CHAPTER 4

PRIMARY DECOMPOSITION IN Γ-SEMIGROUPS

CHAPTER-4

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The theory of primary ideals played an important role in commutative rings and later it was extended to commutative semigroups by SATYANARAYANA[42]. MANNEPALLI and NAGORE [38] obtained a primary decomposition theorem for generalized commutative semigroups which can be favourbly compared with the primary decomposition theorem in commutative noetherian rings. ANJANEYULU [1], [2] introduced the notion of one sided primary ideals in an arbitrary semigroup via ideals rather than elements and obtain a primary decomposition theorem for duo noetherian semigroups. He obtained a necessary condition to have a unique primary decomposition for an ideal in an arbitrary semigroup. A study of semiprimary ideals in commutative semigroups was made by HARBANSLAL [23] and ANJANEYULU [1], [2] extended those results to arbitrary semigroups. In this thesis we study primary and semiprimary Γ-ideals in Γ-semigroups and deduce the results regarding semigroups to Γ-semigroups and also obtain a primary decomposition theorem for duo noetherian Γ-semigroups.

This chapter is divided into 3 sections. In section 1, the terms; left primary Γ-ideal, right primary Γ-ideal, primary Γ-ideal, left primary Γ-semigroup, right primary Γ-semigroup and primary Γ-semigroup are introduced. Equivalent conditions for left primary Γ-ideal, right primary Γ-ideal in a Γ-semigroup are obtained. In a commutative Γ-semigroup, equivalent conditions for a primary Γ-ideal are obtained. It is proved that in a Γ -semigroup S with identity, if $\sqrt{A} = M$ for some Γ-ideal A of S, where M is the unique maximal Γ-ideal of S, then A is a primary Γ -ideal. It is also proved that if S is a Γ-semigroup with identity, then for any natural number *n*, $(M\Gamma)^{n-1}M$ is a primary Γ-ideal of S, where M is the unique maximal Γ-ideal of S. Further it is proved that in a quasi commutative **Γ**-semigroup S, a Γ-ideal A of S is left primary iff A is right primary.

In section 2, the terms; semiprimary Γ-ideal of a Γ-semigroup and semiprimary Γ-semigroup are introduced. If A is a semiprime **Γ**-ideal of a **Γ**-semigroup S, then it is proved that (1) A is a prime **Γ**-ideal, (2) A is a primary **Γ**-ideal, (3) A is a left primary **Γ**-ideal, (4) A is a right primary **Γ**-ideal, (5) A is a semiprimary **Γ**-ideal, are equivalent. It is proved that in semiprimary duo Γ-semigroup, globally idempotent principal Γ-ideals form a chain under set inclusion. In a semisimple Γ-semigroup S it is proved that the conditions; (1) every ideal of S is prime, (2) S is a primary Γ-semigroup, (3) S is a left primary Γ-semigroup, (4) S is a right primary Γ-semigroup, (5) S is a semiprimary Γ-semigroup, (6) prime Γ-ideals of S form a chain, (7) principal Γ-ideals of S form a chain and (8) Γ-ideals of S form a chain, are equivalent.

In section 3, the terms; P-primary, primary decomposition of a **Γ**-ideal, reduced primary decomposition of a **Γ**-ideal in a **Γ**-semigroup S are introduced. If $A_1, A_2, \ldots, A_n$ are P-primary **Γ**-ideals in a **Γ**-semigroup S, then it is proved that $\bigcap_{i=1}^{n} A_i$ is also a P-primary **Γ**-ideal. If a **Γ**-ideal A in a **Γ**-semigroup S has a primary decomposition, then it is proved that A has a reduced primary decomposition. Further it is proved that every Γ-ideal in a (left, right) duo noetherian Γ-semigroup S has a reduced (right, left) primary decomposition. It is proved that (1) if A is a left primary **Γ**-ideal of a **Γ**-semigroup S, then A^l (B) is a left primary **Γ**-ideal, (2) if A is a right primary **Γ**-ideal of a **Γ**-semigroup S, then A^r(B) is a right primary **Γ**-ideal. It is proved that if Q is a P-primary **Γ**-ideal and if $A \not\subseteq P$, then $Q^l(A) = Q^r(A) = Q$ and also if $A \subseteq P$ and $A \not\subseteq Q$, then $\sqrt{(Q^l(A))} = \sqrt{(Q^r(A))} = \sqrt{Q}$. If $A_1, A_2, \ldots A_n$, B are **Γ**-ideals of a **Γ**-semigroup S, then it is proved that $\left(\bigcap_{i=1}^{n} A_i\right)^l (B) = \bigcap_{i=1}^{n} (A_i)^l(B)$. Further if a **Γ**-ideal A in a **Γ**-semigroup S has two reduced (one sided) primary decompositions; $A = A_1 \cap A_2 \cap \ldots \cap A_k = B_1 \cap B_2 \cap \ldots B_s$, where A_i is P_i -primary and B_j is Q_j-primary, then it is proved that $k = s$ and after reindexing if necessary $P_i = Q_i$ for $i = 1, 2, \ldots, k$.

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4.1. PRIMAY Γ-IDEALS

In this section the terms; left primary Γ-ideal, right primary Γ-ideal, primary Γ-ideal, left primary Γ-semigroup, right primary Γ-semigroup and primary Γ-semigroup are introduced. Equivalent conditions for left primary Γ-ideal, right primary Γ-ideal in a Γ-semigroup are obtained. In a commutative Γ-semigroup, equivalent conditions for a primary Γ-ideal are obtained. It is proved that in a Γ -semigroup S with identity, if $\sqrt{A} = M$ for some Γ-ideal A of S, where M is the unique maximal Γ-ideal of S, then A is a primary Γ -ideal. It is also proved that if S is a Γ-semigroup with identity, then for any natural number n, $(M\Gamma)^{n-1}M$ is a primary Γ-ideal of S, where M is the unique maximal

Γ-ideal of S. Further it is proved that in a quasi commutative **Γ**-semigroup S, a Γ-ideal A of S is left primary iff A is right primary.

We now introduce the notions of a left primary Γ-ideal, a right primary Γ-ideal and a primary Γ-ideal of a Γ-semigroup.

DEFINITION 4.1.1 : A Γ-ideal A of a Γ-semigroup S is said to be a ***left primary* Γ*-ideal*** provided

i) if X, Y are two Γ-ideals of S such that $X\Gamma Y \subseteq A$ and $Y \not\subseteq A$ then $X \subseteq \sqrt{A}$.

ii) $\sqrt{A}$ is a prime Γ-ideal of S.

