CHAPTER 3

IDEALS IN DUO Γ-SEMIGROUPS

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KRULL [29] proved that the nil-radical of an ideal A in a commutative ring is equal to the intersection of all minimal prime ideals containing A. SATYANARAYANA [43] obtained KRULL's theorem [29] for commutative semigroups. ANJANEYULU [4] introduced the notions of ideals in duo semigroups and exhibit some examples and some classes of duo semigroups. He obtained KRULL's theorem [29] for pseudo symmetric semigroups which includes duo semigroups. MADHUSUDHANA RAO, ANJANEYULU and GANGADHARA RAO[32], [33], [34] and [35] introduced the notions of duo Γ-semigroups and obtained KRULL's theorem for pseudo and semipseudo symmetric Γ-semigroups. In this thesis we introduce and made a study on ideals in duo Γ-semigroups and obtained an analogue of KRULL's theorem [29] in duo Γ-semigroups

This chapter is divided into 5 sections. In section 1, the terms; left duo Γ-semigroup, right duo Γ-semigroup, duo Γ-semigroup are introduced. It is proved that a Γ- semigroup S is a duo Γ- semigroup if and only if $x\Gamma S^1 = S^1\Gamma x$ for all $x \in S$. Further it is proved that (1) every commutative Γ- semigroup is a duo Γ- semigroup (2) every normal Γ-semigroup is a duo Γ- semigroup (3) every quasi commutative Γ- semigroup is a duo Γ-semigroup (4) every generalized Γ- semigroup is a left duo Γ- semigroup.

In section 2, it is proved that (1) if A is a Γ-ideal in a left duo Γ-semigroup S, then $A_l(a) = \{ x \in S : x\Gamma a \subseteq A \}$ is a Γ-ideal of S for all $a \in S$, (2) if A is a Γ-ideal in a right duo Γ-semigroup S, then $A_r(a) = \{ x \in S : a\Gamma x \subseteq A \}$ is a Γ-ideal of S for all $a \in S$, (3) if A is a Γ-ideal in a duo Γ-semigroup S, then $A_l(a) = \{ x \in S : x\Gamma a \subseteq A \}$ and $A_r(a) = \{ x \in S : a\Gamma x \subseteq A \}$ are Γ-ideals of S for all $a \in S$. Further it is proved that (1) if A is a Γ-ideal in a left duo Γ-semigroup S and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$ for all $s \in S$, (2) if A is a Γ-ideal in a right duo Γ-semigroup S and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$ for all $s \in S$, (3) if A is a Γ-ideal in a duo Γ-semigroup S and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$. It is proved that if A is a Γ-ideal in a duo Γ-semigroup S and $a, b \in S$, then (1) $a\Gamma b \in A$ iff $< a > \Gamma <b> \subseteq A$, (2) $a_1\Gamma a_2\Gamma ..... a_{n-1}\Gamma a_n \subseteq A$ iff $< a_1 > \Gamma < a_2 > ..... \Gamma < a_n > \subseteq A$, (3) for any natural number n, $(a\Gamma)^{n-1}a \subseteq A$ iff $(< a > \Gamma)^{n-1} < a > \subseteq A$. It is also proved that in a duo Γ-semigroup S, a Γ-ideal P is prime Γ-ideal if and only if P is a completely prime Γ-ideal. Further it is proved that a

Γ-ideal A of a duo Γ-semigroup S is a completely semiprime Γ-ideal of S if and only if A is a semiprime Γ-ideal.

In section 3, it is proved that, if A_1 = the intersection of all completely prime Γ-ideals of S containing A, $A_2 = \{x \in S : (x\Gamma)^{n-1}x \subseteq A$ for some natural number $n\}$, A_3 = the intersection of all prime Γ-ideals of S containing A, $A_4 = \{x \in S : (<x>\Gamma)^{n-1}<x> \subseteq A$ for some natural number $n\}$ for a **Γ**- ideal A of a **Γ**-semigroup S, then $A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$. If A is a Γ- ideal of a commutative/duo Γ-semigroup then it is proved that $A_1 = A_2 = A_3 = A_4$. It is proved that if A is a Γ-ideal in a duo Γ-semigroup S, then (1) A_2 is the minimal completely semiprime Γ-ideal of S containing A, (2) A_4 is the minimal semiprime Γ-ideal of S containing A. It is proved that if $a \in \sqrt{A}$, then there exist a positive integer n such that $(a\Gamma)^{n-1}a \subseteq A$. Further if A is a **Γ**-ideal of a duo Γ-semigroup S then it is proved that (1) A_1 = the intersection of all completely prime Γ-ideals of S containing A, (2) A_1' = the intersection of all minimal completely prime Γ-ideals of S containing A, (3) A_1'' = the minimal completely semiprime Γ-ideal of S containing A, (4) $A_2 = \{x \in S : (x\Gamma)^{n-1}x \subseteq A$ for some natural number $n\}$, (5) A_3 = the intersection of all prime Γ-ideals of S containing A, (6) A_3' = the intersection of all minimal prime Γ-ideals of S containing A, (7) A_3'' = the minimal semiprime Γ-ideal of S containing A, (8) $A_4 = \{x \in S : (<x>\Gamma)^{n-1}<x> \subseteq A$ for some natural number $n\}$ are equal.

In section 4, the terms; Archimedean Γ-semigroup and strongly Archimedean Γ-semigroup are introduced. It is proved that if S is a duo **Γ**-semigroup, then the conditions (1) S is strongly Archimedean, (2) S is Archimedean, (3) S has no proper completely prime **Γ**-ideals and (4) S has no proper prime **Γ**-ideals; are equivalent.

In section 5, the terms; left simple Γ-semigroup, right simple Γ-semigroup, simple Γ-semigroup are introduced. It is proved that (1) a **Γ**-semigroup S is a left simple **Γ**-semigroup if and only if $S\Gamma a = S$ for all $a \in S$, (2) a **Γ**-semigroup S is a right simple **Γ**-semigroup if and only if $a\Gamma S = S$ for all $a \in S$, (3) a **Γ**-semigroup S is a simple **Γ**-semigroup if and only if $S\Gamma a\Gamma S = S$ for all $a \in S$. It is also proved that if S is a left simple **Γ**-semigroup or a right simple **Γ**-semigroup then S is a simple **Γ**-semigroup. Further it is proved that if S is a duo **Γ**-semigroup and $a \in S$ then (1) a is regular, (2) a is left regular, (3) a is right regular, (4) a is intra regular and (5) a is semisimple are equivalent.

