denote the set of all integers of the form $4n+3$. If $a\gamma b$ is $a+\gamma+b$, for all $a, b \in$ S and $\gamma \in \Gamma$, then S is a Γ-semigroup.

EXAMPLE 1.1.7 : Let S = { $\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$, $\{a, c\}$, $\{a, b, c\}$} and $\Gamma = \{\emptyset, \{a\}, \{a, b, c\}\}$. If for all A, C ∈ S and B ∈ Γ, ABC = A ∩ B ∩ C, then S is a Γ-semigroup.

EXAMPLE 1.1.8 : Let S = {$\emptyset$, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{b, c\}$, $\{a, c\}$, $\{a, b, c\}$} and $\Gamma = \{\{a, b, c\}\}$. If for all A, C ∈ S and B ∈ Γ, ABC = A ∩ B ∩ C, then S is a Γ-semigroup.

EXAMPLE 1.1.9 : Let S be the set of all 2 × 3 matrices over Q, the set of rational numbers and Γ be the set of all 3 × 2 matrices over Q. Define AαB = usual matrix product of A, α, B; for all A, B ∈ S and for all α ∈ Γ. Then S is a Γ-semigroup. Note that S is not a semigroup.

EXAMPLE 1.1.10 : Let S = { - i , 0, i } and Γ = S. Then S is a Γ-semigroup under the multiplication of complex numbers, while S is not a semigroup under multiplication of complex numbers.

EXAMPLE 1.1.11 : Let S be a Γ-semigroup and α a fixed element in Γ. We define $a.b = a\alpha b$ for all $a,b \in$ S. We can show that (S,.) is a semigroup and we denote this semigroup by S_α .

EXAMPLE 1.1.12 : Let S be a semigroup and Γ be a nonempty set. Define a mapping from S × Γ × S to S as $a\alpha b = ab$, for all $a, b \in$ S and $\alpha \in \Gamma$. Then S is a Γ-semigroup.

Verification : Let $a, b, c \in$ S and $\alpha, \beta \in \Gamma$.
Then $(a\alpha b)\beta c = (ab)\beta c = (ab)c = a(bc) = a\alpha(bc) = a\alpha(b\beta c)$. Therefore S is a Γ-semigroup.

Note 1.1.13 : Every semigroup can be considered to be a Γ-semigroup. Thus the class of all Γ-semigroups includes the class of all semigroups.

EXAMPLE 1.1.14 (FREE Γ-SEMIGROUP) : Let X and Γ be two nonempty sets. A sequence of elements $a_1\alpha_1a_2\alpha_2 \ldots\ldots a_{n-1}\alpha_{n-1}a_n$ where $a_1,a_2,a_3, \ldots\ldots,a_n \in$ X and $\alpha_1,\alpha_2,\alpha_3,\ldots\ldots\alpha_n \in \Gamma$ is called a ***word*** over the alphabet X relative to Γ. The set S of all words with the operation defined from S × Γ × S to S as

$(a_1\alpha_1a_2\alpha_2 \ldots a_{n-1}\alpha_{n-1}a_n)\ \gamma(b_1\beta_1b_2\beta_2 \ldots b_{m-1}\beta_{m-1}b_m) = a_1\alpha_1a_2\alpha_2 \ldots a_{n-1}\alpha_{n-1}a_n\ \gamma b_1\beta_1b_2\beta_2 \ldots b_{m-1}\beta_{m-1}b_m$ is a Γ-semigroup. This Γ-semigroup is called ***free Γ-semigroup*** over the alphabet X relative to Γ.

In the following we introduce the notion of a commutative Γ-semigroup.

DEFINITION 1.1.15 : A Γ-semigroup S is said to be ***commutative*** provided $a\gamma b = b\gamma a$ for all $a,b \in S$ and $\gamma \in \Gamma$.

NOTE 1.1.16 : If S is a commutative Γ-semigroup then $a\,\Gamma b = b\,\Gamma a$ for all $a, b \in S$.

NOTE 1.1.17 : Let S be a Γ-semigroup and $a, b \in$ S and $\alpha \in \Gamma$. Then $a\alpha a\alpha b$ is denoted by $(a\alpha)^2b$ and consequently $a\ \alpha\ a\ \alpha\ a\ \alpha \ldots (n\ terms)b$ is denoted by $(a\alpha)^nb$.

In the following we introduce a quasi commutative Γ-semigroup.

DEFINITION 1.1.18 : A Γ-semigroup S is said to be ***quasi commutative*** provided for each $a,b \in$ S, there exists a natural number n such that $a\gamma b = (b\gamma)^n a\ \forall \gamma \in \Gamma$.

NOTE 1.1.19 : If a Γ-semigroup S is ***quasi commutative*** then for each $a,b \in$ S , there exists a natural number n such that, $a\Gamma b = (b\,\Gamma)^n a$.

THEOREM 1.1.20 : If S is a commutative Γ-semigroup then S is a quasi commutative Γ-semigroup.

***Proof* :** Suppose that S is a commutative Γ-semigroup. Let $a,b \in S$.
Now $a\alpha b = b\alpha a \Rightarrow a\alpha b = (b\ \alpha)^1 a$. Therefore S is a quasi commutative Γ-semigroup.

In the following we introduce the notion of a normal Γ-semigroup.

DEFINITION 1.1.21 : A Γ-semigroup S is said to be ***normal*** provided $a\alpha\text{S} = \text{S}\alpha a\ \forall \alpha \in \Gamma$ and $\forall a \in S$.