DEFINITION 4.1.2 : A Γ-ideal A of a Γ-semigroup S is said to be a ***right primary* Γ*-ideal*** provided

i) if X, Y are two Γ-ideals of S such that $X\Gamma Y \subseteq A$ and $X \not\subseteq A$ then $Y \subseteq \sqrt{A}$.

ii) $\sqrt{A}$ is a prime Γ-ideal of S.

DEFINITION 4.1.3 : A Γ-ideal A of a Γ-semigroup S is said to be a ***primary* Γ*-ideal*** provided A is both a left primary Γ-ideal and a right primary Γ-ideal.

THEOREM 4.1.4 : Let A be a Γ-ideal of a Γ-semigroup S. Then X, Y are two Γ-ideals of S such that $X\Gamma Y \subseteq A$ and $Y \not\subseteq A \Rightarrow X \subseteq \sqrt{A}$ if and only if $x, y \in S$, $<x>\Gamma<y> \subseteq A$ and $y \notin A \Rightarrow x \in \sqrt{A}$.

***Proof*:** Suppose that X, Y are two Γ-ideals of S such that $X\Gamma Y \subseteq A$, $Y \not\subseteq A \Rightarrow X \subseteq \sqrt{A}$.
Let $x, y \in S$, $<x>\Gamma<y> \subseteq A$ and $y \notin A$. Now $y \notin A \Rightarrow <y> \not\subseteq A$.
By supposition $<x>\Gamma<y> \subseteq A$ and $<y> \not\subseteq A \Rightarrow <x> \subseteq \sqrt{A}$. Therefore $x \in \sqrt{A}$.

Conversely suppose that $x, y \in S$, $<x>\Gamma<y> \subseteq A$ and $y \notin A \Rightarrow x \in \sqrt{A}$.
Let X, Y be two Γ-ideals of S such that $X\Gamma Y \subseteq A$ and $Y \not\subseteq A$.
Suppose if possible $X \not\subseteq \sqrt{A}$. Then there exists $x \in X$ such that $x \notin \sqrt{A}$.
Since $Y \not\subseteq A$, let $y \in Y$ so that $y \notin A$.
Now $<x>\Gamma<y> \subseteq X\Gamma Y \subseteq A$ and $y \notin A \Rightarrow x \in \sqrt{A}$. It is a contradiction.
Therefore $X \subseteq \sqrt{A}$.

THEOREM 4.1.5 : Let A be a Γ-ideal of a Γ-semigroup S. Then X, Y are two Γ-ideals of S such that $X\Gamma Y \subseteq A$ and $X \not\subseteq A \Rightarrow Y \subseteq \sqrt{A}$ if and only if $x, y \in S$, $<x>\Gamma<y> \subseteq A$ and $x \notin A \Rightarrow y \in \sqrt{A}$.

Proof **:** Suppose that X, Y are two Γ-ideals of S such that $X\Gamma Y \subseteq A$, $X \not\subseteq A \Rightarrow Y \subseteq \sqrt{A}$.

Let $x, y \in S$, $<x>\Gamma<y> \subseteq A$ and $x \notin A$. Now $x \notin A \Rightarrow <x> \not\subseteq A$.

By supposition $<x>\Gamma<y> \subseteq A$ and $<x> \not\subseteq A \Rightarrow <y> \subseteq \sqrt{A}$. Therefore $y \in \sqrt{A}$.

Conversely suppose that $x, y \in S$, $<x>\Gamma<y> \subseteq A$ and $x \notin A \Rightarrow y \in \sqrt{A}$.

Let X, Y be two Γ-ideals of S such that $X\Gamma Y \subseteq A$ and $X \not\subseteq A$.

Suppose if possible $Y \not\subseteq \sqrt{A}$. Then there exists $y \in Y$ such that $y \notin \sqrt{A}$.

Since $X \not\subseteq A$, let $x \in X$ so that $x \notin A$.

Now $<x>\Gamma<y> \subseteq X\Gamma Y \subseteq A$ and $x \notin A \Rightarrow y \in \sqrt{A}$. It is a contradiction.

Therefore $Y \subseteq \sqrt{A}$.

THEOREM 4.1.6 : Let S be a commutative Γ-semigroup and A be a Γ-ideal of S. Then the following conditions are equivalent.

1) A is a primary Γ-ideal.

2) X, Y are two Γ-ideals of S, $X\Gamma Y \subseteq A$ and $Y \not\subseteq A \Rightarrow X \subseteq \sqrt{A}$.

3) $x, y \in S$, $x\Gamma y \subseteq A$, $y \notin A \Rightarrow x \in \sqrt{A}$.

***Proof*:** (1) ⇒ (2) : Suppose that A is a primary Γ-ideal of S.

Then A is a left primary Γ-ideal of S.

So by definition 4.1.1, we get X, Y are two Γ-ideals of S, $X\Gamma Y \subseteq A$, $Y \not\subseteq A \Rightarrow X \subseteq \sqrt{A}$.

(2) ⇒ (3): Suppose that X, Y are two **Γ**-ideals of S, $X\Gamma Y \subseteq A$ and $Y \not\subseteq A \Rightarrow X \subseteq \sqrt{A}$.

Let $x, y \in S$, $x\Gamma y \subseteq A$ and $y \notin A$.

$x\Gamma y \subseteq A \Rightarrow <x>\Gamma<y> \subseteq A$. Also $y \notin A \Rightarrow <y> \not\subseteq A$.

Now $<x>\Gamma<y> \subseteq A$ and $<y> \not\subseteq A$. Therefore by assumption $<x> \subseteq \sqrt{A} \Rightarrow x \in \sqrt{A}$.

(3) ⇒ (1) : Suppose assume that $x, y \in S$, $x\Gamma y \subseteq A$, $y \notin A$ then $x \in \sqrt{A}$.

Let X, Y be two **Γ**-ideals of S, $X\Gamma Y \subseteq A$ and $Y \not\subseteq A$.

$Y \not\subseteq A \Rightarrow$ there exists $y \in Y$ such that $y \notin A$.

Suppose if possible $X \not\subseteq \sqrt{A}$. Then there exists $x \in X$ such that $x \notin \sqrt{A}$.

Now $x\Gamma y \subseteq X\Gamma Y \subseteq A$. Therefore $x\Gamma y \subseteq A$ and $y \notin A$, $x \notin \sqrt{A}$. It is a contradiction.

Therefore $X \subseteq \sqrt{A}$.

Let $x, y \in S$, $x\Gamma y \subseteq \sqrt{A}$. Suppose that $y \notin \sqrt{A}$.

Now $x\Gamma y \subseteq \sqrt{A} \Rightarrow (x\Gamma y\Gamma)^{m-1}(x\Gamma y) \subseteq A \Rightarrow (x\Gamma)^{m-1}x\Gamma(y\Gamma)^{m-1}y \subseteq A$.