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3.1. DUO Γ-SEMIGROUPS

Duo Γ-semigroups played an important role in the theory of Γ-semigroups. In this section the terms; left duo Γ-semigroup, right duo Γ-semigroup, duo Γ-semigroup are introduced. It is proved that a Γ- semigroup S is a duo Γ- semigroup if and only if $x\Gamma S^1 = S^1\Gamma x$ for all $x \in S$. Further it is proved that (1) every commutative Γ- semigroup is a duo Γ- semigroup (2) every normal Γ-semigroup is a duo Γ- semigroup (3) every quasi commutative Γ- semigroup is a duo Γ-semigroup (4) every generalized Γ- semigroup is a left duo Γ- semigroup.

We now introduce a left duo Γ-semigroup, right duo Γ-semigroup and duo Γ-semigroup.

DEFINITION 3.1.1 : A Γ- semigroup S is said to be a ***left duo* Γ- *semigroup*** provided every left Γ- ideal of S is a two sided Γ- ideal of S.

DEFINITION 3.1.2 : A Γ- semigroup S is said to be a ***right duo* Γ- *semigroup*** provided every right Γ-ideal of S is a two sided Γ- ideal of S.

DEFINITION 3.1.3 : A Γ- semigroup S is said to be a ***duo* Γ- *semigroup*** provided it is both a left duo Γ- semigroup and a right duo Γ- semigroup.

THEOREM 3.1.4 : A Γ-semigroup S is a duo Γ- semigroup if and only if $x\Gamma S^1 = S^1\Gamma x$ for all $x \in S$.

Proof : Suppose that S is a duo Γ-Semigroup and $x \in S$.
Let $t \in x\Gamma S^1$. Then $t=x\gamma s$ for some $s \in S^1$, $\gamma \in \Gamma$.
Since $S^1\Gamma x$ is a left Γ-ideal of S, $S^1\Gamma x$ is a Γ-ideal of S.
So $x \in S^1\Gamma x$, $\gamma \in \Gamma$, $s \in S$, $S^1\Gamma x$ is a Γ-ideal $\Rightarrow x\gamma s \in S^1\Gamma x \Rightarrow t \in S^1\Gamma x$.
Therefore $x\Gamma S^1 \subseteq S^1\Gamma x$. Similarly we can prove that $S^1\Gamma x \subseteq x\Gamma S^1$ .Therefore $S^1\Gamma x = x\Gamma S^1$.

Conversely suppose that $S^1\Gamma x = x\Gamma S^1$ for all $x \in S$. Let A be a left Γ-ideal of S.
Let $x \in A$, $s \in S$ and $\alpha \in \Gamma$. Then $x\alpha s \in x\Gamma S^1 = S^1\Gamma x \Rightarrow x\alpha s = t\beta x$ for some $t \in S^1$, $\beta \in \Gamma$.

$x \in A,\ t \in S,\ \beta \in \Gamma$, A is a left Γ-ideal of S $\Rightarrow t\beta x \in A \Rightarrow x\alpha s \in A$.

Therefore A is a right Γ-ideal of S and hence A is a Γ-ideal of S.

Therefore S is left duo Γ-semigroup.

Similarly we can prove that S is a right duo Γ-semigroup. Hence S is duo Γ-semigroup.

THEOREM 3.1.5 : Every commutative Γ-semigroup is a duo Γ-semigroup.

Proof : Suppose that S is a commutative Γ-semigroup. Therefore $x\Gamma S^1 = S^1\Gamma x$ for all $x \in S$. By theorem 3.1.4, S is a duo Γ-semigroup.

THEOREM 3.1.6 : Every normal Γ- semigroup is a duo Γ- semigroup.

Proof : Suppose that S is normal Γ-semigroup.

Then $a\Gamma S = S\Gamma a$ for all $a \in S \Rightarrow a\Gamma S^1 = S^1\Gamma a$ for all $a \in S$.

By theorem 3.1.4, S is a duo Γ-semigroup.

THEOREM 3.1.7 : Every quasi commutative Γ-semigroup is a duo Γ-semigroup.

Proof : Suppose that S is a quasi commutative Γ-semigroup. Then for $a, b \in S$, there exists $n \in N$ such that $a\gamma b = (b\gamma)^n a$ for all $\gamma \in \Gamma$. Let A be a left Γ-ideal of S. Therefore $S\Gamma A \subseteq A$. Let $a \in A$ and $s \in S$. Since S is a quasi commutative Γ-semigroup, there exists a natural number n such that $a\Gamma s = (s\Gamma)^n a \subseteq S\Gamma A \subseteq A$. Therefore $a\Gamma s \subseteq A$. for all $a \in A$ and $s \in S$ and hence $A\Gamma S \subseteq A$. Thus A is right Γ-ideal of S. Therefore S is a left duo Γ-semigroup. Similarily we can prove that S is a right duo Γ-semigroup. Therefore every quasi commutative Γ-semigroup is a duo Γ-semigroup.

THEOREM 3.1.8 : Every generalized commutative Γ-semigroup is a left duo Γ-semigroup.

Proof : Let S be a generalized commutative Γ-semigroup. Therefore 1 is an r-element.

Let A be a left Γ-ideal of S. Let $x \in A$ and $s \in S$.

Now $x\Gamma s = 1\Gamma x\Gamma s = b\Gamma s\Gamma x = (b\Gamma s)\Gamma x \subseteq A$. Therefore A is a Γ-ideal of S.

Therefore S is a left duo Γ-semigroup.