NOTE 1.1.22 : If a Γ-semigroup S is *normal* then $a\Gamma\text{S} = \text{S}\Gamma a$ for all $a \in$ S.

THEOREM 1.1.23 : If S is a quasi commutative Γ-semigroup then S is a normal Γ-semigroup.

***Proof* :** Suppose that S is a quasi commutative Γ- semigroup.
Let $a \in$ S and $\alpha \in \Gamma$.
Let $x \in a\alpha$S. Then $x = a\alpha b$ where $b \in$ S.

Since S is quasi commutative, $a\alpha b = (b\alpha)^n a$ for some $n \in N$

$\therefore\ x = a\alpha b = (b\alpha)^n a = (b\alpha)^{n-1} b\alpha a \in S\alpha a$

$\therefore a\alpha S \subseteq S\alpha a \rightarrow (1)$

Let $x \in S\alpha a$. Then $x = t\alpha a$ for some $t \in S$.

Since S is quasi commutative $t\alpha a = (a\alpha)^n t$ for some $n \in N$

Now $x = t\alpha a = (a\alpha)^n t = a\alpha(a\alpha)^{n-1} t \in a\alpha S$.

$\therefore S\alpha a \subseteq a\alpha S \rightarrow (2)$

From (1) & (2), $a\alpha S = S\alpha a\ \forall \alpha \in \Gamma, \forall a \in S$ and hence S is a normal Γ-semigroup.

COROLLARY 1.1.24 : Every commutative Γ-semigroup is a normal Γ-semigroup.

Proof : Let S be a commutative Γ-semigroup. By theorem 1.1.20, S is a quasi commutative Γ-semigroup. By theorem 1.1.23, S is a normal Γ-semigroup. Therefore every commutative Γ-semigroup is a normal Γ-semigroup.

In the following we are introducing left identity, right identity and identity of a Γ-semigroup.

DEFINITION 1.1.25 : An element ***a*** of a Γ-semigroup S is said to be a ***left identity*** of S provided $a\alpha s = s$ for all $s \in S$ and $\alpha \in \Gamma$.

DEFINITION 1.1.26 : An element '***a***' of a Γ-semigroup S is said to be a ***right identity*** of S provided $s\alpha a = s$ for all $s \in S$ and $\alpha \in \Gamma$.

DEFINITION 1.1.27 : An element '***a***' of a Γ-semigroup S is said to be a ***two sided identity*** or an ***identity*** provided it is both a left identity and a right identity of S.

THEOREM 1.1.28 : If *a* is a left identity and *b* is a right identity of a Γ-semigroup S, then *a* = *b*.

Proof : Since a is a left identity of S, $a\alpha s = s$ for all $s \in S$ and $\alpha \in \Gamma$ and hence $a\alpha b = b$ for all $\alpha \in \Gamma$. Since b is a right identity of S, $s\alpha b = s$ for all $s \in S$ and $\alpha \in \Gamma$ and hence $a\alpha b = a$ for all $\alpha \in \Gamma$. Now $a = a\alpha b = b$.

THEOREM 1.1.29 : Any Γ-semigroup S has at most one identity.

Proof : Let a, b be two identity elements of the Γ-semigroup S. Now a can be considered as a left identity and b can be considered as a right identity of S. By theorem 1.1.28, $a = b$. Then S has atmost one identity.

NOTE 1.1.30 : The identity (if exists) of a Γ-semigroup is usually denoted by e or 1.

DEFINITION 1.1.31 : A Γ-semigroup S with identity is called a ***Γ-monoid.***

In the following we are introducing left zero, right zero and zero of a Γ-semigroup.

DEFINITION 1.1.32 : An element a of a Γ-semigroup S is said to be a ***left zero*** of S provided $a\alpha s = a$ for all $s \in$ S and $\alpha \in \Gamma$.

DEFINITION 1.1.33 : An element a of a Γ-semigroup S is said to be a ***right zero*** of S provided $s\alpha a = a$ for all $s \in$ S and $\alpha \in \Gamma$.

DEFINITION 1.1.34 : An element a of a Γ-semigroup S is said to be a ***two sided zero*** or ***zero*** provided it is both a left zero and a right zero of S.

We are now introducing left zero Γ-semigroup, right zero Γ-semigroup and zero Γ-semigroup.

DEFINITION 1.1.35 : A Γ-semigroup in which every element is a left zero is called a ***left zero Γ-semigroup.***

DEFINITION 1.1.36 : A Γ-semigroup in which every element is a right zero is called a ***right zero Γ-semigroup.***

DEFINITION 1.1.37 : A Γ-semigroup with 0 in which the product of any two elements equals to 0 is called a ***zero Γ-semigroup*** or a ***null Γ-semigroup.***

THEOREM 1.1.38 : If a is a left zero and b is a right zero of a Γ-semigroup S, then $a = b$.

***Proof* :** Since a is a left zero of S, $a\alpha s = a$ for all s $\in$ S, $\alpha \in \Gamma$ and hence $a\alpha b = a$ for all $\alpha \in \Gamma$. Since b is a right zero of S, $s\alpha b = b$ for all s $\in$ S and $\alpha \in \Gamma$ and hence $a\alpha b = b$ for all $\alpha \in \Gamma$. Now $a = a\alpha b = b$.

THEOREM 1.1.39 : Any Γ-semigroup S has at most one zero element.