Since $y \notin \sqrt{A}$, $(y\Gamma)^{m-1}y \not\subseteq A$.

Now $(x\Gamma)^{m-1}x\Gamma(y\Gamma)^{m-1}y \subseteq A$, $(y\Gamma)^{m-1}y \not\subseteq A \Rightarrow (x\Gamma)^{m-1}x \subseteq \sqrt{A} \Rightarrow x \in \sqrt{(\sqrt{A})} = \sqrt{A}$.

$\sqrt{A}$ is a completely prime Γ-ideal and hence $\sqrt{A}$ is a prime Γ-ideal.

Therefore A is a left primary Γ-ideal. Similarly A is a right primary Γ-ideal.

Hence A is a primary Γ-ideal.

NOTE 4.1.7 : In an arbitrary Γ-semigroup a left primary Γ-ideal is not necessarily a right primary Γ-ideal.

EXAMPLE 4.1.8 : Let S = {a, b, c} and Γ = {x, y, z}. Define a binary operation **.** in S as shown in the following table.

.	a	b	c
a	a	a	a
b	a	a	a
c	a	b	c

Define a mapping from S × Γ × S ⟶ S by $a\alpha b = ab$, for all $a, b \in S$ and $\alpha \in \Gamma$.

It is easy to see that S is a Γ-semigroup. Now consider the Γ-ideal, $< a > = S^1\Gamma a\Gamma S^1 = \{a\}$.

Let $p\Gamma q \subseteq < a >, p \notin < a > \Rightarrow q \in \sqrt{< a >} \Rightarrow (q\Gamma)^{n-1}q \subseteq < a >$ for some $n \in N$.

Since $b\Gamma c \subseteq < a >, c \notin < a > \Rightarrow b \in \sqrt{< a >}$. Therefore $< a >$ is left primary.

If $b \notin < a >$ then $(c\Gamma)^{n-1}c \nsubseteq < a >$ for any $n \in N \Rightarrow c \notin \sqrt{< a >}$.

Therefore $< a >$ is not right primary.

THEOREM 4.1.9 : Every Γ-ideal A in a Γ-semigroup S is left primary if and only if every Γ-ideal A satisfies condition (i) of definition 4.1.1.

***Proof*:** If every Γ-ideal A of S is left primary, then clearly every Γ-ideal satisfies condition (i) of definition 4.1.1.

Conversely suppose that every Γ-ideal of S satisfies condition (i) of definition 4.1.1.

Let A be any Γ-ideal of S. Suppose that $x, y \in S$ and $< x > \Gamma < y > \subseteq \sqrt{A}$.

If $y \notin \sqrt{A}$, then by our supposition $x \in \sqrt{(\sqrt{A})} = \sqrt{A}$.

Therefore $\sqrt{A}$ is a prime Γ-ideal. Hence A is left primary.

THEOREM 4.1.10 : Every Γ-ideal A in a Γ-semigroup S is right primary if and only if every Γ-ideal A satisfies condition (i) of definition 4.1.2.

***Proof* :** If every Γ-ideal A of S is right primary, then clearly every Γ-ideal satisfies condition (i) of definition 4.1.2.

Conversely suppose that every Γ-ideal of S satisfies condition (i) of definition 4.1.2.

Let A be any Γ-ideal of S. Suppose that $x, y \in$ S and $<x> \Gamma <y> \subseteq \sqrt{A}$.
If $x \notin \sqrt{A}$ then by our supposition $y \in \sqrt{(\sqrt{A})} = \sqrt{A}$.
Therefore $\sqrt{A}$ is a prime Γ-ideal. Hence A is left primary.

We now introduce the terms, a left primary Γ-semigroup, a right primary Γ-semigroup and a primary Γ-semigroup.

DEFINITION 4.1.11 : A Γ-semigroup S is said to be ***left primary*** provided every Γ-ideal of S is a left primary Γ-ideal of S.

DEFINITION 4.1.12 : A Γ-semigroup S is said to be ***right primary*** provided every Γ-ideal of S is a right primary Γ-ideal of S.

DEFINITION 4.1.13 : A Γ-semigroup S is said to be ***primary*** provided every Γ-ideal of S is a primary Γ-ideal of S.

THEOREM 4.1.14 : Let S be a Γ-semigroup with identity and let M be the unique maximal Γ-ideal of S. If $\sqrt{A}$ = M for some Γ-ideal of S, then A is a primary Γ-ideal.

***Proof*:** suppose that $x, y \in$ S, $<x> \Gamma <y> \subseteq$ A and $y \notin$ A.
If $x \notin \sqrt{A}$ then $<x> \nsubseteq \sqrt{A}$ = M.
By theorem 2.1.27, M is the union of all proper Γ-ideals of S, we have $<x>$ = S and hence $<y> = <x> \Gamma <y> \subseteq$ A. It is a contradiction. Therefore $x \in \sqrt{A}$.
Let $x, y \in$ S, $<x> \Gamma <y> \subseteq \sqrt{A}$ and $<y> \nsubseteq \sqrt{A}$.
Since M is the unique maximal Γ-ideal, we have $<x>$ = S.
Hence $<y> = <x> \Gamma <y> \subseteq \sqrt{A}$.
It is a contradiction. Therefore $<x> \subseteq \sqrt{A}$.
Similarly if $<x> \nsubseteq \sqrt{A}$, then $<y> \subseteq \sqrt{A}$ and hence $\sqrt{A}$ = M is a prime Γ-ideal.
Thus A is left primary. By symmetry it follows that A is right primary.
Therefore A is a primary Γ-ideal.

NOTE 4.1.15 : If a Γ-semigroup S has no identity, then the theorem 4.1.14, is not true, even if the Γ-semigroup S has a unique maximal Γ-ideal. In example 4.1.8, $\sqrt{<a>}$ = M where M = {*a*, *b*} is the unique maximal Γ-ideal. But $<a>$ is not a primary Γ-ideal.

THEOREM 4.1.16 : If S is a Γ-semigroup with identity, then for any natural number *n*, $(M\Gamma)^{n-1}M$ is primary Γ-ideal of S where M is the unique maximal Γ-ideal of S.

Proof **:** Since M is the only prime Γ-ideal containing $(M\Gamma)^{n-1}M$, we have $\sqrt{((M\Gamma)^{n-1}M)} = M$ and hence by theorem 4.1.14, $(M\Gamma)^{n-1}M$ is a primary Γ-ideal.

NOTE 4.1.17 : If S has no identity then theorem 4.1.16, is not true. In example 4.1.8, $M = \{a, b\}$ is the unique maximal Γ-ideal, but $M\Gamma M = \{a\}$ is not primary.

THEOREM 4.1.18 : In quasi commutative Γ-semigroup S, a Γ-ideal A of S is left primary iff right primary.