3.2. <u>Γ-IDEALS IN DUO Γ-SEMIGROUPS</u>

In this section, it is proved that (1) if A is a Γ-ideal in a left duo Γ-semigroup S, then $A_l(a) = \{ x \in S : x\Gamma a \subseteq A \}$ is a Γ-ideal of S for all $a \in S$, (2) if A is a Γ-ideal in a right duo Γ-semigroup S, then $A_r(a) = \{ x \in S : a\Gamma x \subseteq A \}$ is a **Γ**-ideal of S for all $a \in S$, (3) if A is a Γ-ideal in a duo Γ-semigroup S, then $A_l(a) = \{ x \in S : x\Gamma a \subseteq A \}$ and $A_r(a) = \{ x \in S : a\Gamma x \subseteq A \}$ are **Γ**-ideals of S for all $a \in S$. Further it is proved that (1) if A is a Γ-ideal in a left duo **Γ**-semigroup S and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$ for all $s \in S$, (2) if A is a Γ-ideal in a right duo **Γ**-semigroup S and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$ for all $s \in S$, (3) if A is a Γ-ideal in a duo **Γ**-semigroup S and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$. It is proved that if A is a Γ-ideal in a duo **Γ**-semigroup S and $a, b \in S$, then (1) $a\Gamma b \in A$ iff $< a > \Gamma <b> \subseteq A$, (2) $a_1\Gamma a_2\Gamma \ldots\ldots a_{n-1}\Gamma a_n \subseteq A$ iff $< a_1 > \Gamma < a_2 > \ldots\ldots \Gamma < a_n > \subseteq A$, (3) for any natural number n, $(a\Gamma)^{n-1}a \subseteq A$ iff $(< a > \Gamma)^{n-1} < a > \subseteq A$. It is also proved that in a duo Γ-semigroup S, a Γ-ideal P is prime Γ-ideal if and only if P is a completely prime Γ-ideal. Further it is proved that a Γ-ideal A of a duo Γ-semigroup S is a completely semiprime Γ-ideal of S if and only if A is a semiprime Γ-ideal.

We now characterize left duo Γ-semigroups.

THEOREM 3.2.1 : If A is a Γ-ideal in a left duo Γ-semigroup S, then $A_l(a) = \{ x \in S : x\Gamma a \subseteq A \}$ is a Γ-ideal of S for all $a \in S$.

Proof: Let $x \in A_l(a)$ and $s \in S$. $x \in A_l(a) \Rightarrow x\Gamma a \subseteq A$.

$x\Gamma a \subseteq A$, $s \in S$, A is a Γ-ideal $\Rightarrow s\Gamma x\Gamma a \subseteq A \Rightarrow s\Gamma x \subseteq A_l(a)$.

Therefore $A_l(a)$ is a left Γ-ideal of S. Since S is a left duo Γ-semigroup, $A_l(a)$ is a Γ-ideal of S.

THEOREM 3.2.2 : If A is a Γ-ideal in a left duo Γ-semigroup S and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$ for all $s \in S$.

Proof: Suppose that $x\Gamma y \subseteq A$. Let $s \in S$.

$x\Gamma y \subseteq A \Rightarrow x \in A_l(y)$.

$x \in A_l(y)$, $s \in S$, $A_l(y)$ is a Γ-ideal of S $\Rightarrow x\Gamma s \subseteq A_l(y) \Rightarrow x\Gamma s\Gamma y \subseteq A$.

We now characterize right duo Γ-semigroups.

THEOREM 3.2.3: If A is a Γ-ideal in a right duo Γ-semigroup S, then $A_r(a) = \{ x \in S : a\Gamma x \subseteq A \}$ is a Γ-ideal of S for all $a \in S$.

Proof: Let $x \in A_r(a)$ and $s \in S$. $x \in A_r(a) \Rightarrow a\Gamma x \subseteq A$.

$a\Gamma x \subseteq A$, $s \in S$, A is a Γ-ideal $\Rightarrow a\Gamma x\Gamma s \subseteq A \Rightarrow x\Gamma s \subseteq A_r(a)$.

Therefore $A_r(a)$ is a right Γ-ideal of S.

Since S is a right duo Γ-semigroup, $A_r(a)$ is a Γ-ideal of S.

THEOREM 3.2.4 : If A is a Γ-ideal in a right duo Γ-semigroup S and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$.

Proof: Suppose that $x\Gamma y \subseteq A$. Let $s \in S$. $x\Gamma y \subseteq A \Rightarrow y \in A_r(x)$.

$y \in A_r(x)$, $s \in S$, $A_r(x)$ is a Γ-ideal of S $\Rightarrow s\Gamma y \subseteq A_r(x) \Rightarrow x\Gamma s\Gamma y \subseteq A$.

We now characterize duo Γ-semigroups.

COROLLARY 3.2.5 : If A is a Γ-ideal in a duo Γ-semigroup S and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$.

THEOREM 3.2.6 : If A is a Γ-ideal in a duo Γ-semigroup S, then $A_l(a) = \{ x \in S : x\Gamma a \subseteq A \}$ and $A_r(a) = \{ x \in S : a\Gamma x \subseteq A \}$ are Γ-ideals of S for all $a \in S$.

Proof : Since S is a duo Γ-semigroup, S is left duo Γ-semigroup and hence by theorem 3.2.1, $A_l(a) = \{ x \in S : x\Gamma a \subseteq A \}$ is a Γ-ideal of S. Again S is right duo Γ-semigroup and hence by theorem 3.2.3, $A_r(a) = \{ x \in S : a\Gamma x \subseteq A \}$ is a Γ-ideal of S.

THEOREM 3.2.7 : Let A be a Γ-ideal in a duo Γ-semigroup S and $a, b \in S$. Then $a\Gamma b \in A$ if and only if $< a > \Gamma < b > \subseteq A$.

Proof: Suppose that $< a > \Gamma < b > \subseteq A$. Then $a\Gamma b \subseteq < a > \Gamma < b > \subseteq A$.

Conversely suppose that $a\Gamma b \subseteq A$. Since S is a duo Γ-semigroup. By corollary 3.2.5, $a\Gamma b \subseteq A \Rightarrow a\Gamma s\Gamma b \subseteq A$ for all $s \in S \Rightarrow a\Gamma S^1\Gamma b \subseteq A$. Since A is a Γ-ideal, $a\Gamma S^1\Gamma b \subseteq A$ $\Rightarrow S^1\Gamma a\Gamma S^1\Gamma b\Gamma S^1 \subseteq A \Rightarrow < a > \Gamma <b> \subseteq A$.

THEOREM 3.2.8 : Let A be a Γ-ideal in a duo Γ-semigroup S. Then $a_1\Gamma a_2\Gamma \ldots a_{n-1}\Gamma a_n \subseteq A$ if and only if $< a_1 > \Gamma < a_2 > \ldots \Gamma < a_n > \subseteq A$.