***Proof* :** Let a, b be two zeros of the Γ-semigroup S. Now a can be considered as a left zero and b can be considered as a right zero. By theorem 1.1.38, $a = b$. Thus S has at most one zero.

NOTE 1.1.40 : The zero (if exists) of a Γ-semigroup is usually denoted by 0.

NOTATION 1.1.41 : Let S be a Γ-semigroup. If S has an identity, let $S^1 = S$ and if S does not have an identity, let S^1 be the Γ-semigroup S with an identity adjoined usually denoted by the symbol 1. Similarly if S has a zero, let $S^0 = S$ and if S does not have a zero, let S^0 be the Γ-semigroup S with zero adjoined usually denoted by the symbol 0.

In the following we are introducing left α-inverse, right α-inverse of an element in a Γ-semigroup.

DEFINITION 1.1.42 : An element b of a Γ-semigroup S with identity e is said to be a ***left α-inverse*** of a of a Γ-semigroup S provided, $b\alpha a = e$ for $\alpha \in \Gamma$.

DEFINITION 1.1.43 : An element b of a Γ-semigroup S with identity e is said to be a ***right α-inverse*** of a of a Γ-semigroup S provided, $a\alpha b = e$ for $\alpha \in \Gamma$.

THEOREM 1.1.44 : If *b* is a left α-inverse and *c* is a right α-inverse of an element *a* of a Γ-semigroup S with identity *e*, then *b* = *c*.

***Proof* :** Since b is a left α-inverse of an element a in S, $b\alpha a = e$
and c is a right α- inverse of an element a in S, $a\alpha c = e$ for $\alpha \in \Gamma$.
Now $b = b\alpha e = b\alpha(a\alpha c) = (b\alpha a)\alpha c = e\alpha c = c$.

In the following we are introducing α-inverse of an element in a Γ-semigroup with identity .

DEFINITION 1.1.45 : An element b of a Γ-semigroup S with identity e is said to be a ***α-inverse*** of a of a Γ-semigroup S provided, $a\alpha b = b\alpha a = e$ for $\alpha \in \Gamma$.

THEOREM 1.1.46 : The α-inverse of an element *a* in a Γ-semigroup S with identity *e* (if exists) is unique.

***Proof* :** Let b, c be two α-inverse elements of an element a in a Γ-semigroup S. If b is a $\alpha-$ inverse of a then $a\alpha b = b\alpha a = e$ and if c is a α-inverse of a then $a\alpha c = c\alpha a = e$.
Now $b = b\alpha e = b\alpha(a\alpha c) = (b\alpha a)\alpha c = e\alpha c = c$.

In the following we are introducing left Γ-inverse, right Γ-inverse of an element in a Γ-semigroup.

DEFINITION 1.1.47 : An element b of a Γ-semigroup S with identity e is said to be a ***left* Γ*-inverse*** of a of a Γ-semigroup S provided, $b\alpha a = e$ for all $\alpha \in \Gamma$.

NOTE 1.1.48 : An element b of a Γ-semigroup S with identity e is said to be a ***left* Γ*-inverse*** of a of a Γ-semigroup S provided $b\Gamma a = e$.

DEFINITION 1.1.49 : An element b of a Γ-semigroup S with identity e is said to be a ***right* Γ*-inverse*** of a of a Γ-semigroup S provided, $a\alpha b = e$ for all $\alpha \in \Gamma$.

NOTE 1.1.50 : An element b of a Γ-semigroup S with identity e is said to be a ***right* Γ*-inverse*** of a of a Γ-semigroup S provided $a\Gamma b = e$.

THEOREM 1.1.51 : If b is a left Γ-inverse and c is a right Γ-inverse of an element a of a Γ-semigroup S with identity e, then $b = c$.

***Proof*:** Since b is a left Γ-inverse of an element a in S, $b\Gamma a = e$
and c is a right Γ- inverse of an element a in S, $a\Gamma c = e$.
Now $b = b\Gamma e = b\Gamma(a\Gamma c) = (b\Gamma a)ac = e\Gamma c = c$.

In the following we are introducing Γ-inverse of an element in Γ-semigroup with identity.

DEFINITION 1.1.52 : An element b of a Γ-semigroup S with identity e is said to be a **Γ*-inverse*** of a of a Γ-semigroup S provided it is both left Γ-inverse and right Γ-inverse of a.

NOTE 1.1.53 : An element b of a Γ-semigroup S with identity e is said to be a **Γ*-inverse*** of a of a Γ-semigroup S provided $a\Gamma b = b\Gamma a = e$.

THEOREM 1.1.54 : The Γ-inverse of an element a in a Γ-semigroup S with identity e (if exists) is unique.

***Proof*:** Let b, c be two Γ-inverse elements of an element a in a Γ-semigroup S. If b is a Γ– inverse of a then $a\Gamma b = b\Gamma a = e$ and if c is a Γ-inverse of a then $a\Gamma c = c\Gamma a = e$.
Now $b = b\Gamma e = b\Gamma(a\Gamma c) = (b\Gamma a)\Gamma c = e\Gamma c = c$.

In the following we introduce *unit* of a Γ-semigroup

DEFINITION 1.1.55 : An element a of a Γ-semigroup S is said to be a ***unit*** if it has Γ- inverse.

In the following we introduce the notion of a Γ-group.

DEFINITION 1.1.56 : A Γ-semigroup S is said to be a **Γ-*group*** if

(1) $\exists\, e \in S \ni a\Gamma e = e\Gamma a = a$ for all $a \in S$.