Proof **:** Suppose that A is a left primary Γ-ideal. Let $x\Gamma y \subseteq A$ and $x \notin A$.
Since S is a quasi commutative Γ-semigroup, we have $x\Gamma y = (y\Gamma)^n x$ for some $n \in N$.
So $(y\Gamma)^n x = (y\Gamma)^{n-1} y\Gamma x \subseteq A$ and $x \notin A$.
Since A is left primary, we have $(y\Gamma)^{n-1}y \subseteq \sqrt{A}$ and since $\sqrt{A}$ is a prime Γ-ideal, $y \in \sqrt{A}$.
Therefore A is a right primary Γ-ideal.
Similarly we can prove that if A is a right primary Γ-ideal then A is a left primary Γ-ideal.

COROLLARY 4.1.19 : If A is a Γ-ideal of a quasi commutative Γ-semigroup S, then the following are equivalent.

1) **A is primary**
2) **A is left primary**
3) **A is right primary**

4.2. SEMIPRIMARY Γ-IDEALS

In this section , the terms; semiprimary Γ-ideal of a Γ-semigroup and semiprimary Γ-semigroup are introduced. If A is a semiprime Γ-ideal of a Γ-semigroup S, then it is proved that (1) A is a prime Γ-ideal, (2) A is a primary Γ-ideal, (3) A is a left primary Γ-ideal, (4) A is a right primary Γ-ideal, (5) A is a semiprimary Γ-ideal, are equivalent. It is proved that in semiprimary duo Γ-semigroup, globally idempotent principal Γ-ideals form a chain under set inclusion. In a semisimple Γ-semigroup S it is proved that the conditions; (1) every ideal of S is prime, (2) S is a primary Γ-semigroup, (3) S is a left primary Γ-semigroup, (4) S is a right primary Γ-semigroup, (5) S is a semiprimary Γ-semigroup, (6) prime Γ-ideals of S form a chain, (7) principal Γ-ideals of S form a chain and (8) Γ-ideals of S form a chain, are equivalent.

We now introduce the notions of a semiprimary Γ-ideal of a Γ-semigroup and a semiprimary Γ-semigroup.

DEFINITION 4.2.1 : A Γ-ideal A of a Γ-semigroup S is said to be ***semiprimary*** provided $\sqrt{A}$ is a prime Γ-ideal of S.

DEFINITION 4.2.2 : A Γ-semigroup S is said to be a ***semiprimary Γ-semigroup*** provided every Γ-ideal of S is a semiprimary Γ-ideal.

We now exhibit some classes of semiprimary Γ-ideals.

THEOREM 4.2.3 : (1) Every left primary Γ-ideal of a Γ-semigroup is a semiprimary Γ-ideal (2) Every right primary Γ-ideal of a Γ-semigroup is a semiprimary Γ-ideal.

Proof **:** By the definition of a left primary Γ-ideal of a Γ-semigroup, every left primary Γ-ideal is a semiprimary Γ-ideal. By the definition of a right primary Γ-ideal of a Γ-semigroup, every right primary Γ-ideal is a semiprimary Γ-ideal

THEOREM 4.2.4 : Let S is a Γ-semigroup (not necessarily with identity) and let A be a Γ-ideal of S with $\sqrt{A}$ is a maximal Γ-ideal of S. Then A is a semiprimary Γ-ideal.

Proof **:** If there is no proper prime Γ-ideal P containing A, then every prime Γ-ideal equal to S. Then the intersection of all prime Γ-ideals of S = $\sqrt{A}$ = S.
Since $\sqrt{A}$ is a maximal Γ-ideal , $\sqrt{A}$ must be a proper Γ-ideal.
Therefore there exists a proper prime Γ-ideal P containing A.
Now $\sqrt{A} \subseteq P \subset S$ and $\sqrt{A}$ is maximal, we have $\sqrt{A} = P$.
Therefore $\sqrt{A}$ is a prime Γ-ideal and hence A is a semiprimary Γ-ideal.

THEOREM 4.2.5 : If A is a semiprime Γ-ideal of a Γ-semigroup S, then the following are equivalent.

1) **A is a prime Γ-ideal.**
2) **A is a primary Γ-ideal.**
3) **A is a left primary Γ-ideal.**
4) **A is a right primary Γ-ideal.**
5) **A is a semiprimary Γ-ideal.**

Proof **:** (1) ⇒ (2) ⇒ (3) ⇒ (5) and (2) ⇒ (4) ⇒ (5) are clear.
(5) ⇒ (1) : Suppose that A is a semiprimary Γ-ideal. Then $\sqrt{A}$ is a prime Γ-ideal.
Since A is semiprime, by theorem 2.4.6, A = $\sqrt{A}$ which is a prime Γ-ideal of S.

We now characterize semiprimary Γ-semigroups.

THEOREM 4.2.6 : A Γ-semigroup S is semiprimary iff prime Γ-ideals of S form a chain under set inclusion.

Proof : Suppose that S is a semiprimary Γ-semigroup.
Let A and B are two prime Γ-ideals of S. Now $\sqrt{(A \cap B)} = \sqrt{A} \cap \sqrt{B} = A \cap B$.
Therefore by theorem 2.4.6, $A \cap B$ is semiprime.
By theorem 4.2.5, Since S is a semiprimary Γ-semigroup, it follows that $A \cap B$ is prime.
Suppose if possible $A \not\subseteq B$ and $B \not\subseteq A$. Then there exists $x \in A \backslash B$ and $y \in B \backslash A$.
Now $<x>\Gamma<y> \subseteq A \cap B$ and $x, y \notin A \cap B$. It is a contradiction.
Therefore prime Γ-ideals of S form a chain.

Conversely suppose that prime Γ-ideals of S form a chain under set inclusion.
For every Γ-ideal A, $\sqrt{A} = \cap P_\alpha$, where intersection is over all prime Γ-ideals P_α containing A yields $\sqrt{A} = P_\alpha$ for some α, so that A is a semiprimary Γ-ideal.
Therefore S is a semiprimary Γ-semigroup.

THEOREM 4.2.7 : In a semiprimary duo Γ-semigroup S, globally idempotent principal Γ-ideals form a chain under set inclusion.

Proof : Let $<a>, <b>$ be two globally idempotent principal Γ-ideals of S.
Since S is semiprimary Γ-semigroup, we have $\sqrt{<a>}$ and $\sqrt{<b>}$ are prime Γ-ideals.
By theorem 4.2.6, either $\sqrt{<a>} \subseteq \sqrt{<b>}$ or $\sqrt{<b>} \subseteq \sqrt{<a>}$.
Assume that $\sqrt{<a>} \subseteq \sqrt{<b>}$. Then $a \in <a> \subseteq \sqrt{<a>} \subseteq \sqrt{<b>}$ and hence $a \in \sqrt{<b>}$.
Thus $(<a>\Gamma)^{n-1}<a> \subseteq <b>$ for some $n \in N$.
Since $<a>$ is a globally idempotent principal Γ-ideal,
$<a> = <a>\Gamma<a> = (<a>\Gamma)^{n-1}<a> \subseteq <b>$. Therefore $<a> \subseteq <b>$.

