Chapter 0
Preliminary Notions and Results

We present here all necessary definitions and results which we use in this thesis and most of the definitions and results are taken from Pilz [15].

0.1 Near-Rings

Definition 0.1.1. An algebraic system R = (R, +, .) is called a *right near-ring* if it satisfies the following conditions:

1. (R, +) is a group (not necessarily abelian);

2. (R, .) is a semi group (we write xy instead of x.y , for all x, y in R);

3. (x + y)z = xz + yz for all x, y, z in R.

Similarly, one can define a *left near-ring.*

A left near-ring would satisfy (1), (2) and $(3)'$:

$(3)'$ z(x + y) = zx + zy for all x, y, z $\in$ R.

Throughout this thesis, only right near-rings are considered and R stands for a right near-ring.

We write just *near-ring* for right near-ring.

It can be easily proved that $0.a = 0$ and $(-a).b = -ab$, for all a, b $\in$ R, where 0 is the additive identity in R.

Definition 0.1.2.

1. $R_0 := \{r \in R \mid r0 = 0\}$ is called the *zero-symmetric part of R.*

2. $R_c := \{r \in R \mid r0 = r\} = \{r \in R \mid rs = r \text{ for all } s \in R\}$ is called *the constant part of R.*

3. R is called zero *symmetric* if $R = R_0$.

4. R is called *constant* if $R = R_c$.

In a near-ring, *left (right) identities, left (right) zero-divisors, left (right) invertible elements, idempotent and nilpotent elements* are defined in the usual way as in ring theory.

Definition 0.1.3. Let R and S be near-rings. A map $h : R \to S$ is called a *(near-ring) homomorphism* if for all x, y $\in$ R,

1. $h(x + y) = h(x) + h(y)$, and

2. $h(xy) = h(x)h(y)$.

A map $h : R \to S$ is called a *near-ring isomorphism* or simply an *isomorphism* if h is a one-to-one and onto homomorphism.

If A, B are two non-empty subsets of R, we denote $\{ab \mid a \in A, b \in B\}$ by *AB*.

Definition 0.1.4. A non-empty subset A of R is called

1. *a subnear-ring of R* if A is a subgroup of (R, +) and $AA \subseteq A$.

2. *a left ideal of R* if A is a normal subgroup of (R, +) and r(s + a) - rs $\in$ A for all r, s $\in$ R and a $\in$ A.

3. *a right ideal of R* if A is a normal subgroup of (R, +) and AR $\subseteq$ A.

4. *an ideal of R* if A is both a left ideal and a right ideal of R.

5. *a (left) R-subgroup of R* if A is a subgroup of (R, +) and RA $\subseteq$ A.

6. *an invariant subnear-ring of R* if A is a subnear-ring of R, RA $\subseteq$ A and AR $\subseteq$ A (we call an R-subgroup A of R with AR $\subseteq$ A, an *invariant R-subgroup of R*).

R_0 is a left ideal of R and R_c is an invariant subnear-ring of R. Moreover, R = R_0 + R_c and $R_0 \cap R_c = \{0\}$. So, every element r $\in$ R can be uniquely written as r = r_0 + r_c, where $r_0 \in R_0$ and $r_c \in R_c$.

If R is zero-symmetric, then every left ideal of R is an R-subgroup of R.

Let (G, +) be a group. *M(G)* denotes the set of all mappings of G into G. M(G) is a near-ring under pointwise addition and composition of mappings. The zero-symmetric part of M(G) is denoted by *$M_0(G)$* and the constant part of M(G) is denoted by *$M_c(G)$*. $M_0(G) = \{f \in M(G) \mid f(0) = 0\}$ and $M_c(G) = \{f \in M(G) \mid$ f is a constant map$\}$.

Definition 0.1.5. If (R, +) is an abelian group, then R is called an *abelian near-ring.* If (R, .) is commutative, then R is called a *commutative near-ring.* R is called a simple near-ring if R $\neq \{0\}$ and R has no non-trivial ideals. R is called a *near-field* if the set of all non-zero elements of R is a group under multiplication.