Proof: Suppose that $< a_1 > \Gamma < a_2 > \ldots \Gamma < a_n > \subseteq A$.

Then $a_1 \Gamma a_2 \Gamma \ldots a_{n-1} \Gamma a_n \subseteq < a_1 > \Gamma < a_2 > \ldots \Gamma < a_n > \subseteq A$.

Conversely suppose that $a_1 \Gamma a_2 \Gamma \ldots a_{n-1} \Gamma a_n \subseteq A$.

Then for any $t \in < a_1 > \Gamma < a_2 > \ldots \Gamma < a_n >$, we have

$t = s_1\alpha_1 a_1\beta_1 s_2\alpha_2 a_2\beta_2 \ldots \alpha_n a_n\beta_n s_{n+1}$, where $s_i \in S$ and $\alpha_i, \beta_i \in \Gamma$.

Since $x, y \in S$, $x\Gamma y \subseteq A \Rightarrow x\Gamma s\Gamma y \subseteq A$, we have $t \in A$.

Therefore $< a_1 > \Gamma < a_2 > \ldots\ldots \Gamma < a_n > \subseteq$ A.

COROLLARY 3.2.9 : Let A be a Γ-ideal in a duo Γ-semigroup S. Then for any natural number n, $(a\,\Gamma)^{n-1}a \subseteq A$ if and only if $(< a > \Gamma)^{n-1} < a > \subseteq A$.

Proof : The proof follows from theorem 3.2.8, by taking $a_1 = a_2 = a_3 = \ldots = a_n = a$.

THEOREM 3.2.10 : Let S be a duo Γ-semigroup. A Γ-ideal P of S is prime Γ-ideal if and only if P is a completely prime Γ-ideal.

Proof : Suppose that P is a prime Γ-ideal of Γ-semigroup S. Let $x, y \in S$ and $x\Gamma y \subseteq P$.
Now $x\Gamma y \subseteq P$, P is a Γ-ideal $\Rightarrow x\Gamma y\Gamma S^1 \subseteq P$.
Since S is duo Γ-semigroup, $x\Gamma S^1\Gamma y = x\Gamma y\,\Gamma\, S^1 \subseteq P$.
By corollary 2.2.7, either $x \in P$ or $y \in P$. Hence P is a completely prime Γ-ideal.

Conversely suppose that P is a completely prime Γ-ideal of S.
By theorem 2.2.8, P is a prime Γ-ideal of S.

COROLLARY 3.2.11 : Let S be a commutative Γ-semigroup. A Γ-ideal P of S is prime Γ-ideal if and only if P is a completely prime Γ-ideal.

THEOREM 3.2.12 : Let S be a duo Γ-semigroup. A Γ-ideal A of S is completely semiprime iff semiprime.

Proof : Suppose that A is a completely semiprime Γ-ideal of S.
By theorem 2.3.7, A is a semiprime Γ-ideal of S.

Conversely Suppose that A is a semiprime Γ-ideal of S. Let $x \in S$ and $x\Gamma x \subseteq A$.
Now $x\Gamma x \subseteq A \Rightarrow s\Gamma x\Gamma x \subseteq A$ for all $s \in S \Rightarrow x\Gamma s\Gamma x \subseteq A$ for all $s \in S \Rightarrow x\Gamma S\Gamma x \subseteq A$
$\Rightarrow x \in A$, since A is semiprime. Therefore A is a completely semiprime Γ-ideal of S.

COROLLARY 3.2.13 : Let S be a commutative Γ-semigroup. A Γ-ideal A of S is completely semiprime iff semiprime.

3.3. Γ-RADICALS IN DUO Γ-SEMIGROUPS

In this section, it is proved that, if A_1 = the intersection of all completely prime Γ-ideals of S containing A, $A_2 = \{x \in S : (x\Gamma)^{n-1}x \subseteq A$ for some natural number $n\}$, A_3 = the intersection of all prime Γ-ideals of S containing A, $A_4 = \{x \in S : (< x > \Gamma)^{n-1} < x > \subseteq A$ for some natural number $n\}$ for a Γ- ideal A of a Γ-semigroup S, then

$A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$. If A is a Γ- ideal of a commutative/duo Γ-semigroup then it is proved that $A_1 = A_2 = A_3 = A_4$. It is proved that if A is a Γ-ideal in a duo Γ-semigroup S, then (1) A_2 is the minimal completely semiprime Γ-ideal of S containing A, (2) A_4 is the minimal semiprime Γ-ideal of S containing A. It is proved that if $a \in \sqrt{A}$, then there exist a positive integer n such that $(a\Gamma)^{n-1}a \subseteq A$. Further if A is a Γ-ideal of a duo Γ-semigroup S then it is proved that (1) A_1 = the intersection of all completely prime Γ-ideals of S containing A, (2) A_1' = the intersection of all minimal completely prime Γ-ideals of S containing A, (3) A_1'' = the minimal completely semiprime Γ-ideal of S containing A, (4) $A_2 = \{x \in S : (x\Gamma)^{n-1}x \subseteq A$ for some natural number $n\}$, (5) A_3 = the intersection of all prime Γ-ideals of S containing A, (6) A_3' = the intersection of all minimal prime Γ-ideals of S containing A, (7) A_3'' = the minimal semiprime Γ-ideal of S containing A, (8) $A_4 = \{x \in S : (<x>\Gamma)^{n-1}<x> \subseteq A$ for some natural number $n\}$ are equal.

NOTATION 3.3.1 : If A is a Γ-ideal of a Γ-semigroup S, then we associate the following four types of sets.

A_1 = The intersection of all completely prime Γ-ideals of S containing A.

$A_2 = \{x \in S : (x\Gamma)^{n-1}x \subseteq A$ for some natural number $n\}$

A_3 = The intersection of all prime ideals of S containing A.

$A_4 = \{x \in S : (<x>\Gamma)^{n-1}<x> \subseteq A$ for some natural number $n\}$

NOTE 3.3.2 : If A is a Γ-ideal of a Γ-semigroup S then ***rad*** **A** = A_3 and ***c.rad*** **A** = A_4.

THEOREM 3.3.3 : If A is a Γ- ideal of a Γ-semigroup S, then $A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$.