Similarly we can show that if $\sqrt{<b>} \subseteq \sqrt{<a>}$, then $<b> \subseteq <a>$.

Therefore globally idempotent principal Γ-ideals form a chain under set inclusion.

COROLLARY 4.2.8 : In a semiprimary semisimple duo Γ-semigroup, principal Γ-ideals form a chain.

Proof : In a semisimple Γ-semigroup, every principal Γ-ideal is globally idempotent; by theorem 4.2.7, principal Γ-ideals form a chain.

THEOREM 4.2.9 : In a semisimple Γ-semigroup S, the following are equivalent.

1) **Every Γ-ideal of S is a prime Γ-ideal.**
2) **S is a primary Γ-semigroup.**
3) **S is a left primary Γ-semigroup.**
4) **S is a right primary Γ-semigroup.**
5) **S is a semiprimary Γ-semigroup.**
6) **Prime Γ-ideals of S form a chain.**
7) **Γ-ideals of S form a chain.**
8) **Principal Γ-ideals of S form a chain.**

If in addition S is duo, then the above statements are equivalent to

9) **α-idempotents of S, α ∈ Γ, form a chain under natural ordering.**

Proof**:** Since S is semisimple, $< x > = < x > \Gamma < x >$ for all $x \in$ S.
Let A be any Γ-ideal of S.
$x \in$ S, $< x > \Gamma < x > \subseteq A \Rightarrow < x > \subseteq A \Rightarrow x \in A$. Therefore A is semiprime.
Therefore every Γ-ideal of a semisimple Γ-semigroup is semiprime.
By theorem 4.2.5, (1) to (5) are equivalent.
By theorem 4.2.6, (5) and (6) are equivalent.
By corollary 4.2.8, (5) implies (8).
By theorem 2.1.43, (7) and (8) are equivalent.

(7) ⇒ (6) : Suppose that Γ-ideals of S form a chain. So prime Γ-ideals of S form a chain.
Hence the conditions (1) to (8) are equivalent.

If S is a duo Γ-semigroup, since S is semisimple, then by theorem 1.3.45, S regular.
Hence by theorem 1.3.30, every principal Γ-ideal is generated by an α-idempotent.
So (8) and (9) are equivalent.

THEOREM 4.2.10 : In a Γ-semigroup S, the following conditions are equivalent.

1) **Every Γ-ideal of S is prime.**
2) **S is a semisimple primary Γ-semigroup.**
3) **S is a semisimple semiprimary Γ-semigroup.**

Proof**:** (1) ⇒ (2): Suppose that every Γ-ideal of S is prime.
Let A is a Γ-ideal of S. Then A is a prime Γ-ideal of S.
By theorem 4.2.5, A is a primary Γ-ideal. Therefore S is a primary Γ-semigroup.

Let $a \in S$. Clearly $<a>\Gamma<a> \subseteq <a>$.

Since every Γ-ideal is prime , $<a>\Gamma<a> \subseteq <a>\Gamma<a> \Rightarrow <a> \subseteq <a>\Gamma<a>$ and hence $<a> = <a>\Gamma<a>$. So a is semisimple.

Therefore S is a semisimple primary Γ-semigroup.

(2) ⇒ (3) : Suppose that S is a semisimple primary Γ-semigroup.

Since every primary Γ-semigroup is a semiprimary Γ-semigroup, S is a semisimple semiprimary Γ-semigroup.

(3) ⇒ (1) : Suppose that S is a semisimple semiprimary Γ-semigroup.

By theorem 4.2.9, every Γ-ideal of S is prime.

COROLLARY 4.2.11 : In a Γ-semigroup S, the following conditions are equivalent.

1) **Every Γ-ideal of S is prime.**
2) **S is a semisimple Γ-semigroup.**
3) **Γ-ideals of S form a chain.**

Proof : (1) ⇒ (2) : Suppose that every Γ-ideal of S is prime.

By theorem 4.2.10, S is a semisimple primary Γ-semigroup.

(2) ⇒ (3) : Suppose that S is a semisimple Γ-semigroup.

By theorem 4.2.9, Γ-ideals of S form a chain.

(3) ⇒ (1) : Suppose that S is a semisimple Γ-semigroup and (prime) Γ-ideals of S form a chain. Then by theorem 4.2.9, every Γ-ideal of S is prime.

NOTE 4.2.12 : In an arbitrary Γ-semigroup S, Γ-idempotents need not form a chain under natural ordering, even if S is pseudo symmetric primary Γ-semigroup. For example in a left zero Γ-semigorup, Γ-idempotents do not form a chain under natural ordering.

NOTATION 4.2.13 : For any Γ-semigroup S, let E_s denotes the set of all Γ-idempotents of S together with the binary relation denoted by $e \leq f$ if and only if $e = e\alpha f = f\beta e$ for $e, f \in E_s$, $\alpha, \beta \in \Gamma$. i.e, $e \in e\Gamma f = f\Gamma e$.

THEOREM 4.2.14 : Let S is a duo semiprimary Γ-semigroup, then the Γ-idempotents of S form a chain under natural ordering.

Proof : Let e and f are two Γ-idempotents of S.

Since S is a semiprimary Γ-semigroup, $\sqrt{<e>}$ and $\sqrt{<f>}$ are prime Γ-ideals of S.

By theorem 4.2.6, either $\sqrt{<e>} \subseteq \sqrt{<f>}$ or $\sqrt{<f>} \subseteq \sqrt{<e>}$.

Assume that $\sqrt{<e>} \subseteq \sqrt{<f>}$. Since $e \in <e> \subseteq \sqrt{<e>} \subseteq \sqrt{<f>}$, we have $e \in \sqrt{<f>}$.

Since S is a duo Γ-semigroup, $\sqrt{<f>} = \{x \in S : (x\Gamma)^{n-1}x \subseteq <f> \text{ for some } n \in N\}$.

Thus, since e is a Γ-idempotent, we have $e \in <f>$.

Since S is a duo Γ-semigroup, $e \in <f> = f\Gamma S = S\Gamma f$.

Hence $e = f\alpha s = t\beta f$ for some $s, t \in S$, $\alpha, \beta \in \Gamma$.

So $e\alpha f = t\beta f\alpha f = t\beta f = e$ and $f\beta e = f\beta f\alpha s = f\alpha s = e$. Therefore $e \leq f$.