If F is a near-field, then (F, +) is abelian.

Definition 0.1.6. An element x in R is called *distributive* if $x(y + z) = xy + xz$, for all x, y, z in R. R is called a *distributive near-ring* if each element of R is distributive.

Definition 0.1.7. R is called a distributively generated (d.g.) near-ring if (R, +) is generated by the set of all distributive elements of R.

A d.g. near-ring is zero-symmetric. If R is an abelian d.g. near-ring, then R is a ring.

Definition 0.1.8. An idempotent $e \in R$ is called a central idempotent if $er = re$ for all $r \in R$.

The following definition is due to G. Betsch.

Definition 0.1.9. (Remark 3.49 of Pilz [15]) A near-ring R is called *biregular* if there exists a set E of central idempotents of R such that

1. Re is an ideal of R for all $e \in E$;
2. For each $r \in R$ there exists an $e \in E$ such that $Re = (r)$, the ideal of R generated by r;
3. $e + f = f + e$, for all $e, f \in E$;
4. $ef, e + f - ef \in E$, if $e, f \in E$.

Isomorphism theorems also hold in near-rings.

Definition 0.1.10. Let R_i, $i \in I$, be a family of near-rings.

1. $X_{i \in I} R_i$ with the componentwise defined operations '+' and '.' is called the *direct product* $\Pi_{i \in I} R_i$ of the near-rings R_i, $i \in I$.

2. The subnear-ring of $\Pi_{i\in I}$ R_i consisting of those elements where all components, except a finite number equal to zero, is called the *(external) direct sum* $\oplus_{i\in I}R_i$ of the R_i.

A subgroup K of (R, +) is called a *right R-subgroup of R* if KR $\subseteq$ K.

Lemma 0.1.1. *(Lemma 9 of [6]) Let (G, +) be a group.*

1. *If K is a right $M_0(G)$-subgroup of $M_0(G)$, then $G_K := \{f(g) \mid f \in K, g \in G\}$ is a subgroup of G.*

2. *If K is a right ideal of $M_0(G)$, then $G_K := \{f(g) \mid f \in K, g \in G\}$ is a normal subgroup of G.*

Theorem 0.1.2. *(Theorem 6 of [6]) Let (G, +) be a finite group and let K be a right $M_0(G)$-subgroup of $M_0(G)$. If $G_K := \{f(g) \mid f \in K, g \in G\} = G$, then $K = M_0(G)$.*

Lemma 0.1.3. *(Lemma 10 of [6]) Let N be a subgroup (normal subgroup) of (G, +). Then $K_N := \{f \in M_0(G) \mid f(G) \subseteq N\}$ is a right $M_0(G)$-subgroup (right ideal) of $M_0(G)$.*

Let G be an additive group and let End(G) be the set of all endomorphisms of G. The subnear-ring of M_0(G) generated by End(G) is denoted by *E(G)* and is a d.g. near-ring.

Theorem 0.1.4. *(Corollary 7.48 of Pilz[15]) Let (G, +) be a finite simple non-abelian group. Then $E(G) = M_0(G)$.*

Theorem 0.1.5. *(Theorem 7.30 of Pilz[15]) $M_0(G)$ is simple for every group $G \neq \{0\}$.*

Equiprime near-rings were introduced in [3].

A near-ring R is called an *equiprime near-ring* if $0 \neq a \in R$, $x, y \in R$ and $arx = ary$ for all $r \in R$, implies $x = y$. An ideal I of R is called *equiprime* if R/I is an equiprime near-ring.

It is known that a near-ring R is equiprime if and only if

1. $x, y \in R$ and $xRy = \{0\}$ implies $x = 0$ or $y = 0$.

2. $\{0\} \neq I$ is an invariant subnear-ring of R, $x, y \in R$ and $ax = ay$ for all $a \in I$ implies $x = y$.