Proof : (i) $A \subseteq A_4$: Let $x \in A$. Then $(<x>\Gamma)^0<x> \subseteq A$ and hence $x \in A_4$. $\therefore A \subseteq A_4$.

(ii) $A_4 \subseteq A_3$: Let $x \in A_4$. Then $(<x>\Gamma)^{n-1}<x> \subseteq A$ for some $n \in N$.

Let P be any prime Γ-ideal of S containing A.

Then $(<x>\Gamma)^{n-1}<x> \subseteq A \Rightarrow (<x>\Gamma)^{n-1}<x> \subseteq P$.

Since P is prime, $<x> \subseteq P$ and hence $x \in P$.

Since this is true for all prime Γ-ideals P containing A, $x \in A_3$. Therefore $A_4 \subseteq A_3$.

(iii) $A_3 \subseteq A_2$: Let $x \in A_3$. Suppose if possible $x \notin A_2$. Then $(x\Gamma)^{n-1}x \nsubseteq A$ for all $n \in N$.

Consider $T = \cup(x\Gamma)^{n-1}x$, where $x \in S$ and n is a natural number.

Let $a, b \in T$. Then $a \in (x\Gamma)^{r-1}x$, $b \in (x\Gamma)^{s-1}x$ for some $r, s \in N$.

Therefore $a\Gamma b = (x\Gamma)^{r-1}x\Gamma(x\Gamma)^{s-1}x = (x\Gamma)^{r+s-1}x \subseteq T$.

Therefore T is a Γ-subsemigroup of S and T is a c-system of S and $x \in T$.

By theorem 2.2.4, $P = S\setminus T$ is a completely prime Γ-ideal of S and $x \notin P$.

By theorem 2.2.8, P is prime Γ-ideal of S and $x \notin P$.

Therefore $x \notin A_3$. It is a contradiction. $\therefore x \in A_2$ and hence $A_3 \subseteq A_2$.

(iv) $\mathbf{A_2 \subseteq A_1}$ **:** Let $x \in A_2$. Now $x \in A_2 \Rightarrow (x\Gamma)^{n-1}x \subseteq A$ for some natural number n.

Let P be any completely prime Γ-ideal of S containing A.

Then $(x\Gamma)^{n-1}x \subseteq A \subseteq P \Rightarrow (x\Gamma)^{n-1}x \subseteq P \Rightarrow x \in P$. Therefore $x \in A_1$. Therefore $A_2 \subseteq A_1$.

Hence $A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$.

THEOREM 3.3.4 : If A is a Γ-ideal of a commutative Γ-semigroup S, then $A_1 = A_2 = A_3 = A_4$.

Proof : By theorem 3.3.3, $A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$. By corollary 3.2.11, in a commutative Γ-semigroup S, a Γ-ideal P is a prime Γ-ideal iff P is a completely prime Γ-ideal. So $A_1 = A_3$. By theorem 3.2.13, in a commutative Γ-semigroup S, a Γ-ideal P is a semiprime Γ-ideal iff P is a completely semiprime Γ-ideal. So $A_4 = A_2$.

Therefore $A_1 = A_2 = A_3 = A_4$.

NOTE 3.3.5 : If A is a Γ-ideal in a arbitrary Γ-semigroup, then A_1, A_2, A_3, A_4 need not be equal.

EXAMPLE 3.3.6 : Let S be the free Γ-semigroup generated by two alphabets a, b. It is clear that $A = S\Gamma a\Gamma a\Gamma S$ is a Γ-ideal in S. Since $(a\Gamma)^3 a \subseteq S\Gamma a\Gamma a\Gamma S = A$, We have $a \in A_2$. Evidently $(a\Gamma b\Gamma)^{n-1}a\Gamma b \nsubseteq S\Gamma a\Gamma a\Gamma S$ for all natural number n and thus $a\Gamma b \nsubseteq A_2$. Thus A_2 is not a Γ-ideal in S. Therefore $A_1 \neq A_2$ and $A_2 \neq A_3$.

EXAMPLE 3.3.7 : Let S be the free Γ-semigroup over the countable infinite alphabet $\{x_1, x_2, \ldots\ldots\}$ and Γ as $\{\alpha_1, \alpha_2, \ldots\ldots\}$. Consider the Γ-ideal

$A = \bigcup_{l(s)>1} (<s>\Gamma)^{l(s)-1} <s>$, where $l(s)$ is the length of the word s. For any $s \in S$,

$<x_1\Gamma s\Gamma x_1>^{l(s)+1} <x_1\Gamma s\Gamma x_1> \subseteq A$ and hence $x_1\Gamma s\Gamma x_1 \subseteq A_4$ for all $s \in S$. If $A_3 = A_4$, then A_4 is a semiprime Γ-ideal and hence $x_1 \in A_4$. Therefore $(<x_1>\Gamma)^{n-1} <x_1> \subseteq A$ for some natural number n. Consider the word $t = x_1\alpha_1 x_2\alpha_2 x_1\alpha_3 x_3\alpha_4 x_1 \ldots\ldots\ldots \alpha_{n-1}x_1\alpha_n x_{n+1}$.

Now $t \in <x_1\Gamma>^{n-1} <x_1> \subseteq A$. So $t \in <s\Gamma>^{l(s)-1} <s>$ for some $s \in S$ with $l(s) > 1$.

Thus in t, s occurs at least two times, which is a contradiction. So $A_3 \neq A_4$.

THEOREM 3.3.8 : If A is a Γ-ideal of a duo Γ-semigroup S, then $A_1 = A_2 = A_3 = A_4$.

Proof : By theorem 3.3.3, $A \subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$. By theorem 3.1.10, in a duo Γ-semigroup S, a Γ-ideal P is a prime Γ-ideal iff P is a completely prime Γ-ideal.
So $A_1 = A_3$. By theorem 3.2.12, in a duo Γ-semigroup S, a Γ-ideal P is a semiprime Γ-ideal iff P is a completely semiprime Γ-ideal. So $A_4 = A_2$.
Therefore $A_1 = A_2 = A_3 = A_4$.

THEOREM 3.3.9 : If A is a Γ-ideal of a duo Γ-semigroup S, then *rad* A = *c.rad* A

Proof : By theorem 3.3.8, *rad* A = *c.rad* A.