Similarly we can prove that if $\sqrt{<f>} \subseteq \sqrt{<e>}$ then $f \leq e$.

Hence Γ-idempotens of S form a chain under natural ordering.

THEOREM 4.2.15 : In a duo Γ-semigroup S, the following conditions are equivalent.

1) **Every Γ-ideal of S is prime.**
2) **S is a regular primary Γ-semigroup.**
3) **S is a regular semiprimary Γ-semiprimary.**
4) **S is a regular Γ-semigroup and Γ-idempotents in S form a chain under natural ordering.**

Proof : (1) ⇒ (2): Suppose that every Γ-ideal of S is prime.

By theorem 4.2.10, S is semisimple primary Γ-semigroup.

By theorem 1.3.45, S is regular. Therefore S is regular primary Γ-semigroup.

(2) ⇒ (3) : Suppose that S is a regular primary Γ-semigroup.

Since every primary Γ-semigroup is a semiprimary Γ-semigroup.

Therefore S is a regular semiprimary Γ-semigroup.

(3) ⇒ (4): Suppose that S is regular semiprimary Γ-semigroup.

By theorem 4.2.14, Γ-idempotents in S form a chain under natural ordering.

(4)⇒ (1): Suppose that S is a regular Γ-semigroup and Γ-idempotents in S form a chain under natural ordering. Then by theorem 4.2.9, every Γ-ideal of S is prime.

4.3. PRIMARY DECOMPOSITION

In this section, the terms; P-primary, primary decomposition of a **Γ**-ideal, reduced primary decomposition of a **Γ**-ideal in a **Γ**-semigroup S are introduced. If $A_1, A_2, \ldots, A_n$ are P-primary **Γ**-ideals in a **Γ**-semigroup S, then it is proved that $\bigcap_{i=1}^{n} A_i$ is also a P-primary **Γ**-ideal. If a **Γ**-ideal A in a **Γ**-semigroup S has a primary decomposition, then it is proved that A has a reduced primary decomposition. Further it is proved that every Γ-ideal in a (left, right) duo noetherian Γ-semigroup S has a reduced (right, left) primary decomposition. It is proved that (1) if A is a left primary **Γ**-ideal of a **Γ**-semigroup S, then $A^l(B)$ is a left primary **Γ**-ideal, (2) if A is a right primary **Γ**-ideal of a **Γ**-semigroup S, then $A^r(B)$ is a right primary **Γ**-ideal. It is proved that if Q is a P-primary **Γ**-ideal and if $A \not\subseteq P$, then $Q^l(A) = Q^r(A) = Q$ and also if $A \subseteq P$ and $A \not\subseteq Q$, then $\sqrt{(Q^l(A))} = \sqrt{(Q^r(A))} = \sqrt{Q}$. If $A_1, A_2, \ldots, A_n$, B are **Γ**-ideals of a **Γ**-semigroup S, then it is proved that $\left(\bigcap_{i=1}^{n} A_i\right)^l (B) = \bigcap_{i=1}^{n} (A_i)^l(B)$. Further if a **Γ**-ideal A in a **Γ**-semigroup S has two reduced (one sided) primary decompositions; $A = A_1 \cap A_2 \cap \ldots \cap A_k = B_1 \cap B_2 \cap \ldots B_s$, where A_i is P_i -primary and B_j is Q_j-primary, then it is proved that $k = s$ and after reindexing if necessary $P_i = Q_i$ for $i = 1, 2, \ldots, k$.

We now introduce the notions of P-primary Γ-ideal of a Γ-semigroup S.

DEFINITION 4.3.1 : Let P be any prime Γ-ideal in a Γ-semigroup S. A primary Γ-ideal A in S is said to be ***P-primary*** or P is a ***prime* Γ*-ideal belonging to A*** provided $\sqrt{A} = P$.

THEOREM 4.3.2 : If $A_1, A_2, \ldots, A_n$ are P-primary Γ-ideals in a Γ-semigroup S, then $\bigcap_{i=1}^{n} A_i$ is also a P-primary Γ-ideal.

Proof **:** Let $A = \bigcap_{i=1}^{n} A_i$. Now $\sqrt{A} = \sqrt{\cap A_i} = \cap\sqrt{A_i} = P$. So $\sqrt{A}$ is a prime Γ-ideal. Suppose $\langle a \rangle \Gamma \langle b \rangle \subseteq A$ and $b \notin A$. So $b \notin A_i$ for some i. Now Suppose $\langle a \rangle \Gamma \langle b \rangle \subseteq A_i$ and $b \notin A_i$. Since A_i is a P-primary Γ-ideal, we have $a \in \sqrt{A_i} = P = \sqrt{A}$. So A is a left primary Γ-ideal. Similarly we can show that A is a right primary Γ-ideal. Thus A is a P-primary Γ-ideal.

We now introduce the notions of left primary decomposition, right primary decomposition, primary decomposition, reduced primary decomposition of a Γ-ideal in a Γ-semigroup.

DEFINITION 4.3.3 : A Γ-ideal A in a Γ-semigroup S is said to have a (***left, right***) ***primary decomposition*** if $A = A_1 \cap A_2 \cap \ldots \cap A_n$ where each A_i is a (left, right) primary Γ-ideal. If no A_i contains $A_1 \cap A_2 \cap \ldots \cap A_{i-1} \cap A_{i+1} \cap \ldots \cap A_n$ and the Γ-radicals P_i of the Γ-ideals A_i are all distinct, then the primary decomposition is said to be ***reduced***. If P_i is minimal in the set $\{ P_1, P_2, \ldots, P_n \}$ then P_i is said to be ***isolated prime***.

THEOREM 4.3.4 : If a Γ-ideal A in a Γ-semigroup S has a primary decomposition, then A has a reduced primary decomposition.

Proof : If $A = A_1 \cap A_2 \cap \ldots \cap A_n$ where each A_i is primary and some A_i contains $A_1 \cap A_2 \cap \ldots \cap A_{i-1} \cap A_{i+1} \cap \ldots \cap A_n$, then $A = A_1 \cap A_2 \cap \ldots \cap A_{i-1} \cap A_{i+1} \cap \ldots \cap A_n$ is also a primary decomposition. By thus eliminating the superfluous A_i reindexing we have $A = A_1 \cap A_2 \cap \ldots \cap A_k$ with no A_i containing the intersection of other A_j. Let $P_1, P_2, \ldots, P_r$ be the distinct prime Γ-ideals in the set $\sqrt{A_1}, \sqrt{A_2}, \ldots, \sqrt{A_k}$. Let A_i^1, $1 \leq I \leq r$ be the intersection of all A_j's belonging to the prime P_i. By theorem 4.3.2, each A_i^1 is primary for P_i. Clearly no A_i^1contains the intersection of all other A_j^1. Therefore $A = \bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{r} A_i^1$ and hence A has a reduced primary decomposition.