Moreover, an equiprime near-ring is zero-symmetric.

0.2 R-groups

In this section we give some definitions and results on R-groups.

Definition 0.2.1. An additive group (G, +) is called a *(left) R-group* if there exists a mapping $((r, g) \to rg)$ of $R \times G$ into G such that

1. $(r + s)g = rg + sg$, and

2. $(rs)g = r(sg)$ for all $r, s \in R$ and $g \in G$.

Let G be an R-group and $r \in R$, $g \in G$. Then $0g = 0$ and $(-r)g = -rg$. Let T and H be non-empty subsets of R and G respectively. We denote the subset $\{th \mid t \in T, h \in H\}$ of G by *TH*.

Definition 0.2.2. Let G be an R-group. A non-empty subset H of G is called

1. an *R-subgroup* of G if H is a subgroup of G and $RH \subseteq H$.

2. an *ideal* of G if H is a normal subgroup of (G, +) and r(g + h) - rg $\in$ H for all r $\in$ R, g $\in$ G and h $\in$ H.

R0 is the smallest R-subgroup of the R-group G and when R = R_0 we have R0 = {0}.

Definition 0.2.3. Let G be an R-group and let H_1, H_2 be non-empty subsets of G.

1. Then (H_1 : H_2) denotes the set $\{r \in R \mid rH_2 \subseteq H_1\}$.
2. We write (g : H_1) and (H_1 : g) for ({g} : H_1) and (H_1 : {g}) respectively, g $\in$ G.
3. (0 : H_1) is called the *annihilator* of H_1.
4. G is called *faithful* if (0 : G) = {0}.

Definition 0.2.4. Let G be an R-group. Then G is called

1. *simple* if G $\neq$ {0} and it has no non-zero proper ideals.
2. *R-simple* if G $\neq$ {0} and the only R-subgroups of G are R0 and G.
3. *R_0-simple* if G considered as an R_0-group is R_0-simple.

Definition 0.2.5. An R-group G is called

1. *monogenic* if there is a $g_0 \in$ G such that Rg_0 = G.
2. *strongly monogenic* if G is monogenic and for each g $\in$ G, either Rg = {0} or Rg = G.

Definition 0.2.6. A monogenic R-group G with RG $\neq$ {0} is said to be of

1. *type-0* if G is simple.

2. *type-1* if G is simple and strongly monogenic.

3. *type-2* if G is R_0-simple.

Remark 0.2.1. An R-group of type-2 is an R-group of type-1 and an R-group of type-1 is an R-group of type-0. Moreover, if $R = R_0$ has identity, then an R-group of type-1 is an R-group of type-2.

Definition 0.2.7. A left ideal L of R is called *modular* if there exists $e \in R$, such that $r - re \in L$ for all $r \in R$. In this case, L is said to be *modular by e.*

Proposition 0.2.1. *Let L be a left ideal of R which is modular by e. Then* $(L : R) = \{r \in R \mid rR \subseteq L\}$ *is the largest ideal of R contained in L.*

Each modular left ideal L of R is contained in a modular maximal left ideal. For $z \in R$, denote the left ideal of R generated by the set $\{r - rz \mid r \in R\}$ by L_z.

Definition 0.2.8. $z \in R$ is called *quasi-regular* if $z \in L_z$. A subset T of R is called quasi-regular if each element of T is quasi-regular.

Remark 0.2.2. Let R be a zero-symmetric near-ring. Then $z \in R$ is quasi-regular if and only if $L_z = R$.

Proposition 0.2.2. *Let R be a zero-symmetric near-ring. Then*

1. *each nilpotent element of R is quasi-regular.*

2. *each nil subset of R is quasi-regular.*

3. *a proper left ideal L of R is modular by e implies e is not quasi-regular.*

4. a non-zero idempotent e of R is not quasi-regular.