(2) every element $a \in S$ has a α-inverse in S for some $\alpha \in \Gamma$.

EXAMPLE 1.1.57 : Let S = { 1, 2, 3, 4, } and Γ be a non-empty set. If for all $a, b \in S$ and $\alpha \in \Gamma$, $a\alpha b = a \times_5 b$ then S is a Γ-group.

Verification : Let us form the composition table

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	6	1
3	3	1	4	2
4	4	3	2	1

Now from the table it is easy to see that S is a Γ-group.

EXAMPLE 1.1.58 : Let I is set of all integers and Γ = { 2 }. Define a mapping from I×Γ×I to I by $a\alpha b = a + b + \alpha$ for all $a, b \in I$ and $\alpha \in \Gamma$, then I is Γ-group.

1.2. Γ-SUBSEMIGROUP

In this section, the terms; Γ-subsemigroup, Γ-subsemigroup generated by a subset, cyclic Γ-subsemigroup of a Γ-semigroup and cyclic Γ-semigroup are introduced. It is proved that (1) the nonempty intersection of any two **Γ**-subsemigroups of a **Γ**-semigroup S is a **Γ**-subsemigroup of S, (2) the nonempty intersection of any family of **Γ**-subsemigroups of a **Γ**-semigroup S is a **Γ**-subsemigroup of S. If S is a **Γ**-semigroup and A is a nonempty subset of S, then it is proved that $< A > = \{ a_1\, \alpha_1\, a_2\, \alpha_2 \ldots\ldots a_{n-1}\, \alpha_{n-1}\, a_n : n \in N, a_1, a_2, \ldots\ldots, a_n \in A, \alpha_1, \alpha_2, \ldots\ldots, \alpha_{n-1} \in \Gamma \}$. It is also proved that if A is a nonempty subset of a Γ-semigroup S, then the Γ-subsemigroup of S generated by A is the intersection of all Γ-subsemigroups of S containing A.

We are now introducing Γ-subsemigroup which is due to JIROJKUL. CH, SRIPAKORN. R and CHINRAM. R [26].

DEFINITION 1.2.1 : Let S be a Γ-semigroup. A nonempty subset T of S is said to be a ***Γ-subsemigroup*** of S if $a\gamma b \in$ T, for all $a, b \in$ T and $\gamma \in \Gamma$.

The following note is due to SEN and SAHA [51].

NOTE 1.2.2 : A nonempty subset T of a Γ-semigroup S is a Γ-subsemigroup of S iff $T\Gamma T \subseteq T$.

EXAMPLE 1.2.3 : Let S = [0,1] and Γ = { $1/n$: n is a positive integer}. Then S is a Γ-semigroup under the usual multiplication. Let T = [0, 1/2]. Now T is a nonempty subset of S and $a\gamma b \in$ T, for all $a, b \in$ T and $\gamma \in \Gamma$. Then T is a Γ-subsemigroup of S.

THEOREM 1.2.4 : The nonempty intersection of two Γ-subsemigroups of a Γ-semigroup S is a Γ-subsemigroup of S.

***Proof*:** Let T_1, T_2 be two Γ-subsemigroups of S. Let $a, b \in T_1 \cap T_2$ and $\gamma \in \Gamma$.

$a, b \in T_1 \cap T_2 \Rightarrow a, b \in T_1$ and $a, b \in T_2$

$a, b \in T_1, \gamma \in \Gamma$, T_1 is a Γ-subsemigroup of S $\Rightarrow a\gamma b \in T_1$.

$a, b \in T_2, \gamma \in \Gamma$, T_2 is a Γ-subsemigroup of S $\Rightarrow a\gamma b \in T_2$.

$a\gamma b \in T_1, a\gamma b \in T_2 \Rightarrow a\gamma b \in T_1 \cap T_2$. Therefore $T_1 \cap T_2$ is a Γ-subsemigroup of S.

THEOREM 1.2.5 : The nonempty intersection of any family of Γ-subsemigroups of a Γ-semigroup S is a Γ-subsemigroup of S.

***Proof*:** Let $\{T_\alpha\}_{\alpha\in\Delta}$ be a family of Γ-subsemigroups of S and let $T = \bigcap_{\alpha\in\Delta} T_\alpha$.

Let $a, b \in$ T and $\gamma \in \Gamma$.

$a, b \in T \Rightarrow a, b \in \bigcap_{\alpha\in\Delta} T_\alpha \Rightarrow a, b \in T_\alpha$ for all $\alpha \in \Delta$.

$a, b \in T_\alpha, \gamma \in \Gamma$, T_α is a Γ-subsemigroup of S $\Rightarrow a\gamma b \in T_\alpha$.

$a\gamma b \in T_\alpha$ for all $\alpha \in \Delta \Rightarrow a\gamma b \in \bigcap_{\alpha\in\Delta} T_\alpha \Rightarrow a\gamma b \in T$.

Therefore T is a Γ-subsemigroup of S.

In the following we are introducing a Γ-subsemigroup which is generated by a subset and a cyclic Γ-subsemigroup of Γsemigroup.

DEFINITION 1.2.6 : Let S be a Γ-semigroup and A be a nonempty subset of S. The smallest Γ-subsemigroup of S containing A is called a ***Γ-subsemigroup of S generated by A***. It is denoted by < A >.