THEOREM 3.3.10 : If A is a Γ-ideal in a duo Γ-semigroup S. Then $A_2 = \{x \in S : (x\Gamma)^{n-1} x \subseteq A$ for some $n \in N\}$ is the minimal completely semiprime Γ-ideal of S containing A.

Proof : Clearly $A \subseteq A_2$ and hence A_2 is nonempty subset of S. Let $x \in A_2$ and $s \in S$. Since $x \in A_2$, $(x\Gamma)^{n-1}x \subseteq A$ for some $n \in N$. Now $(x\Gamma s)^{n-1}x\Gamma s \subseteq A$ and $(s\Gamma x)^{n-1}s\Gamma x \subseteq A$ implies $x\Gamma s, s\Gamma x \in A_2$. Therefore A_2 is a Γ-ideal of S containing A. Let $x \in S$ such that $x\Gamma x \subseteq A_2$. Then $(x\Gamma x\Gamma)^{n-1}\ x\,\Gamma\, x \subseteq A$ implies $(x\Gamma)^{2n-1}x \subseteq A \Rightarrow x \in A_2$. Thus A_2 is a completely semiprime Γ-ideal of S containing A. Let P be a completely semi prime Γ-ideal of S containing A. Let $x \in A_2$. Then $(x\Gamma)^{n-1}x \subseteq A$ for some $n \in N$. Since $A \subseteq P$, then $(x\Gamma)^{n-1}x \subseteq P$, for some $n \in N$. Since P is completely semiprime Γ-ideal of S, $(x\Gamma)^{n-1}x \subseteq P \Rightarrow x \in P$. Therefore $A_2 \subseteq P$ and hence A_2 is the minimal completely semiprime Γ-ideal of S containing A.

THEOREM 3.3.11 : If A is a Γ-ideal in a duo Γ-semigroup S, then $A_4 = \{x \in S : (\langle x \rangle\Gamma)^{n-1}\langle x \rangle \subseteq A$ for some $n \in N\}$ is the minimal semiprime Γ-ideal of S containing A.

Proof : Clearly $A \subseteq A_4$ and hence A_4 is nonempty subset of S. Let $x \in A_4$ and $s \in S$.
Since $x \in A_4$, $(\langle x \rangle\Gamma)^{n-1}\langle x \rangle \subseteq A$ for some $n \in N$.
Now $(\langle x\Gamma s \rangle\Gamma)^{n-1}\langle x\Gamma s \rangle \subseteq (\langle x \rangle\Gamma)^{n-1}\langle x \rangle \subseteq A$ and
$(\langle s\Gamma x \rangle\Gamma)^{n-1}\langle s\Gamma x \rangle \subseteq (\langle x \rangle\Gamma)^{n-1}\langle x \rangle \subseteq A$ implies $x\Gamma s, s\Gamma x \subseteq A_4$.
Therefore A_4 is a Γ-ideal of S containing A. Let $x \in S$ such that $(\langle x \rangle\,\Gamma)\langle x \rangle \subseteq A_4$.

Then $(<x>\Gamma<x>\Gamma)^{n-1}<x>\Gamma<x> \subseteq A$ implies $(<x>\Gamma)^{2n-1}<x> \subseteq A \Rightarrow x \in A_4$.

Thus A_4 is semiprime Γ-ideal of S containing A.

Let Q be a semiprime Γ-ideal of S containing A. Let $x \in A_4$. Then $(<x>\Gamma)^{n-1}<x> \subseteq A$ for some $n \in N$. Since $A \subseteq Q$, then $(<x>\Gamma)^{n-1}<x> \subseteq Q$ for some $n \in N$.

Since Q is a semiprime Γ-ideal of S, $(<x>\Gamma)^{n-1}<x> \subseteq Q \Rightarrow x \in Q$.

Therefore $A_4 \subseteq Q$ and hence A_4 is the minimal semiprime Γ-ideal of S containing A.

COROLLARY 3.3.12 : If A is a Γ-ideal of a duo Γ-semigroup S then

(1) A_1 = the intersection of all completely prime Γ-ideals of S containing A,

(2) A_1' = the intersection of all minimal completely prime Γ-ideals of S containing A,

(3) A_1'' = the minimal completely semiprime Γ-ideal of S containing A,

(4) $A_2 = \{x \in S : (x\Gamma)^{n-1}x \subseteq A$ for some natural number n },

(5) A_3 = the intersection of all prime Γ-ideals of S containing A,

(6) A_3' = the intersection of all minimal prime Γ-ideals of S containing A,

(7) A_3'' = the minimal semiprime Γ-ideal of S containing A,

(8) $A_4 = \{x \in S : (<x>\Gamma)^{n-1}<x> \subseteq A$ for some natural number n } are equal.

THEOREM 3.3.13 : If $a \in \sqrt{A}$, then there exist a positive integer n such that $(a\Gamma)^{n-1}a \subseteq A$.

***Proof*:** By theorem 3.3.3, $A_3 \subseteq A_2$ and hence $a \in \sqrt{A} = A_3 \subseteq A_2$.

Therefore $(a\Gamma)^{n-1}a \subseteq A$ for some $n \in N$.

3.4. ARCHIMEDIAN Γ-SEMIGROUPS

In this section, the terms; Archimedean Γ-semigroup and strongly Archimedean Γ-semigroup are introduced. It is proved that if S is a duo Γ-semigroup, then the conditions (1) S is strongly Archimedean, (2) S is Archimedean, (3) S has no proper completely prime Γ-ideals and (4) S has no proper prime Γ-ideals; are equivalent.

We now introduce the notions of archimedian Γ-semigroup and strongly archimedian Γ-semigroup.

DEFINITION 3.4.1 : A Γ- semigroup S is said to be an ***archimedian Γ- semigroup*** provided for any $a, b \in S$, there exists a natural number n such that $(a\Gamma)^{n-1}a \subseteq <b>$.

DEFINITION 3.4.2 : A Γ-semigroup S is said to be a ***strongly archimedean Γ-semigroup*** provided for any a, $b \in$ S, there is a natural number n such that $(<a>\Gamma)^{n-1}<a> \subseteq <b>$.

We now characterize archimedean Γ-semigroups.

THEOREM 3.4.3 : If S is a duo Γ-semigroup, then S is strongly archimedean if and only if archimedean.