Proposition 0.2.3. *Let R be a zero-symmetric near-ring satisfying DCC on R-subgroups of R and let M be an R-subgroup of R. Then the following are equivalent:*

1. *M is quasi-regular.*
2. *M is nilpotent.*
3. *M is nil.*

Definition 0.2.9. Let $\nu \in \{0, 1, 2\}$. A left ideal L of R is called *ν-modular* if L modular and R/L is an R-group of type-ν.

Definition 0.2.10. Let $\nu \in \{0, 1, 2\}$.

1. R is called *ν-primitive* if there is a faithful R-group G of type-ν.
2. An ideal I of R is called a *ν-primitive ideal* of R if R/I is a ν-primitive near-ring.

Proposition 0.2.4. *Let $\nu \in \{0, 1, 2\}$. Then the following are equivalent:*

1. *R is ν-primitive.*
2. *$\{0\}$ is a ν-primitive ideal of R.*
3. *There is a left ideal L of R such that L is ν-modular and $(L : R) = \{0\}$.*

Definition 0.2.11. An ideal P of R is called *a prime ideal* if I and J are ideals of R and $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

Definition 0.2.12. The intersection of all prime ideals of R is called the *prime radical* of R and is denoted by *P(R)*.

Proposition 0.2.5. *P(R) is a nil ideal of R and contains the sum of all nilpotent ideals of R.*

Theorem 0.2.6. *(4.34 of Pilz [15]) A 0-primitive ideal of R is a prime ideal of R.*

Remark 0.2.3. 2-primitivity implies 1-primitivity and this in turn implies 0-primitivity.

Proposition 0.2.7. *Let R contains either a left or a right identity e.*

1. *Then every v-primitive ideal I of R is modular.*

2. *If e is a left identity of R, then R is 1-primitive if and only if R is 2-primitive.*

Proposition 0.2.8. *Let R be a zero-symmetric near-ring satisfying DCC on left ideals. If R is 1-primitive on G, then*

1. *R is simple.*

2. *R is either 2-primitive on G (or) there is no R-group of type-2.*

Proposition 0.2.9. *Let R be a zero-symmetric near-ring satisfying DCC on R-subgroups of R and let $I \neq \{0\}$ be an ideal of R. If R has a left identity, then*

1. *R is 1-primitive if and only if R is 2-primitive if and only if R is simple.*

2. *I is 1-primitive if and only if I is 2-primitive if and only if I is maximal.*

Definition 0.2.13. Let $\nu \in \{0, 1, 2\}$. $J_\nu(R) := \cap \{(0 : G) \mid G$ is an R-group of type-$\nu\}$ is called the Jacobson radical of R of type-ν.

Definition 0.2.14. $J_{1/2}(R)$ is the intersection of all 0-modular left ideals of R.

Proposition 0.2.10. *Let R be a zero-symmetric near-ring. Then*

1. $J_{1/2}(R)$ *is the greatest quasi-regular left ideal of* R.

2. $J_0(R)$ *is the largest ideal of* R *contained in* $J_{1/2}(R)$.

3. $J_0(R)$ *is the largest quasi-regular ideal of* R.

4. $J_{1/2}(R)$ *contains all nil left ideals.*

5. $J_1(R)$ *contains all nilpotent left ideals*

6. $J_2(R)$ *contains all nilpotent* R*-subgroups.*

Proposition 0.2.11. *Let* $\nu \in \{0, 1, 2\}$. *Then*

1. $J_\nu(R) = \cap \{I \mid I$ *is a* ν*-primitive ideal of* $R\}$.

2. $J_\nu(R) = \cap \{(L : R) \mid L$ *is a* ν*-modular left ideal of* $R\}$.

3. $J_0(R) \subseteq J_{1/2}(R) \subseteq J_1(R) \subseteq J_2(R)$.

4. $J_1(R) = J_2(R)$ *if* R *has unity.*

5. *for* $\nu \in \{1, 2\}$, $J_\nu(R) = \cap \{L \mid L$ *is a* ν*-modular left ideal of* $R\}$,

Definition 0.2.15. Let $\nu \in \{0, 1, 2\}$. R is called *J_ν-semisimple* if $J_\nu(R) = \{0\}$.