NOTE 4.3.5 : In an arbitrary Γ-semigroup it is not necessarily true that every Γ-ideal has a primary decomposition even if the Γ-semigroup is finite.

EXAMPLE 4.3.6 : Let $S = \{a, b, c\}$ and $\Gamma = \{x, y, z\}$. Define a binary operation **.** in S as shown in the following table.

.	*a*	*b*	*c*
a	*a*	*a*	*a*
b	*a*	*a*	*a*
c	*a*	*b*	*c*

Define a mapping $S \times \Gamma \times S \to S$ by $a\alpha b = ab$, for all $a, b \in S$ and $\alpha \in \Gamma$.
It is easy to see that S is a Γ-semigroup. Now consider the Γ-ideal $< a > = S^1\Gamma a\Gamma S^1 = \{a\}$.
Let $p\Gamma q \subseteq < a >, p \notin < a > \Rightarrow q \in \sqrt{< a >} \Rightarrow (q\Gamma)^{n-1}q \subseteq < a >$ for some $n \in N$.
Since $b\Gamma c \subseteq < a >, c \notin < a > \Rightarrow b \in < a >$. Therefore $< a >$ is left primary.
If $b \notin < a >$ then $(c\Gamma)^{n-1}c \notin < a >$ for any $n \in N \Rightarrow c \notin \sqrt{< a >}$.
Therefore $< a >$ is not right primary. In the Γ-semigroup S, { *b*, *c* } and { a, b, *c* } are the only primary Γ-ideals and hence { *a* } has no primary decomposition.

We now introduce the notion of a noetherian Γ-semigroup.

DEFINITION 4.3.7 : A Γ-semigroup S is said to be a ***noetherian Γ-semigroup*** if ascending chain if Γ-ideals becomes stationary; i.e., if $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ is an ascending chain of Γ-ideals of S, then there exists a natural number *m* such that $A_m = A_n$ for all natural numbers $n \geq m$.

THEOREM 4.3.8 : Every Γ-ideal in a (left, right) duo noetherian Γ-semigroup S has a reduced (right, left) primary decomposition.

Proof : Let Σ be the collection of all Γ-ideals in S which has no primary decomposition. If Σ is not empty, then since S is noetherian, Σ contains maximal elements. Let C be a maximal element in Σ. Clearly C is not primary. Suppose that C is not left primary. Then there exists elements a, b in S such that $< a > \Gamma < b > \subseteq C, b \notin C$ and $a \notin \sqrt{C}$. Since S is a duo Γ-semigroup and hence by theorem 3.3.4, $\sqrt{C} = \{x \in S : (x\,\Gamma)^{n-1}x \subseteq C \text{ for some natural number } n\}$. Therefore $(a\Gamma)^{n-1}a \nsubseteq C$ and hence $(a\gamma)^{n-1}a \notin C$ for some $\gamma \in \Gamma$. For any natural number n, write $B_n = \{x \in S : (a\gamma)^n x \in C\}$. Let $x \in B_n$ and $s \in S$. $x \in B_n \Rightarrow (a\gamma)^n x \in C$. $(a\gamma)^n x \in C, s \in S \Rightarrow (a\gamma)^n x\gamma s \in C \Rightarrow x\gamma s \in B_n$. Therefore B_n is a right Γ-ideal in S. Since S is duo Γ-semigroup, B_n is a Γ-ideal in S. Now $B_1 \subseteq B_2 \subseteq \cdots$ is an ascending chain of Γ-ideals in S. Since S is noetherian there is a natural number k such that $B_k = B_i$ for all $i \geq k$. Since $b \in B_k$, we have B_k contains C properly. Write $D = (a\gamma)^k S \cup C$. Since S is a duo Γ-semigroup, D is a Γ-ideal in S and containing C properly. Now we prove that $C = B_k \cap D$. Clearly $C \subseteq B_k \cap D$. If $x \in B_k \cap D$ and $x \notin C$, then $x = (a\gamma)^k y$ for some $y \in S$. Sincex $x \in B_k$, we have $(a\gamma)^k x \in C$. Therefore $(a\gamma)^{2k} y = (a\gamma)^k (a\gamma)^k y = (a\gamma)^k x \in C$. Therefore $(a\gamma)^{2k} y \in C$. So $y \in B_{2k} = B_k$. Thus $x = (a\gamma)^k y \in C \Rightarrow x \in C$. It is a contradiction. So $B_k \cap D \subseteq C$ and hence $C = B_k \cap D$. Since B_k and D contains C properly and C is maximal in Σ, B_k and D have primary decompositions and hence C has a primary decomposition. It is a contradiction. Thus C is left primary. Similarily we can prove that C is right primary. Hence C is primary. It is a condiction. Therefore Σ is empty. Thus every Γ-ideal in a duo noetherian Γ-semigroup has a primary decomposition and hence by theorem 4.3.4, every Γ-ideal has a reduced primary decomposition.

COROLLARY 4.3.9 : Every Γ-ideal in a commutative noetherian Γ-semigroup S has a reduced primary decomposition.

THEOREM 4.3.10 : If A is a left primary Γ-ideal of a Γ-semigroup S, then A^l (B) is a left primary Γ-ideal.

Proof : If B ⊆ A, then clearly A^l (B) = S. Suppose B ⊈ A. Let b ∈ B\A. Let x ∈ A^l (B).

Then $< x > \Gamma B \subseteq A$. So $< x > \Gamma < b > \subseteq A$. Since $b \notin A$, We have $x \in \sqrt{A}$ and hence $\sqrt{(A^l(B))} = \sqrt{A}$. Let $< x >\Gamma< y > \subseteq A^l(B)$ and $y \notin A^l(B)$. Now $< x >\Gamma< y > \Gamma B \subseteq A$. If $x \notin \sqrt{(A^l(B))} = \sqrt{A}$, then $< y > \Gamma B \subseteq A$ and hence $y \in A^l(B)$. It is a contradiction. So $x \in \sqrt{(A^l(B))}$. Therefore $A^l(B)$ is a left primary Γ-ideal.

THEOREM 4.3.11 : If A is a right primary Γ-ideal of a Γ-semigroup S, then $A^r(B)$ is a right primary Γ-ideal.