THEOREM 1.2.7 : Let S be a Γ-semigroup and A be a nonempty subset of S. Then $< A > = \{ a_1 \alpha_1 a_2 \alpha_2 \ldots\ldots a_{n-1} \alpha_{n-1} a_n : n \in N, a_1, a_2, \ldots\ldots, a_n \in A, \alpha_1, \alpha_2, \ldots\ldots, \alpha_{n-1} \in \Gamma \}$.

***Proof*:** Let $T = \{ a_1 \alpha_1 a_2 \alpha_2 \ldots a_{n-1} \alpha_{n-1} a_n : n \in N, a_1, a_2, \ldots a_n \in A, \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \Gamma\}$.
Let $a, b \in T$ and $\gamma \in \Gamma$.
$a \in T \Rightarrow a = a_1 \alpha_1 a_2 \alpha_2 \ldots\ldots a_{m-1} \alpha_{m-1} a_m$ where $a_1, a_2, \ldots a_m \in A, \alpha_1, \alpha_2, \ldots, \alpha_{m-1} \in \Gamma$.
$b \in T \Rightarrow b = b_1 \beta_1 b_2 \beta_2 \ldots\ldots b_{n-1} \beta_{n-1} b_n$ where $b_1, b_2, \ldots b_n \in A, \beta_1, \beta_2, \ldots, \beta_{n-1} \in \Gamma$.
Now $a\gamma b = (a_1 \alpha_1 a_2 \alpha_2 \ldots\ldots a_{m-1} \alpha_{m-1} a_m) \gamma (b_1 \beta_1 b_2 \beta_2 \ldots\ldots b_{n-1} \beta_{n-1} b_n) \in T$.
Therefore T is a Γ-subsemigroup of S.
Let K be a Γ-subsemigroup of S such that $A \subseteq K$.
Let $a \in T$. Then $a = a_1 \alpha_1 a_2 \alpha_2 \ldots\ldots a_{n-1} \alpha_{n-1} a_n$ where $a_1, a_2, \ldots a_n \in A, \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \Gamma$
$a_1, a_2, \ldots. a_n \in A, A \subseteq K \Rightarrow a_1, a_2, \ldots. a_n \in K$.
$a_1, a_2, \ldots. a_n \in K, \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \Gamma$, K is a Γ-subsemigroup
$\Rightarrow a_1 \alpha_1 a_2 \alpha_2 \ldots\ldots a_{n-1} \alpha_{n-1} a_n \in K \Rightarrow a \in K$. Therefore $T \subseteq K$.
So T is the smallest Γ-subsemigroup of S containing A. Hence < A > = T.

THEOREM 1.2.8 : Let S be a Γ-semigroup and A be a nonempty subset of S. Then < A > = The intersection of all Γ-subsemigroups of S containing A.

***Proof*:** Let Δ be the set of all Γ-subsemigroups of S containing A.
Since S is a Γ-subsemigroup of S containing A, $S \in \Delta$. So $\Delta \neq \emptyset$.
Let $T^* = \bigcap_{T \in \Delta} T$. Since $A \subseteq T$ for all $T \in \Delta$, $A \subseteq T^*$.
By theorem 1.2.5, T^* is a Γ-subsemigroup of S.
Since $T^* \subseteq T$ for all $T \in \Delta$, T^* is the smallest Γ-subsemigroup of S containing A.
Therefore $T^* = < A >$.

DEFINITION 1.2.9 : Let S be a Γ-semigroup. A Γ-subsemigroup T of S is said to be ***cyclic Γ-subsemigroup*** of S if T is generated by a single element subset of S.

NOTE 1.2.10 : Let T be a Γ-subsemigroup of Γ-semigroup S. Then T is cyclic iff $T = \bigcup_{n \in N} (a\Gamma)^{n-1} a$ for some $a \in S$.

DEFINITION 1.2.11 : A Γ-semigroup S is said to be a ***cyclic* Γ*-semigroup*** if S is a cyclic Γ-subsemigroup of S itself.

1.3. SPECIAL ELEMENTS OF A Γ-SEMIGROUP

In this section, the terms; α-idempotent, Γ-idempotent, strongly Γ-idempotent, midunit, *r*-element, regular element, left regular element, right regular element, completely regular element, (α, β)-inverse of an element, semisimple element, intra regular element, left α-cancellative element, right α-cancellative element, α-cancellative element, left Γ-cancellative element, right Γ-cancellative element, Γ-cancellative element, strongly left Γ-cancellative element, strongly right Γ-cancellative element and strongly Γ-cancellative element in a Γ-semigroup are introduced. Also the terms; idempotent Γ-semigroup and generalized commutative Γ-semigroup are introduced. It is proved that every ***α***-idempotent element of a Γ-semigroup is regular. It is proved that every **Γ**-ideal of a regular **Γ**-semigroup S is a regular **Γ**-subsemigroup of S. It is proved that if a Γ-semigroup S is a regular Γ-semigroup then every principal Γ-ideal is generated by an idempotent. Further it is also proved that, in a Γ-semigroup, *a* is a regular element if and only if *a* has an (*α*, *β*)-inverse. It is proved that, (1) if '*a*' is a completely regular element of a Γ-semigroup S, then *a* is regular and semisimple, (2) if *a* is a completely regular element of Γ-semigroup then *a* is both left regular and right regular, (3) if '*a*' is a left regular element of a Γ- semigroup S, then *a* is semisimple, (4) if '*a*' is a right regular element of a Γ- semigroup S, then *a* is semisimple, (5) if '*a*' is a regular element of a Γ- semigroup S, then *a* is semisimple and (6) if '*a*' is a intra regular element of a Γ-semigroup S, then *a* is semisimple.