Theorem 0.2.12. *(Theorem 5.40 of Pilz [15]) Let* R *be a zero-symmetric near-ring satisfying DCC on* R*-subgroups of* R. *Then* $J_0(R)$ *is nilpotent and each prime ideal* $P \neq R$ *is 0-primitive. Moreover, the following are equivalent:*

1. R *is 0-semisimple.*

2. R *has no non-zero quasi-regular ideal.*

3. R has no non-zero nil ideal.

4. R has no non-zero nilpotent ideal.

0.3 Matrix Near-Rings

Matrix near-rings were introduced in [11].

Let R be a near-ring with identity. R^n denotes the direct sum of n-copies of the group, (R, +). I_i, Π_i denote the i-th coordinate injection and projection functions respectively. For r ∈ R, $f^r : R \to R$ is defined by $f^r(s) = rs$, for all s ∈ R. For $1 \le i, j \le n$, r ∈ R, $f^r_{ij} : R \to R$ is defined by $f^r_{ij} = I_i f^r \Pi_j$. The *near-ring of n×n matrices over R*, denoted by *$M_n(R)$*, is the subnear-ring $M(R^n)$ generated by $\{f^r_{ij} \mid r \in R, 1 \le i, j \le n\}$. $M_n(R)$ is a right near-ring with identity and if R is zero-symmetric, then $M_n(R)$ is also zero-symmetric. If R is a ring, then $M_n(R)$ is (isomorphic to) the usual ring of n×n matrices over R.

More general properties of matrix near-rings can be found in [11] and [12].

The following results which were proved for zero-symmetric near-rings with identity in [16] are also valid for general right near-rings with (left) identity.

Proposition 0.3.1. *(Proposition 1 of [16]) If R has a left identity e and $R = R_1 \oplus R_2 \oplus ... \oplus R_n$, a direct sum of right ideals of R, then R has a set $\{e_i \mid i = 1, 2, ..., n\}$ of orthogonal idempotents which are distributive such that $e = e_1 + e_2 + ... + e_n$ and $R_i = e_iR$, for i = 1, 2, ..., n.*

Proposition 0.3.2. *If R has a left identity e and $\{e_1, e_2, ..., e_n\}$ is a set of orthogonal idempotents in R which are distributive such that $e = e_1 + e_2 + ... + e_n$, then R is a direct sum of the right ideals e_1R, e_2R, ..., e_nR.*

Proposition 0.3.3. *(Proposition 3 of [16]) Let e_1, e_2 be distributive idempotents in R. Then the right R-groups e_1R and e_2R are R-isomorphic if and only if there exist distributive elements e_{12} and e_{21} in R such that $e_1e_{12}e_2 = e_{12}$, $e_2e_{21}e_1 = e_{21}$ and $e_{12}e_{21} = e_1$, $e_{21}e_{12} = e_2$.*

Definition 0.3.1. A set of distributive elements $\{e_{ij} \mid 1 \leq i, j \leq n\}$ in a near-ring R with identity is said to be a set of matrix units in R if and only if $e_{11} + e_{22} + ... + e_{nn} = 1$ and $e_{rs}e_{pq} = \delta_{sp}\, e_{rq}$, where

$$\delta_{sp} = \begin{cases} 1 & if\ s = p \\ 0 & if\ s \neq p \end{cases}$$

Proposition 0.3.4. *(Proposition 5 of [16]) Let R be a near-ring with identity. Then $R = R_1 \oplus R_2 \oplus ... \oplus R_n$, a direct sum of n, R-isomorphic right ideals of R (as right R-groups) if and only if R has a set of matrix units $\{e_{ij} \mid 1 \leq i, j \leq n\}$. In this case, $R_i = e_{ii}R$ for all $1 \leq i \leq n$.*

