# Chapter 4

# Right Semisimple Right Near-Rings

Near-rings considered are right near-rings. In [16] and [17], R. Srnivasa Rao has shown that the right ideals are relevant for the identification of Meldrum-van der Walt matrix near-rings using matrix units. So, in general one can't obtain an extension of the Wedderburn-Artin theorem of rings to near-rings in terms of matrix near-rings by replacing the Jacobson radical of rings with a left Jacobson radical of near-rings. In order to extend the Wedderburn-Artin theorem of rings to near-rings and to develop a relevant structure theory for near-rings, we have already introduced and studied the right Jacobson radicals $J^r_\nu$, $\nu \in \{0, 1, 2\}$ in chapters 1, 2 and 3. In this chapter semisimple near-rings corresponding to these right Jacobson radicals are studied. It is shown that the right Jacobson radicals developed so far in fact extend a form of the Wedderburn-Artin theorem to near-rings in terms of matrix near-rings. It is also shown that a right $\nu$-primitive d.g. near-ring R satisfying DCC on right ideals is isomorphic to a matrix near-ring $M_n(B)$, where n = dim R and the near-ring B is a right B-group of type-$\nu$, $\nu \in \{1, 2\}$.

## 4.1 Introduction

Throughout this chapter R stands for a right near-ring and all near-rings considered are right near-rings. In [16], R. Srinivasa Rao gave a necessary and sufficient condition for a zero-symmetric near-ring with identity to be a matrix near-ring in terms of matrix units. It was also shown that a zero-symmetric near-ring with identity has matrix units if and only if it is a finite direct sum of isomorphic right ideals. So, in general one can't obtain an extension of the Wedderburn-Artin theorem of rings to near-rings in terms of matrix near-rings by replacing the Jacobson radical of a ring with a left Jacobson radical of near-rings. In order to extend the Wedderburn-Artin theorem of rings to near-rings and to develop a relevant structure theory for near-rings, we introduced and studied the right Jacobson radicals $J^r_\nu$ for general near-rings in chapters 1 and 2. In this chapter, the corresponding semisimple near-rings are studied. Using these right Jacobson radicals, a form of the Wedderburn-Artin theorem of rings is extended to near-rings in terms of matrix near-rings. Unlike in rings there is a simple near-ring with identity satisfying DCC on right ideals which is a direct sum of minimal non-isomorphic right ideals. So, not all forms of Wedderburn-Artin theorem of rings can be extended to general near-rings. Corresponding to each right Jacobson radical we get a generalization of Wedderburn-Artin theorem to near-rings. It is shown that a right $\nu$-primitive d.g. near-ring R satisfying DCC on right ideals is isomorphic to a matrix near-ring $M_n(B)$, where n = dim R and the near-ring B is a right B-group of type-$\nu$, $\nu \in \{1, 2\}$. It is also shown that if R satisfies DCC on right ideals and $D^r_2(R) = \{0\}$, then R is a direct product of a finite number of simple near-rings $T_i$ each of which is a right 2-primitive near-ring with $D^r_2(T_i) = \{0\}$. If R is a simple near-ring with $D^r_2(R) = \{0\}$ and any two right R-groups of type-2 are

R-isomorphic, then R has a subnear-ring isomorphic to $M_n(S)$, where S is a near-field and n = dim R. If, in addition, R is finite and eRe is a non-ring for some right 2-primitive idempotent $e \in R$, then R is isomorphic to $M_n(S)$. Also, some equivalent conditions are developed for a near-ring R satisfying DCC on right ideals of R having $D^r_1(R) = \{0\}$.

This chapter is divided into three sections. In section 2, completely reducible right R-groups and right completely reducible near-rings are introduced and studied. If R has DCC on right ideals and $D^r_\nu(R) = \{0\}$, then it is shown that R is right completely reducible and some of its properties are studied. Some equivalent conditions are developed for a near-ring R satisfying DCC on right ideals with $D^r_1(R) = \{0\}$.

In section 3, right semisimple near-rings with matrix units are studied. Near-rings satisfying DCC on right ideals are considered in this section. Conditions are developed under which a right semisimple near-ring has matrix units. Some generalizations of the Wedderburn-Artin theorem of rings to near-rings are developed.

## 4.2 Right Semisimple Right Near-Rings

Throughout this section R stands for a non-zero right near-ring.

In this section, completely reducible right R-groups and right completely reducible near-rings are introduced and studied. If R has DCC on right ideals and $D_{\nu}^{r}(R) = \{0\}$, then it is shown that R is right completely reducible and some of its properties are studied. Some equivalent conditions are developed for a near-ring R satisfying DCC on right ideals with $D_{1}^{r}(R) = \{0\}$.