Proof : Suppose that S is strongly Archimedean.
Then for any $a, b \in$ S, there is a natural number n such that $(<a>\Gamma)^{n-1}<a> \subseteq <b>$.
Therefore $(a\Gamma)^{n-1}a \subseteq (<a>\Gamma)^{n-1}<a> \subseteq <b>$ and hence S is Archimedean.

Conversely suppose that S is archemedian. Let $a, b \in$ S. Since S is archemedian, there exists a natural number n such that $(<a>\Gamma)^{n-1}<a> \subseteq <b> \subseteq S\Gamma b\Gamma S$. Since $S\Gamma b\Gamma S$ is a Γ-ideal of a duo Γ-semigroup S, by corollary 3.2.5, $(a\Gamma)^{n-1}a \subseteq S\Gamma b\Gamma S$
$\Rightarrow (<a>\Gamma)^{n-1}<a> \subseteq S\Gamma b\Gamma S$. Therefore S is a strongly Archimedean duo **Γ**-semigroup.

THEOREM 3.4.4 : If S is a duo Γ-semigroup, then S is archimedean if and only if S has no proper prime Γ-ideals.

Proof : Suppose that S is archimedean Γ-semigroup. Let P be prime Γ-ideal of S. Let $a, b \in$ S. Since P is Γ-ideal, $S\Gamma a\Gamma S \subseteq P$. Since S is archemedian, $(b\Gamma)^{n-1} \subseteq S\Gamma a\Gamma S$ for some natural number n. Thus $(b\Gamma)^{n-1} \subseteq S\Gamma a\Gamma S \subseteq P$. Since S is a duo Γ-semigroup, by theorem 3.2.10, P is completely prime. Thus $(b\Gamma)^{n-1}b \subseteq P \Rightarrow b \in$ P. Hence $S = P$. Therefore S has no proper prime Γ-ideals.

Conversely suppose that S has no proper prime Γ-ideals. Then for any $b \in$ S, the intersection of all prime Γ-ideals of S containing $B = <b>$ is S itself. Therefore $B_3 = S$.
We have $B_4 = \{x \in S : (<x>\Gamma)^{n-1} <x> \subseteq <b> \text{ for some } n \in N\} = S$.
Therefore for any $a \in$ S, $(<a>\Gamma)^{n-1}<a> \subseteq <b>$ for some natural number n.
So $(<a>\Gamma)^{n-1}<a> \subseteq S\Gamma b\Gamma S$. Thus S is strongly archimedian.
Hence by theorem 3.4.3, S is archimedian.

COROLLARY 3.4.5 : If S is a duo Γ-semigroup, then the conditions (1) S is strongly Archimedean, (2) S is Archimedean, (3) S has no proper completely prime Γ-ideals and (4) S has no proper prime Γ-ideals are equivalent.

3.5. SIMPLE Γ-SEMIGROUPS

In this section, the terms; left simple Γ-semigroup, right simple Γ-semigroup, simple Γ-semigroup are introduced. It is proved that (1) a **Γ**-semigroup S is a left simple **Γ**-semigroup if and only if S**Γ**a = S for all $a \in$ S, (2) a **Γ**-semigroup S is a right simple **Γ**-semigroup if and only if a**Γ**S = S for all $a \in$ S, (3) a **Γ**-semigroup S is a simple **Γ**-semigroup if and only if SΓaΓS = S for all $a \in$ S. It is also proved that if S is a left simple **Γ**-semigroup or a right simple **Γ**-semigroup then S is a simple **Γ**-semigroup. Further it is proved that if S is a duo **Γ**-semigroup and $a \in$ S then (1) a is regular, (2) a is left regular, (3) a is right regular, (4) a is intra regular and (5) a is semisimple are equivalent.

We now introduce a left simple Γ-semigroup.

DEFINITION 3.5.1 : A Γ-semigroup S is said to be a ***left simple* Γ*-semigroup*** if S is its only left Γ-ideal.

We now characterize left simple Γ-semigroups.

THEOREM 3.5.2 : A Γ-semigroup S is a left simple Γ-semigroup if and only if SΓa = S for all $a \in$ S.

***Proof*:** Suppose that S is a left simple Γ-semigroup and $a \in$ S.

Let $t \in$ SΓa, $s \in$ S, $\gamma \in \Gamma$.

$t \in S\Gamma a \Rightarrow t = s_1\alpha a$ where $s_1 \in S$ and $\alpha \in \Gamma$.

Now $s\gamma t = s\gamma(s_1\alpha a) = (s\gamma s_1)\alpha a \in S\Gamma a \Rightarrow$ SΓa is a left Γ-ideal of S.

Since S is a left simple Γ-semigroup, SΓa = S.

Therefore SΓa = S for all $a \in$ S.

Conversely suppose that SΓa = S for all $a \in$ S. Let L be a left Γ-ideal of S.

Let $l \in$ L. Then $l \in$ S. By assumption SΓl = S.

Let $s \in$ S. Then $s \in S\Gamma l \Rightarrow s = t\alpha l$ for some $t \in$ S, $\alpha \in \Gamma$.

$l \in$ L, $t \in$ S, $\alpha \in \Gamma$ and L is a left Γ-ideal $\Rightarrow t\alpha l \in$ L $\Rightarrow s \in$ L.

Therefore S ⊆ L. Clearly L ⊆ S and hence S = L.

Therefore S is the only left Γ-ideal of S. Hence S is left simple Γ-semigroup.

We now introduce a right simple Γ-semigroup.

DEFINITION 3.5.3 : A Γ-semigroup S is said to be a ***right simple* Γ*-semigroup*** if S is its only right Γ-ideal.

We now characterize right simple Γ-semigroups.

THEOREM 3.5.4 : A Γ-semigroup S is a right simple Γ-semigroup if and only if *a*ΓS= S for all *a* ∈ S.

***Proof*:** Suppose that S is a right simple Γ-semigroup and $a \in$ S. Let $t \in a\Gamma$S, $s \in$ S, $\gamma \in \Gamma$.
$t \in a\Gamma S \Rightarrow t = a\alpha s_1$ where $s_1 \in S$ and $\alpha \in \Gamma$.

Now $t\gamma s = (a\alpha s_1)\gamma s = a\alpha(s_1\gamma s) \in a\Gamma S \Rightarrow a\Gamma$S is a right Γ-ideal of S.