Remark 0.3.1. Suppose R has a set of matrix units $\{e_{ij} \mid 1 \leq i, j \leq n\}$. B denotes the set $\{a \in R \mid e_{ij}a = ae_{ij}$, for all $1 \leq i, j \leq n\}$. B is a subnear-ring of R with identity. C_i denotes the set $\{a \in R \mid a = e_{1i}b_1 + e_{2i}b_2 + ... + e_{ni}b_n$, where $b_1, b_2, ..., b_n \in B\}$, for $1 \leq i \leq n$. C_i is a subgroup of (R, +). D denotes the set $\{a \in R \mid a = e_{ij}b$, where $1 \leq i, j \leq n$ and $b \in B\}$. The subnear-ring of R generated by D is denoted by S. S contains the identity element. Moreover, C_i is a right S-subgroup of S.

Let B, S and C_j be as defined in Remark 0.3.1. We have the following.

Lemma 0.3.5. *(Lemma 8 of [16]) Let $\{e_{ij} \mid 1 \leq i, j \leq n\}$ be a set of matrix units in R. Then $Re_{ij} \subseteq C_j$ and hence $Re_{ij} = C_j$.*

Corollary 0.3.6. *(Corollary 9 of [16]) Let* $\{e_{ij} \mid 1 \leq i, j \leq n\}$ *be a set of matrix units in R. If R is d.g., then* $R = S$.

Lemma 0.3.7. *Let* $\{e_{ij} \mid 1 \leq i, j \leq n\}$ *be a set of matrix units in R. Then* $Be_{ii} = \{be_{ii} \mid b \in B\}$ *is a subnear-ring of R and the mapping* $b \to be_{ii}$ *of B into* Be_{ii} *is a near-ring isomorphism of B onto* Be_{ii}, $1 \leq i \leq n$. *Moreover,* $e_{ii}Re_{ii} = Be_{ii}$, $1 \leq i \leq n$.

Theorem 0.3.8. *(Theorem 13 of [16]) A near-ring R with identity is isomorphic to a matrix near-ring if and only if R has a set of matrix units* $\{e_{ij} \mid 1 \leq i, j \leq n\}$, $R = S$ *and* C_i *is a faithful right S-subgroup of S. In this case, R is isomorphic to* $M_n(B)$.

Corollary 0.3.9. *(Corollary 14 of [16]) If R is a d.g. near-ring with identity, then R is isomorphic to a matrix near-ring if and only if R contains a set of matrix units* $\{e_{ij} \mid 1 \leq i, j \leq n\}$ *and* C_i *is a faithful right R-subgroup of R. In this case, R is isomorphic to* $M_n(B)$.

As stated soon after Corollary 15 of [16], we have the following.

Theorem 0.3.10. *Let R be a simple and d.g. near-ring with identity. Then R is isomorphic to a matrix near-ring* $M_n(S)$ *if and only if R has a set of matrix units* $\{e_{ij} \mid 1 \leq i, j \leq n\}$.

Corollary 0.3.11. *(Corollary 17 of [16]) Let* $(G, +)$ *be a finite group with* $|G| \geq 3$. *Then* $M_0(G^n)$ *is isomorphic to the matrix near-ring* $M_n(M_0(G))$.

Corollary 0.3.12. *(Corollary 19 of [16]) If* $(G, +)$ *is a finite simple group, then* $E(G^n)$ *is isomorphic to the matrix near-ring* $M_n(E(G))$.

The following Definitions are from [17].

Definition 0.3.2. A right R-group G is called *strictly irreducible* if gR = G, for all $0 \neq g \in G$.

Definition 0.3.3. A right ideal K of R is called a *strictly minimal right ideal* of R if K is strictly irreducible as a right R-group.

We also need the following Theorems of [17]. The following Theorem which was proved for a zero-symmetric near-ring with identity is also valid for a zero-symmetric near-ring with left identity.