We now introduce α-idempotent element in a Γ-semigroup which is due to SEN [47].

DEFINITION 1.3.1 : An element *a* of Γ-semigroup S is said to be a ***α-idempotent*** provided $a\alpha a = a$.

NOTE 1.3.2 : The set of all *α*-idempotent elements in a Γ- semigroup S is denoted by E_α

We now introduce Γ-idempotent element in a Γ-semigroup.

DEFINITION 1.3.3 : An element *a* of Γ- semigroup S is said to be an ***idempotent*** or ***Γ-idempotent*** if $a\alpha a = a$ for all $\alpha \in \Gamma$.

NOTE 1.3.4 : In a Γ-semigroup S, ***a*** is an idempotent iff a is an α-idempotent for all $\alpha \in \Gamma$.

NOTE 1.3.5 : If an element ***a*** of Γ- semigroup S is ***an idempotent,*** then $a\Gamma a = a$.

We now introduce an idempotent Γ-semigroup and a strongly idempotent Γ-semigroup.

DEFINITION 1.3.6 : A Γ-semigroup S is said to be an ***idempotent* Γ*-semigroup*** provided every element of S is α-idempotent for some $\alpha \in \Gamma$.

DEFINITION 1.3.7 : A Γ-semigroup S is said to be a ***strongly idempotent Γ- semigroup*** provided every element in S is an idempotent.

We now introduce a special element which is known as midunit in a Γ-semigroup.

DEFINITION 1.3.8 : An element ***a*** of Γ-semigroup S is said to be a ***midunit*** provided $x\Gamma a\Gamma y = x\Gamma y$ for all $x, y \in S$.

NOTE 1.3.9 : Identity of a Γ- semigroup S is a midunit.

We now introduce an r-element in a Γ-semigroup and also a generalized commutative Γ-semigroup.

DEFINITION 1.3.10 : An element 'a' of Γ- semigroup S is said to be ***an r-element*** provided $a\Gamma s = s\Gamma a$ for all $s \in S$ and if $x, y \in$ S, then $a\Gamma x\Gamma y = b\Gamma y\Gamma x$ for some $b \in$ S.

DEFINITION 1.3.11 : A Γ- semigroup S with identity 1 is said to be ***a generalized commutative* Γ*- semigroup*** provided 1 is an r-element in S.

We now introduce a regular element in a Γ-semigroup and regular Γ-semigroup which are due to CHINRAM and SIAMMAI [11] and SEN and SAHA [49].

DEFINITION 1.3.12 : An element a of a Γ-semigroup S is said to be ***regular*** provided $a = a\alpha x\beta a$, for some $x \in$ S and $\alpha, \beta \in \Gamma$. i.e, $a \in a\Gamma S\Gamma a$.

DEFINITION 1.3.13 : A Γ- semigroup S is said to be *a* ***regular* Γ*- semigroup*** provided every element is regular.

The following example is due to SEN and SAHA [49].

EXAMPLE 1.3.14 : Let S be the set of 3×2 matrices and Γ be a set of some 2×3 matrices over of field. Then S is a regular Γ-semigoup.

Verification : Let A ∈ S, where $A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$.

Then we chose B ∈ Γ according to the following cases such that ABABA = ABA = A.

CASE 1 : When the submatrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non-singular, then $ad - bc \neq 0$.

e, f may both be 0 or one of them is 0 or both of them are non-zero.

Then $B = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} & 0 \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} & 0 \end{pmatrix}$ and we find ABA = A.

CASE 2 : $af - be \neq 0$. Then $B = \begin{pmatrix} \frac{f}{af-be} & 0 & \frac{-b}{af-be} \\ \frac{-e}{af-be} & 0 & \frac{a}{af-be} \end{pmatrix}$ and ABA = A.

CASE 3 : $cf - de \neq 0$. Then $B = \begin{pmatrix} 0 & \frac{f}{cf-de} & \frac{-d}{cf-de} \\ 0 & \frac{-e}{cf-de} & \frac{c}{cf-de} \end{pmatrix}$ and ABA = A.

CASE 4 : When the submatrices are singular,

then either $\begin{cases} ad - bc = 0 \\ cf - be = 0 \end{cases}$ or $\begin{cases} ad - bc = 0 \\ af - de = 0 \end{cases}$.

If all the elements of A are 0, then the case is trivial. Next we consider at least one of the elements of A is non-zero, say $a_{ij} \neq 0$, $i = 1,2,3$ and $j = 1,2$. Then we take the b_{ji}th element of B as $(a_{ij})^{-1}$ and the other elements of B are zero and we find that ABA = A. Thus A is regular. Hence S is a regular Γ-semigroup.

EXAMPLE 1.3.15 : Let S = {0, *a*, *b*} and Γ be any nonempty set. If we define a binary operation on S as the following Cayley table, then S is a semigroup.

.	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

Define a mapping from S × Γ × S to S as $a\alpha b = ab$ for all $a, b \in S$ and $\alpha \in \Gamma$. Then S is regular Γ-semigroup.

THEOREM 1.3.16 : Every α-idempotent element in a Γ-semigroup is regular

***Proof*:** Let a be an α-idempotent element in a Γ-semigroup S.
Then $a = a\alpha a$ for some $\alpha \in \Gamma$. Hence $a = a\alpha a\alpha a$. Therefore a is a regular element.

We now introduce a regular Γ-ideal of a Γ-semigroup. .