Since S is a right simple Γ-semigroup, $a\Gamma S = S$.
Therefore $a\Gamma S = S$ for all $a \in S$.

Conversely suppose that $a\Gamma S = S$ for all $a \in S$.
Let R be a right Γ-ideal of a Γ-semigroup S.
Let $r \in$ R. Then $r \in$ S. By assumption $r\Gamma S = S$.
Let $s \in$ S. Then $s \in r\Gamma S \Rightarrow s = rat$ for some $t \in$ S, $\alpha \in \Gamma$.
$r \in$ R, $t \in$ S, $\alpha \in \Gamma$ and R is a right Γ-ideal $\Rightarrow rat \in R \Rightarrow s \in$ R.
Therefore $S \subseteq R$. Clearly $R \subseteq S$ and hence $S = R$.
Therefore S is the only right Γ-ideal of S. Hence S is right simple Γ-semigroup.

We now introduce a simple Γ-semigroup.

DEFINITION 3.5.5 : A Γ-semigroup S is said to be ***simple* Γ*-semigroup*** if S is its only two-sided Γ-ideal.

We now characterize simple Γ-semigroups

THEOREM 3.5.6 : If S is a left simple Γ-semigroup or a right simple Γ-semigroup then S is a simple Γ-semigroup.

***Proof*:** Suppose that S is a left simple Γ-semigroup. Then S is the only left Γ-ideal of S.
If A is a Γ-ideal of S, then A is a left Γ-ideal of S and hence A = S.
Therefore S itself is the only Γ-ideal of S and hence S is a simple Γ-semigroup.
Suppose that S is a right simple Γ-semigroup. Then S is the only right Γ-ideal of S.
If A is a Γ-ideal of S, then A is a right Γ-ideal of S and hence A = S.
Therefore S itself is the only Γ-ideal of S and hence S is a simple Γ-semigroup.

THEOREM 3.5.7 : A Γ-semigroup S is simple Γ-semigroup if and only if SΓ*a*ΓS = S for all *a* ∈ S.

***Proof*:** Suppose that S is a simple Γ-semigroup and $a \in S$.

Let $t \in S\Gamma a\Gamma S$, $s \in S$ and $\gamma \in \Gamma$.

$t \in S\Gamma a\Gamma S \Rightarrow t = s_1\alpha a\beta s_2$ where $s_1, s_2 \in S$ and $\alpha, \beta \in \Gamma$.

Now $t\gamma s = (s_1\alpha a\beta s_2)\gamma s = s_1\alpha a\beta(s_2\gamma s) \in S\Gamma a\Gamma S$

and $s\gamma t = s\gamma(s_1\alpha a\beta s_2) = (s\gamma s_1)\alpha a\beta s_2 \in S\Gamma a\Gamma S$. Therefore SΓ*a*ΓS is a Γ-ideal of S.

Since S is a simple Γ-semigroup, S itself is the only Γ-ideal of S and hence SΓ*a*ΓS = S .

Conversely suppose that SΓ*a*ΓS = S for all $a \in S$. Let I be a Γ-ideal of S.

Let $a \in I$. Then $a \in S$. So SΓ*a*ΓS = S.

Let $s \in S$. Then $s \in S\Gamma a\Gamma S \Rightarrow s = t_1\alpha a\beta t_2$ for some $t_1, t_2 \in S$, $\alpha, \beta \in \Gamma$.

$a \in I$, $t_1, t_2 \in S$, $\alpha, \beta \in \Gamma$, I is a Γ-ideal of S $\Rightarrow t_1\alpha a\beta t_2 \in I \Rightarrow s \in I$.

Therefore $S \subseteq I$. Clearly $I \subseteq S$ and hence S = I.

Therefore S is the only Γ-ideal of S. Hence S is a simple Γ-semigroup.

THEOREM 3.5.8 : If S is a duo Γ-semigroup, then the following are equivalent for any element *a* ∈ S.

1) *a* is regular.
2) *a* is left regular.
3) *a* is right regular.
4) *a* is intra regular.
5) *a* is semisimple.

***Proof*:** Since S is duo Γ-semigroup, $a\Gamma S^1 = S^1\Gamma a$.

We have $a\Gamma S^1\Gamma a = a\Gamma a\Gamma S^1 = S^1\Gamma a\Gamma a = \langle a\Gamma a\rangle = \langle a\rangle\Gamma\langle a\rangle$.

(1) ⇒ (2) : Suppose that a is regular. Then $a = a\alpha x\beta a$ for some $x \in S$ and $\alpha, \beta \in \Gamma$.

Therefore $a \in a\Gamma S^1\Gamma a = a\Gamma a\Gamma S^1 \Rightarrow a = a\gamma a\delta y$ for some $y \in S^1$, $\gamma, \delta \in \Gamma$.

Therefore a is left regular.

(2) ⇒ (3) : Suppose that a is left regular. Then $a = a\alpha a\beta x$ for some $x \in S$ and $\alpha, \beta \in \Gamma$.

Therefore $a \in a\Gamma a\Gamma S^1 = S^1\Gamma a\Gamma a \Rightarrow a = y\gamma a\delta a$ for some $y \in S^1$, $\gamma, \delta \in \Gamma$.

Therefore a is right regular.

(3) ⇒ (4): Suppose that a is right regular. Then for some $x \in S$, $\alpha, \beta \in \Gamma$; $a = x\alpha a\beta a$.

Therefore $a \in S^1\Gamma a\Gamma a = \langle a\Gamma a\rangle \Rightarrow a = x\alpha a\beta a\gamma y$ for some $x, y \in S^1$ and $\alpha, \beta, \gamma \in \Gamma$.

Therefore a is intra regular.

(4) ⇒ (5): Suppose that a is intra regular.

Then $a = x\alpha a\beta a\gamma y$ for some $x, y \in S^1$ and $\alpha, \beta, \gamma \in \Gamma$. Therefore $a \in \langle a \rangle \Gamma \langle a \rangle$.

Therefore a is semisimple.

(5) ⇒ (1): Suppose that a is semisimple. Then $a \in \langle a \rangle \Gamma \langle a \rangle = a\Gamma S^1 \Gamma a$

⇒ $a \in a\alpha x\beta a$ for some $x \in S^1$ and $\alpha, \beta \in \Gamma$.

Therefore a is a regular element.

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