Theorem 0.3.13. *(Theorem 3.5 of [17]) Let R be a near-ring with left identity e. Suppose that R is the direct sum of strictly minimal right ideals e_1R, e_2R, ... , e_nR and $e = e_1 + e_2 + ... + e_n$, e_i are distributive idempotents. Then each minimal ideal I of R contains a e_kR for some $1 \leq k \leq n$ and is the direct sum of all those strictly minimal right ideals e_jR, $1 \leq j \leq$ for which $e_kRe_j \neq \{0\}$. Moreover, R is the direct sum of its minimal ideals.*

Theorem 0.3.14. *(Theorem 3.13 of [17]) Let R be a finite near-ring and let R be the direct sum of the strictly minimal right R-isomorphic right ideals $e_{11}R$, $e_{22}R$, ... , $e_{nn}R$, where e_{11}, e_{22}, ... ,e_{nn} are distributive idempotents of R, with $1 = e_{11} + e_{22} + ... + e_{nn}$. Let $F := e_{11}Re_{11}$ be a non-ring. Then F is a near-field and $M_n(F)$, the near-ring of $n \times n$-matrices over F, is isomorphic to R.*

Theorem 0.3.15. *(Theorem 3.15 of [17]) Let F be a near-field and let n be a positive integer. Then the matrix near-ring $M_n(F)$ over the field F is simple and is a direct sum of right $M_n(F)$-isomorphic strictly minimal right ideals of $M_n(F)$ and hence $M_n(F)$ is a right completely reducible simple near-ring.*

0.4 Radical Theory

Let Q be a mapping which assigns to each near-ring R an ideal Q(R) of R. Such mappings are called ideal-mappings. We consider the following properties which Q may satisfy:

(H1) $h(Q(R)) \subseteq Q(h(R))$ for all homomorphisms h of R;

(H2) $Q(R/Q(R)) = \{0\}$ for all R;

Q is *r-hereditary* if $I \cap Q(R) \subseteq Q(I)$ for all ideals I of R;

Q is *s-hereditary* if $Q(I) \subseteq I \cap Q(R)$ for all ideals I of R;

Q is *ideal-hereditary* if it is both r-hereditary and s-hereditary, that is, if $Q(I) = I \cap Q(R)$ for all ideals I of R;

Q is *idempotent* if $Q(Q(R)) = Q(R)$ for all R;

Q is *complete* if $Q(I) = I$ and I is an ideal of R implies $I \subseteq Q(R)$.

With Q we associate two classes of near-rings $\mathbb{R}_Q$ and $\mathbb{S}_Q$ defined by $\mathbb{R}_Q := \{R \mid Q(R) = R\}$, $\mathbb{S}_Q := \{R \mid Q(R) = \{0\}\}$ and are called Q-radical class and Q-semisimple class respectively.

An ideal-mapping Q is a *Hoehnke* radical (H-radical) or a *radical map* if it satisfies conditions (H1) and (H2).

An ideal-mapping Q is a *Kurosh-Amitsur* radical (KA-radical) if it is a complete idempotent H-radical.

Let $\mathbb{M}$ be a class of near-rings. Classes of near-rings are always assumed to be abstract, that is, they contain the one element near-ring and are closed under isomorphic copies. With every near-ring R we associate two ideals of R, depending on $\mathbb{M}$. These ideals are defined by:

$\mathbb{M}(R) := \Sigma \{I \mid I \text{ is an ideal of } R \text{ and } I \in \mathbb{M}\}$ and

$(R)\mathbb{M} := \cap \{I \mid I \text{ is an ideal of } R \text{ and } R/I \in \mathbb{M}\}$.

The mapping P defined by $P(R) := (R)\mathbb{M}$ is always an H-radical and is called the H-radical corresponding to $\mathbb{M}$.