Proof : If $B \subseteq A$, then clearly $A^r(B) = S$. Suppose $B \not\subseteq A$. Let $b \in B \setminus A$. Let $x \in A^r(B)$. Then $B\Gamma< x > \subseteq A$. So $< x > \Gamma < b > \subseteq A$. Since $b \notin A$, We have $x \in \sqrt{A}$ and hence $\sqrt{(A^r(B))} = \sqrt{A}$. Let $< x >\Gamma< y > \subseteq A^r(B)$ and $x \notin A^r(B)$. Now $< y >\Gamma< x > \Gamma B \subseteq A$. If $y \notin \sqrt{(A^r(B))} = \sqrt{A}$, then $< x > \Gamma B \subseteq A$ and hence $x \in A^r(B)$, a contradiction. So $y \in \sqrt{(A^r(B))}$. Therefore $A^r(B)$ is a right primary Γ-ideal.

THEOREM 4.3.12 : If Q is a P-primary Γ-ideal and if $A \not\subseteq P$, then $Q^l(A) = Q^r(A) = Q$ and also if $A \subseteq P$ and $A \not\subseteq Q$, then $\sqrt{(Q^l(A))} = \sqrt{(Q^r(A))} = \sqrt{Q}$.

Proof : Clearly $Q \subseteq Q^l(A)$. Let $x \in Q^l(A)$. Then $< x >\Gamma A \subseteq Q$. Since $A \not\subseteq P$, there exists $a \in A \setminus P$. Now $< x >\Gamma A \subseteq Q$ and $a \notin \sqrt{Q}$. So $x \in Q$. Therefore $Q^l(A) = Q$. Similarly we can show that $Q^r(A) = Q$. The proof of the second part is evident.

THEOREM 4.3.13 : If $A_1, A_2, \ldots\ldots, A_n$ B are Γ-ideals of a Γ-semigroup S, then $\left(\bigcap_{i=1}^{n} A_i\right)^l(B) = \bigcap_{i=1}^{n}(A_i)^l(B)$.

Proof : $x \in \left(\bigcap_{i=1}^{n} A_i\right)^l(B) \Leftrightarrow < x > \Gamma B \subseteq \cap A_i \Leftrightarrow < x > \Gamma B \subseteq A_i$ for $i = 1, 2, 3\ldots\ldots n$. $\Leftrightarrow x \in A_i^l(B)$ for $i = 1, 2, 3\ldots\ldots n \Leftrightarrow x \in \bigcap_{i=1}^{n} A_i(B)$. Similarly we can show that if $x \in \cap A_i^l(B)$. Then $x \in (\cap A_i)^l(B)$. Therefore $\left(\bigcap_{i=1}^{n} A_i\right)^l(B) = \bigcap_{i=1}^{n}(A_i)^l(B)$.

THEOREM 4.3.14 : Suppose a Γ-ideal A in a Γ-semigroup S has two reduced (one sided) primary decompositions $A = A_1 \cap A_2 \cap \ldots\ldots \cap A_k = B_1 \cap B_2 \cap \ldots\ldots B_s$, where A_i is P_i-primary and B_j is Q_j-primary. Then k = s and after reindexing if necessary $P_i = Q_i$ for $i = 1, 2, \ldots, k$. Further if each P_i is an isolated prime, then $A_i = B_i$ for $i = 1, 2, \ldots, n$.

Proof : Let P_k be the maximal lement in the set $P_1, P_2, \ldots, P_k, Q_1, Q_2, \ldots\ldots, Q_s$. Now we show that P_k occurs among $Q_1, Q_2, \ldots\ldots, Q_s$.

For this it is enough to show that $P_k \subseteq Q_j$ for some j. If $A_k \subseteq Q_j$ for some j, $P_k = \sqrt{A_k} \subseteq Q_j$.

Suppose $A_k \nsubseteq Q_j$ for all j. Then by theorem 3.3.10, $B_j^{l}(A_k) = B_j$ for all j.

Now $A^{l}(A_k) = (B_1 \cap B_2 \cap \ldots \cap B_s)^{l}(A_k)$

$= B_1^{l}(A_k) \cap B_2^{l}(A_k) \cap \ldots \cap B_s^{l}(A_k)$, by using theorem 3.3.10,

$= B_1 \cap B_2 \cap \ldots \cap B_s = A$.

But on the other hand if $1 \leq i < k$, then $P_k \nsubseteq P_i$ and therefore $A_k \nsubseteq P_i$, so that $A_i^{l}(A_k) = A_i$ and $A_k^{l}(A_k) = S$.

So we have $A^{l}(A_k) = (A_1 \cap A_2 \cap \ldots \cap A_k)^{l}(A_k) = A_1^{l}(A_k) \cap A_2^{l}(A_k) \cap \ldots \cap A_k^{l}(A_k)$ $= A_1 \cap A_2 \cap \ldots \cap A_{k-1}$. Therefore $A = A_1 \cap A_2 \cap \ldots \cap A_{k-1}$.

It is a contradiction to the fact that given decomposition is reduced.

Thus $A_k \subseteq Q_j$ for some j and hence $P_k \subseteq Q_j$. Therefore $P_k = Q_j$.

Without loss of generality we may assume that $P_k = Q_s$.

Let $B = A_k \cap B_s$. By theorem 4.3.2, B is a primary Γ-ideal and $P_k = Q_s (= P$ say) is a prime Γ-ideal belonging to B. Since $P \nsubseteq P_i$ for all i, $1 \leq i < k$ and $B \subseteq A_k$, we have $A_i^{l}(B) = A_i$ and $A_k^{l}(B) = S$. Therefore $A^{l}(B) = A_1 \cap A_2 \cap \ldots \cap A_{k-1}$.

Similarly we can show that $A^{l}(B) = B_1 \cap B_2 \cap \ldots \cap B_{s-1}$.

Hence $A^{l}(B) = A_1 \cap A_2 \cap \ldots \cap A_{k-1} = B_1 \cap B_2 \cap \ldots \cap B_{s-1}$ are two reduced primary decompositions for $A^{l}(B)$.

By continuing the above process, we get $k = s$ and $P_i = Q_i$ for $i = 1, 2, \ldots, k$.

Suppose P_i's are isolated primes.

If $A_1 \nsubseteq B_1$ then since B_1 is primary and $A_1 \cap A_2 \cap \ldots \cap A_k \subseteq B_1 \cap B_2 \cap \ldots \cap B_k \subseteq B_1$, we have $A_2 \cap A_3 \cap \ldots \cap A_k \subseteq \sqrt{B_1} = P_i$.

Now $P_2 \cap P_3 \cap \ldots \cap P_k = \sqrt{A_1 \cap A_2 \cap \ldots \cap A_k} = P_1$.

Since P_1 is a prime Γ-ideal, $P_i \subseteq P_1$ for some $1 < i \leq k$.

It is a contradiction to the fact that P_1 is an isolated prime.

So $A_1 \subseteq B_1$. Similarly we can show that $B_1 \subseteq A_1$. Therefore $A_1 = B_1$.

By continuing in this way we get $A_i = B_i$ for some $i = 1, 2, \ldots, k$.

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