DEFINITION 1.3.17 : A Γ-ideal A of a Γ-semigroup S is said to be ***regular*** if every element of A is regular in A.

THEOREM 1.3.18 : Every Γ-ideal of a regular Γ-semigroup S is a regular Γ-ideal of S.

Proof : Let A be a Γ-ideal of S and $a \in A$. Then $a \in S$ and hence a is regular in S.
Therefore $a = a\alpha b\beta a$ where $b \in S$ and $\alpha, \beta \in \Gamma$.
Hence $a = a\alpha b\beta a = (a\alpha b\beta)(a\alpha b\beta a) = a\alpha[(b\beta a)\alpha b]\beta a$.
Let $b_1 = (b\beta a)\alpha b \in S\Gamma A\Gamma S \subseteq A$.
Now $a\alpha b_1\beta a = a\alpha[(b\beta a)\ \alpha b]\ \beta a = a$.
Therefore a is regular in A and hence A is a regular Γ-ideal.

THEOREM 1.3.19 : If a Γ-semigroup S is a regular Γ-semigroup then every principal Γ-ideal is generated by a β-idempotent for some $\beta \in \Gamma$.

Proof : Suppose that S is a regular Γ-semigroup. Let $< a >$ be a principal Γ-ideal of S.
Since S is a regular Γ-semigroup, there exists $x \in S$, $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.
Let $a\alpha x = e$. Then $e\beta e = (a\alpha x)\beta(a\alpha x) = (a\alpha x\beta a)\alpha x = a\alpha x = e$.
Thus e is a β-idempotent element of S.
Now $a = a\alpha x\beta a = e\beta a \in < e > \Rightarrow < a > \subseteq < e >$.
Also $e = a\alpha x \in < a > \Rightarrow < e > \subseteq < a >$.
Therefore $< a > = < e >$ and hence every principal Γ-ideal is generated by an idempotent.

We now introduce left regular element, right regular element, completely regular element in a Γ-semigroup and completely regular Γ-semigroup.

DEFINITION 1.3.20 : An element a of a Γ-semigroup S is said to be ***left regular*** provided $a = a\alpha a\beta x$, for some $x \in$ S and $\alpha, \beta \in \Gamma$. i.e, $a \in a\Gamma a\Gamma S$.

DEFINITION 1.3.21 : An element a of a Γ-semigroup S is said to be ***right regular*** provided $a = x\alpha a\beta a$, for some $x \in$ S and $\alpha, \beta \in \Gamma$. i.e, $a \in S\Gamma a\Gamma a$.

DEFINITION 1.3.22 : An element a of a Γ-semigroup S is said to be ***completely regular*** provided, there exists an element $x \in$ S such that $a = a\alpha x\beta a$ for some $\alpha, \beta \in \Gamma$ and $a\alpha x = x\beta a$ i.e., $a \in a\Gamma x\Gamma a$ and $a\Gamma x = x\Gamma a$.

DEFINITION 1.3.23 : A Γ-semigroup S is said to be ***completely regular Γ-semigroup*** provided every element of S is completely regular.

We now introduce (α, β)-inverse of an element in a Γ-semigroup which is due to CHINRAM and SIAMMAI [11].

DEFINITION 1.3.24 : Let S be a Γ-semigroup, $a \in$ S and $\alpha, \beta \in \Gamma$. An element $b \in$ S is said to be an ***(α, β)-inverse*** of a if $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$.

THEOREM 1.3.25 : Let S be a Γ-semigroup and $a \in$ S. Then a is a regular element if and only if a has an (α, β)-inverse.

Proof : Suppose that a is a regular element. Then $a = a\alpha b\beta a$ for some $b \in$ S and $\alpha, \beta \in \Gamma$.
Let $x = b\beta a\alpha b \in$ S.
Now $a\alpha x\beta a = a\alpha(b\beta a\alpha b)\beta a = (a\alpha b\beta a)\alpha b\beta a = a\alpha b\beta a = a$ and
$x\beta a\alpha x = (b\beta a\alpha b)\beta a\alpha(b\beta a\alpha b) = b\beta(a\alpha b\beta a)\alpha(b\beta a\alpha b) = b\beta a\alpha(b\beta a\alpha b) = b\beta(a\alpha b\beta a)\alpha b$
$= b\beta a\alpha b = x$. Therefore $x = b\beta a\alpha b$ is the (α, β)-inverse of a.
Conversely suppose that b is an (α, β)-inverse of a.
Then $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$. Therefore $a = a\alpha b\beta a$ and hence a is regular.

We now introduce a semisimple element of a Γ-semigroup and a semisimple Γ-semigroup.

DEFINITION 1.3.26 : An element a of Γ- semigroup S is said to be ***semisimple*** provided $a \in \langle a \rangle \Gamma \langle a \rangle$, that is, $\langle a \rangle \Gamma \langle a \rangle = \langle a \rangle$.

DEFINITION 1.3.27 : A Γ- semigroup S is said to be ***semisimple* Γ- *semigroup*** provided every element of S is a semisimple element.

We now introduce an intra regular element of a Γ-semigroup.

DEFINITION 1.3.28 : An element a of a Γ-semigroup S is said to be ***intra regular*** provided $a = x\alpha a\beta a\gamma y$ for some $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$.

EXAMPLE 1.3.29 : The Γ-semigroup given in example 1.3.15, is an intra regular Γ-semigroup.

THEOREM 1.3.30 : If '*a*' is a completely regular element of a Γ- semigroup S, then *a* is regular and semisimple.

