Introduction

The Jacobson radical is a very significant radical in rings. It is an ideal-hereditary Kurosh-Amitsur radical (KA-radical) in the class of all rings and it gives several important structure theorems in rings. Wedderburn-Artin theorem is one of the basic structure theorems in rings. Wedderburn-Artin theorem states that:
"A ring R is a non-zero Jacobson semisimpe right (left) Artinian if and only if it is isomorphic to a direct product of a finite number of full matrix rings of finite ranks over division rings."
It is well known in ring theory that even though the right and the left Jacobson radicals are the same for rings which we call the Jacobson radical, a right (left) primitive ideal need not be a left (right) primitive ideal.

To extend the Wedderburn-Artin theorem of rings to near-rings, we need both matrix near-rings and Jacobson radicals of near-rings. We also should know whether the existing Jacobson radicals are useful for the extension of the Wedderburn-Artin theorem to near-rings.

In this thesis we consider only right near-rings and R is a right near-ring. So far, only left Jacobson radicals were developed and studied for near-rings. The (left)

Jacobson radicals J_ν, $\nu \in \{0, 1, 2\}$ were developed and studied by G. Betsch [1] in 1963 and the (left) Jacobson radical J_3 was introduced and studied by W. M. L. Holcombe [7] in 1982. Using the radicals J_ν, $\nu \in \{0, 1, 2\}$, many structure theorems were developed for near-rings including a density-like structure theorem for zero-symmetric 0-primitive near-rings based on the density theorem for primitive rings. Almost all the structure theorems developed based on these radicals, as expected, decomposes certain near-rings as direct sum of minimal left ideals and not as direct sum of minimal right ideals. It is well known that J_2 and J_3 are ideal-hereditary KA-radicals in the class of all zero-symmetric near-rings. But J_2 is not a KA-radical in the class of all near-rings. It is not known whether J_3 is a KA-radical in the class of all near-rings. Moreover, J_0 and J_1 are not KA-radicals in the class of all zero-symmetric near-rings. S. Veldsman [27] introduced the (left) Jacobson radicals $J_{2(0)}$ and $J_{3(0)}$ for near-rings. $J_{2(0)}$ and $J_{3(0)}$ are KA-radicals in the class of all near-rings and are ideal-hereditary in the class of all zero-symmetric near-rings.

Initially only for certain special classes of near-rings R, the near-ring of $n \times n$-matrices over R were defined. In 1986, treating a matrix as mapping, matrix near-rings over general near-rings were introduced by J.D.P. Meldrum and A.P.J. van der Walt [11]. A lot of research has been done on the ideals and radicals of the matrix near-rings (see for example [10], [14], [26], [13] and [25]). J.H. Meyer [12], has shown that if R is an infinite near-field which is not a division ring, then the matrix near-ring $M_n(R)$ does't satisfy DCC on left ideals, for $n \geq 2$. This is in contrast to the ring case where $M_n(D)$ is a finite direct sum of minimal left ideals satisfying both DCC and ACC on left ideals, D is a division ring.

Structure of matrix near-rings was studied in R. Srinivasa Rao [16] and [17]. In [16], it is shown that a zero-symmetric near-ring R with identity is a matrix near-ring if and only if R has a set of matrix units and satisfies two other conditions. These matrix units decomposes a near-ring as a direct sum of right ideals and not necessarily as a direct sum of left ideals. In [17], it is shown that if R is a near-field, then $M_n(R)$ is a direct sum of n minimal right ideals. So, right ideals are relevant for the extension of the Wedderburn-Artin theorem to near-rings.

It is clear from these papers that decomposition of $M_n(R)$ as a direct sum of right ideals is a natural decomposition and $M_n(R)$ can't be decomposed all the time as a direct sum of left ideals and even if it is decomposed it can't be a natural decomposition. So, the right Jacobson-type radicals have an important role to play in the study of Meldrum-van der Walt matrix near-rings. With this motivation, in this thesis we develop the right Jacobson-type radicals of near-rings. The aim of this thesis is to give a good beginning in this direction of investigation. We show that the right Jacobson radicals provide us a decomposition for certain classes of near-rings as a direct sum of minimal right ideals. Also, we show that these right Jacobson radicals provide an analogue of Wedderburn-Artin theorem in near-rings.
In 2005, J.F.T. Hartney and D.S. Rusznyak [5] introduced the right Jacobson radical of type-0 for distributively generated (d.g.) near-rings.
Let $\nu \in \{0, 1, 2\}$. In this thesis, we are introducing right Jacobson radicals J^r_ν, $J^r_{\nu(e)}$ for general near-rings which generalize the Jacobson radical of rings. These are all distinct and are distinct from the existing left Jacobson radicals of near-rings. Let