If I is an ideal of R, then we denote it by $I \lhd R$. A subset S of R is *left invariant* if $RS \subseteq S$. By a radical class, we mean a radical class in the sense of Kurosh-Amitsur. Let $\mathcal{E}$ be a class of near-rings. $\mathcal{E}$ is called *regular* if $\{0\} \neq I \lhd R \in \mathcal{E}$ implies that $0 \neq I/K \in \mathcal{E}$ for some $K \lhd I$. It is known that, if $\mathcal{E}$ is a regular class, then $\mathcal{U}\mathcal{E} = \{R \mid R \text{ has no non-zero homomorphic image in } \mathcal{E}\}$ is a radical class, called the *upper radical* determined by $\mathcal{E}$. The *subdirect closure* of a class of near-rings $\mathcal{E}$ is the class $\overline{\mathcal{E}} = \{R \mid R \text{ is a subdirect sum of near-rings from } \mathcal{E}\}$. A class $\mathcal{E}$ is called hereditary if $I \lhd R \in \mathcal{E}$ implies $I \in \mathcal{E}$. $\mathcal{E}$ is called *c-hereditary* if I is a left invariant ideal of $R \in \mathcal{E}$ implies $I \in \mathcal{E}$. It is clear that a hereditary class is a regular class. If $I \lhd R$ and for every non-zero ideal J of R, $J \cap I \neq \{0\}$, then I is called an *essential ideal* of R and is denoted by $I \lhd\cdot R$. A class of near-rings $\mathcal{E}$ is called closed *under essential extensions (essential left invariant extensions)* if $I \in \mathcal{E}$, $I \lhd\cdot R$ (I is an essential ideal of R which is left invariant) implies $R \in \mathcal{E}$. A class of near-rings $\mathcal{E}$ is said to satisfy condition (F_l), if $K \lhd I \lhd R$, and I is left invariant in R and $I/K \in \mathcal{E}$, then $K \lhd R$.

In [2], G. L. Booth and N. J. Groenewald defined special radicals for near-rings. A class $\mathcal{E}$ consisting of equiprime near-rings is called a *special class* if it is hereditary and closed under left invariant essential extensions. If $\mathcal{R}$ is the upper radical in the class of all near-rings determined by a special class of near-rings, then $\mathcal{R}$ is called a special radical. If $\mathcal{R}$ is a radical class, then the class $\mathcal{S}\mathcal{R} = \{R \mid \mathcal{R}(R) = \{0\}\}$ is called the *semisimple class* of $\mathcal{R}$.

We also need the following Theorem:

Theorem 0.4.1. *(Theorem 2.4 of [27]) Let $\mathcal{E}$ be a class of zero-symmetric near-rings. If $\mathcal{E}$ is regular, closed under essential left invariant extensions and satisfies condition (F_l), then $\mathcal{R} := \mathcal{U}\mathcal{E}$ is c-hereditary radical class in the variety of all near-rings, $\mathcal{SR}$ $= \overline{\mathcal{E}}$ and $\mathcal{SR}$ is hereditary. So, $\mathcal{R}(R) = \cap \{I \lhd R \mid R/I \in \mathcal{E}\}$.*

Remark 0.4.1. Since all ideals in a zero-symmetric near-ring are left invariant, under the hypothesis of Theorem 0.4.1, in the variety of zero-symmetric near-rings both $\mathcal{R}$ and $\mathcal{SR}$ are hereditary and hence the radical is ideal-hereditary, that is, if I $\lhd$ R, then $\mathcal{R}$(I) = I $\cap\mathcal{R}$(R).

Proposition 0.4.2. *(Proposition 3.3 of [3]) The class of all equiprime near-rings is closed under essential left invariant extensions.*

Proposition 0.4.3. *(Corollary 2.4 of [3]) The class of all equiprime near-rings satisfy condition (F_l).*

$\mathcal{F}$ denotes the class of all near-rings R in which the constant part R_c of R is an ideal of R. In [4], Fuchs has shown that the class of all near-rings $\mathcal{F}$ is a variety. Obviously, $\mathcal{F}$ contains all zero-symmetric, constant and abstract affine near-rings.