Proof : Since a is a completely regular element in the Γ- smigroup S, $a = a\alpha x\beta a$ for some $\alpha, \beta \in \Gamma$ and $x \in S$. Therefore a is regular.
Now $a = a\alpha x\beta a \in a\Gamma x\Gamma a \subseteq < a > \Gamma < a >$. Therefore a is semisimple.

THEOREM 1.3.31 : If '*a*' is a completely regular element of a Γ- semigroup S, then *a* is both a left regular element and a right regular element.

Proof : Suppose that a is completely regular. Then $a \in a\Gamma S\Gamma a$ and $a\Gamma S = S\Gamma a$.
Now $a \in a\Gamma S\Gamma a = a\Gamma a\Gamma S$. Therefore a is left regular.
Also $a \in a\Gamma S\Gamma a = S\Gamma a\Gamma a$. Therefore a is right regular.

THEOREM 1.3.32 : If '*a*' is a left regular element of a Γ-semigroup S, then *a* is semisimple.

Proof : Suppose that a is left regular. Then $a \in a\Gamma a\Gamma x$ and hence $a \in < a > \Gamma < a >$. Therefore a is semisimple.

THEOREM 1.3.33 : If '*a*' is a right regular element of a Γ-semigroup S, then *a* is semisimple.

Proof : Suppose that a is right regular. Then $a \in a\Gamma a\Gamma x$ and hence $a \in < a > \Gamma < a >$. Therefore a is semisimple.

THEOREM 1.3.34 : If '*a*' is a regular element of a Γ-semigroup S, then *a* is semisimple.

Proof : Suppose that a is regular element of Γ-semigroup S.

Then $a = a\alpha x\beta a$, for some $x \in S$, $\alpha, \beta \in \Gamma$ and hence $a \in < a > \Gamma < a >$.
Therefore a is semisimple.

THEOREM 1.3.35 : If 'a' is a intra regular element of a Γ- semigroup S, then a is semisimple.

Proof : Suppose that a is intra regular. Then $a \in x\Gamma a\Gamma a\Gamma y$ for $x, y \in S$ and hence $a \in < a > \Gamma < a >$ Therefore a is semisimple.

We now introduce some more special elements, namely left α-cancellative element, right α-cancellative element, α-cancellative element, left Γ-cancellative element, right Γ-cancellative element, Γ-cancellative element, strongly left **Γ**-cancellative element, strongly right **Γ**-cancellative element and strongly **Γ**-cancellative element.

DEFINITION 1.3.36 : An element a of a Γ-semigroup S is said to be ***left α-cancellative provided*** for $\alpha \in \Gamma$, $a\alpha b = a\alpha c$ implies $b = c$.

DEFINITION 1.3.37 : An element a of a Γ-semigroup S is said to be ***right α-cancellative*** provided for $\alpha \in \Gamma$, $b\alpha a = c\alpha a$ implies $b = c$.

DEFINITION 1.3.38 : An element a of a Γ-semigroup S is said to be ***α-cancellative*** provided a is both a left α-cancellative element and a right α-cancellative element.

DEFINITION 1.3.39 : An element a of a Γ-semigroup S is said to be ***left Γ-cancellative*** provided a is left α-cancellative for all $\alpha \in \Gamma$.

DEFINITION 1.3.40 : An element a of a Γ-semigroup S is said to be ***right Γ-cancellative*** provided a is right α-cancellative for all $\alpha \in \Gamma$.

DEFINITION 1.3.41 : An element a of a Γ-semigroup S is said to be ***Γ-cancellative*** provided a is a both left Γ-cancellative and Γ-cancellative.

DEFINITION 1.3.42 : An element a of a Γ-semigroup S is said to be ***strongly left Γ-cancellative*** provided $a\Gamma b = a\Gamma c$ implies $b = c$.

NOTE 1.3.43 : An element a of a Γ-semigroup S is said to be ***strongly left Γ-cancellative*** provided $a\alpha b = a\beta c$, $\alpha, \beta \in \Gamma \Rightarrow b = c$.

DEFINITION 1.3.44 : An element a of a Γ-semigroup S is said to be ***strongly right Γ-cancellative*** provided $b\Gamma a = c\Gamma a$ implies $b = c$.

NOTE 1.3.45 : An element a of a Γ-semigroup S is said to be ***strongly right Γ-cancellative*** provided $b\alpha a = c\beta a$, $\alpha, \beta \in \Gamma \Rightarrow b = c$.

DEFINITION 1.3.46 : An element a of a Γ-semigroup S is said to be ***strongly Γ-cancellative*** provided a is a both strongly left **Γ**-*cancellative* and strongly right **Γ**-*cancellative.*

THEOREM 1.3.47 : Every Γ-group is a strongly Γ-cancellative Γ-semigroup.

Proof : Let S be a Γ-group with the identity element ***e***. Let $a,b,c \in S$.
i) Suppose that $a\Gamma b = a\Gamma c$. Now $b = b\Gamma c = b\Gamma(a\Gamma b) = b\Gamma(a\Gamma c) = (b\Gamma a)\Gamma c = e\Gamma c = c$.
ii) Suppose that $b\Gamma a = c\Gamma a$. Now $b = e\Gamma b = (b\Gamma a)\Gamma b = (c\Gamma a)\Gamma b = c\Gamma(a\Gamma b) = c\Gamma e = c$.
Since every Γ-group is a Γ-semigroup and hence every Γ-group is a strongly Γ-cancellative Γ-semigroup.

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