(G, +) be a non-zero finite group. We know that for the simple near-ring $M_0(G)$, $J_\nu(M_0(G)) = \{0\}$, $\nu \in \{0, 1, 2, 3\}$. The right Jacobson radicals of $M_0(G)$ depend on the structure of the group G. Even though $J_0^r(M_0(G)) = \{0\}$, for many classes of groups G, we have $J_\nu^r(M_0(G)) = M_0(G)$, $\nu \in \{1, 2\}$. For the d.g. near-ring $M_0(G)$, we have $J_2^r(M_0(G)) = M_0(G)$ and $J_3(M_0(G)) = \{0\}$, where G is a finite simple non-abelian additive group. The radicals J_ν^r are introduced in [21], [22] and [19].
Moreover, unlike in the left Jacobson radicals, almost all the near-rings with trivial multiplication are J_ν^r-radical near-rigs, $\nu \in \{0, 1, 2\}$. Right quasi-regular elements of type-ν, right ν-modular right ideals, right ν-primitive ideals are introduced in near-rings. $J_0^r(R)$ is the largest right quasi-regular ideal of R. $J_\nu^r(R)$ is the largest right quasi-regular ideal of R of type-ν, $\nu \in \{1, 2\}$. There are right quasi-regular near-rings which are J_2-semisimple. So, there is no much comparison between right and left quasi-regularity and hence between the left and right Jacobson radicals in near-rings. We have $P(R) \subseteq N(R) \subseteq J_0^r(R) \subseteq J_1^r(R) \subseteq J_2^r(R)$, where P and N are prime and nil radicals of near-rings. Also, we have $P(R) \subseteq N(R) \subseteq J_{0(e)}^r(R) \subseteq J_{1(e)}^r(R) \subseteq J_{2(e)}^r(R)$ and $J_\nu^r(R) \subseteq J_{\nu(e)}^r(R)$, $\nu \in \{0, 1, 2\}$. If R is a zero-symmetric near-ring satisfying DCC on (left) R-subgroups of R, then $J_0(R) \subseteq J_0^r(R)$.
Semisimple classes of the right Jacobson radicals of near-rings are also studied. Some analogues of the Wedderburn-Artin theorem to near-rings are obtained. It is shown that a right ν-primitive d.g. near-ring R satisfying DCC on right ideals is isomorphic to a matrix near-ring $M_n(B)$, where n = dim R and the near-ring B is a right B-group of type-ν, $\nu \in \{1, 2\}$. If R is a non-zero d.g. J_ν^r-semisimple near-ring satisfying DCC on right ideals of R, then R is a direct sum of a finite number of minimal ideals which are right ν-primitive d.g. near-rings satisfying DCC on right ideals, $\nu \in \{1, 2\}$. It

is also shown that if R $\neq \{0\}$ satisfies DCC on right ideals and $D_2^r(R) = \{0\}$, then R is a direct product of a finite number of simple near-rings T_i each of which is a right 2-primitive near-ring with $D_2^r(T_i) = \{0\}$, $D_\nu^r(R)$ is the intersection of all right ν-modular right ideals of R.

It is shown that J_ν^r is a KA-radical in the class of all near-rings R in which the constant part of R is an ideal of R. Hence, they are KA-radicals in the class of all zero-symmetric near-rings. But, they are not ideal-hereditary radicals in that class. Moreover, $J_{\nu(e)}^r$ is a KA-radical in the class of all near-rings and is an ideal-hereditary radical in the class of all zero-symmetric near-rings.

This thesis is divided into five chapters. In chapter 0, existing literature relevant to the work of this thesis is presented.
In Chapter 1, we introduce and study J_0^r, the right Jacobson radical of type-0, for general near-rings. Left quasi-regularity was introduced and studied in near-rings. To facilitate the study of J_0^r, in this chapter, we introduce right quasi-regularity, right R-groups of type-0, right modular right ideal and right 0-primitive ideal in near-rings. It is shown that J_0^r is a radical map and $J_0^r(R)$ is the largest right quasi-regular ideal of R. A right 0-primitive ideal of R is a prime ideal of R. Moreover, $P(R) \subseteq N(R) \subseteq J_0^r(R)$, where P and N are the prime and nil radicals of R. If (G, +) is a group, then $J_0^r(M_0(G)) = J_\nu(M_0(G)) = \{0\}$, where ν = 0, 1, 2. Let (R, +) be a group and S be a non-empty subset of R and $0 \notin S$. Let (R, +, .) be the trivial near-ring determined by S. If S = R - {0}, then $J_0^r(R) = R$ and $J_\nu(R) = \{0\}$, ν = 0, 1, 2. So, R is right quasi-regular and {0} is the largest left quasi-regular ideal of R. Hence, left

and right quasi-regularities differ sharply in near-rings. Moreover, $J_0^r(R) = R$ if and only if R - S is not a normal subgroup of (R, +) of index 2. Therefore, unlike the left Jacobson radical classes, the right Jacobson radical class of type-0 contains almost all the classes of near-rings with trivial multiplication.

The content of this chapter forms the paper [21], 'A radical for right near-rings: The right Jacobson radical of type-0' which was published in the 'International Journal of Mathematics and Mathematical Sciences' in 2006.

In chapter 2, the right Jacobson radicals of type-1 and 2 for near-rings are introduced and studied. It is proved that both of them are radicals for near-rings. Let (G, +) be a finite group. We know that the left ideals of $M_0(G)$ do not depend on the structure of the group G and hence left Jacobson radicals also do not depend on the structure of G. We know that $J_\nu(M_0(G)) = \{0\}$, $\nu \in \{0, 1, 2\}$. But, we see in this chapter that the right Jacobson radicals of $M_0(G)$ depend on the nature of G and many of them are right Jacobson radical near-rings of type-2, in contrast to the left Jacobson radicals. Moreover, $M_0(G)$ is a right Jacobson radical near-ring of type-2 if and only if G has no subgroup of index 2. Also, $M_0(G)$ is a right Jacobson radical near-ring of type-1 if and only if G has no maximal normal subgroup which is also a maximal subgroup. Unlike left Jacobson-type radicals, the constant part of R is contained in every right 1-modular (2-modular) right ideal of R. For any family of near-rings R_i, $i \in I$, $J_\nu^r(\oplus_{i\in I} R_i) = \oplus_{i\in I} J_\nu^r(R_i)$, where J_ν^r is the right Jacobson radical of type-ν, $\nu \in \{0, 1, 2\}$. In a near-ring, right quasi-regular elements of type-ν are introduced and shown that $J_\nu^r(R)$ is the largest right quasi-regular ideal of R of type-ν, $\nu \in \{0, 1, 2\}$. It is also shown that if R is a d.g. near-ring, then $J_\nu^r(R)$ is the largest

right quasi-regular right ideal of R of type-ν, $\nu \in \{1, 2\}$.
Most of the results of sections 2.2 and 2.3 of this chapter form a part of the paper [22], 'Two more radicals for right near-rings: The right Jacobson radicals of type-1 and 2' which was published in the 'Kyungpook Mathematical Journal' in 2006.

Let $\nu \in \{0, 1, 2\}$. In chapter 3, properties of the right Jacobson radical of type-ν are studied. It is known that the left Jacobson radicals of type-0 and 1 are not KA-radicals in the class of all zero-symmetric near-rings and only the left Jacobson radicals of type-2 and 3 are KA-radicals in the class of all zero-symmetric near-rings. Surprisingly, J_0^r, the right Jacobson radical of type-0, is a KA-radical in the class of all zero-symmetric near-rings. It is also shown that J_ν^r is a KA-radical even in a bigger class of near-rings namely in the variety of all near-rings R in which the constant part of R is an ideal of R. But J_ν^r is not s-hereditary and hence not an ideal-hereditary radical in the class of all zero-symmetric near-rings.
The content of section 3.3 of this chapter forms the paper [24], 'Kurosh-Amitsur right Jacobson radicals of type-1 and 2 for right near-rings' which was ~~Communicated to~~ accepted '~~Results in Mathematics~~' for publication in 'Results in Mathematics'.

In [16], R.Srinivasa Rao gave a necessary and sufficient condition for a zero-symmetric near-ring with identity to be a matrix near-ring in terms of matrix units. It was also shown that a zero-symmetric near-ring with identity has matrix units if and only if it is a finite direct sum of isomorphic right ideals. In order to extend the Wedderburn-Artin theorem of rings to near-rings and to develop a relevant structure theory for near-rings, we introduced and studied the right Jacobson radicals

J^r_ν for general near-rings in chapters 1, 2 and 3. In chapter 4, the corresponding semisimple near-rings are studied. Using these right Jacobson radicals, a form of the Wedderburn-Artin theorem of rings is extended to near-rings in terms of matrix near-rings. Unlike in rings there is a simple near-ring with identity satisfying DCC on right ideals which is a direct sum of minimal non-isomorphic right ideals. So, not all forms of Wedderburn-Artin theorem of rings can be extended to general near-rings. Corresponding to each right Jacobson radical we get a generalization of Wedderburn-Artin theorem to near-rings. It is shown that a right ν-primitive d.g. near-ring R satisfying DCC on right ideals is isomorphic to a matrix near-ring $M_n(B)$, where n = dim R and the near-ring B is a right B-group of type-ν, $\nu \in \{1, 2\}$. If R is a non-zero d.g. J^r_ν-semisimple near-ring satisfying DCC on right ideals of R, then R is a direct sum of a finite number of minimal ideals which are right ν-primitive d.g. near-rings satisfying DCC on right ideals, $\nu \in \{1, 2\}$. It is also shown that if R $\neq \{0\}$ satisfies DCC on right ideals and $D^r_2(R) = \{0\}$, then R is a direct product of a finite number of simple near-rings T_i each of which is a right 2-primitive near-ring with $D^r_2(T_i) = \{0\}$. If R has DCC on right ideals and $D^r_2(R) = \{0\}$ and any two right R-groups of type-2 are R-isomorphic, then R is simple and has a subnear-ring isomorphic to $M_n(eRe)$, where e is a right 2-primitive idempotent in R, eRe is a near-field and n = dim R. If, in addition, R is finite and eRe is a non-ring for some right 2-primitive idempotent e $\in$ R, then R is isomorphic to $M_n(eRe)$. Also, some equivalent conditions are developed for a near-ring R satisfying DCC on right ideals of R having $D^r_1(R) = \{0\}$.

The content of this chapter forms the paper [20], 'Right semisimple right near-rings' which was communicated to 'Novi Sad Journal of Mathematics' for publication.

The right Jacobson radicals of type-ν, $\nu \in \{0, 1, 2\}$ were introduced and studied in the previous chapters. In chapters 3, we have shown that the right Jacobson radicals of type-0, 1 and 2 are Kurosh-Amitsur radicals (KA-radicals) in the class of all zero-symmetric near-rings but they are not ideal-hereditary in that class. If G is a right R-group of type-ν, then its annihilator $(0 : G) = \{r \in R \mid Gr = \{0\}\}$ need not be an ideal of R, $\nu \in \{0, 1, 2\}$. Now we develop some more right Jacobson radicals for near-rings which eliminate these problems. Let $\nu \in \{0, 1, 2\}$. In chapter 5, right R-groups of type-ν(e), right ν(e)-primitive ideals and right ν(e)-primitive near-rings are introduced. Using them the right Jacobson radical of type-ν(e) is introduced for right near-rings and is denoted by $J^r_{\nu(e)}$. If G is a right R-group of type-$\nu(e)$, then $(0 : G)$ is an ideal of R. A right 0(e)-primitive ideal of R is an equiprime ideal of R. It is shown that $J^r_{\nu(e)}$ is a Kurosh-Amitsur radical in the class of all near-rings and is an ideal-hereditary radical in the class of all zero-symmetric near-rings. Moreover, for any ideal I of R, $J^r_{\nu(e)}(I) \subseteq J^r_{\nu(e)}(R) \cap I$ with equality, if I is left invariant.

The content of section 5.2 of this chapter forms the paper [23], 'Hereditary right Jacobson radical of type-0(e) for right near-rings' which was communicated to the 'Journal of the Australian Mathematical Society' for publication.

The content of section 5.3 of this chapter forms the paper [18], 'Hereditary right Jacobson radicals of type-1(e) and 2(e) for right near-rings' which was communicated to 'Algebra Colloquium' for publication.