**Definition 4.2.1.** Let G be a right R-group. G is said to be *completely reducible* if it is a direct sum of simple ideals. *An R-subgroup H of G is said to be completely reducible* if H is completely reducible as a right R-group. R is said to be *right completely reducible* if the right R-group R is completely reducible.

**Proposition 4.2.1.** *Let G be a right R-group and let $\{H_i \mid i \in I\}$ be a non-empty collection of maximal ideals of G. If G has DCC on ideals and $\cap_{i\in I} H_i = \{0\}$, then G is completely reducible.*

*Proof.* Let $\mathbb{A} := \{H_i \mid i \in I\}$ and $i_1 \in I$. $H_{i_1}$ is a maximal ideal of G. Since $\cap_{i\in I} H_i = \{0\}$, if $H_{i_1} \neq \{0\}$, then we get an $i_2 \in I$ such that $H_{i_1} \cap H_{i_2} \neq H_{i_1}$. If $H_{i_1} \cap H_{i_2} \neq \{0\}$, then as $\cap_{i\in I} H_i = \{0\}$, we get an $i_3 \in I$ such that $H_{i_3} \cap (H_{i_1} \cap H_{i_2}) \neq H_{i_1} \cap H_{i_2}$. This way we get $H_{i_1} \supset H_{i_1} \cap H_{i_2} \supset H_{i_1} \cap H_{i_2} \cap H_{i_3} \ldots$ , a descending chain of distinct ideals. Since G has DCC on ideals, we get an n such that $\cap_{k=1}^{n} H_{i_k} = \{0\}$. So, we get a finite number of maximal ideals $H_1, H_2, \ldots, H_r$ in $\mathbb{A}$ such that $\cap_{i=1}^{r} H_i = \{0\}$ and $K_j := \cap_{i=1, i\neq j}^{r} H_i \neq \{0\}$, j = 1, 2, ... , r. We show now that $G = K_1 \oplus K_2 \oplus \ldots \oplus K_r$. If r = 1, then it is clear that $G = K_1$ as desired. Suppose that r > 1. For any

$1 \leq j \leq r$, $K_j \cap H_j = \{0\}$. Since $K_j \neq \{0\}$, $K_j \not\subseteq H_j$. So, we have $G = K_j \oplus H_j$ and hence $K_j$ is R-isomorphic to $G/H_j$. Therefore, $K_j$ is a minimal ideal of G. Let $T_s = \cap_{i=1}^{s} H_i$, s = 1, 2, ... , r. We claim that $G = K_1 \oplus K_2 \oplus ... \oplus K_s \oplus T_s$, s = 1, 2, ... , r. We prove this by induction on s. If s = 1, then $G = K_1 \oplus H_1 = K_1 \oplus T_1$ is clear. Assume now that it is true for $1 \leq s < r$. We prove it for s + 1. Now $H_{s+1} + T_s = G$, as $H_{s+1}$ is a maximal ideal of G and $T_s \not\subseteq H_{s+1}$. Also, $G/H_{s+1} = (H_{s+1} + T_s)/H_{s+1} \simeq T_s/(H_{s+1} \cap T_s) = T_s/T_{s+1}$. Since $G/H_{s+1}$ is a simple right R-group, $T_s/T_{s+1}$ is also a simple right R-group. So, $T_{s+1}$ is maximal in $T_s$. Now $K_{s+1} \cap T_{s+1} = (\cap_{i=1, i\neq s+1}^{r} H_i) \cap (\cap_{i=1}^{s+1} H_i) = \{0\}$ and $K_{s+1} \subseteq T_s$. Therefore, $T_s = K_{s+1} \oplus T_{s+1}$. Hence $G = K_1 \oplus K_2 \oplus ... \oplus K_{s+1} \oplus T_{s+1}$. Since $T_r = \{0\}$, we have $G = K_1 \oplus K_2 \oplus ... \oplus K_r$. □

We immediately have the following Corollary.

**Corollary 4.2.2.** *Let $\{M_i \mid i \in I\}$ be a non-empty collection of maximal right ideals of R. If R has DCC on right ideals of R and $\cap_{i \in I} M_i = \{0\}$, then R is right completely reducible.*

In [9], Jordan-Holder-theory for Ω-groups was presented. Therefore, as a special case, Jordan-Holder theorem also holds for right R-groups. So, we have the following result.

**Proposition 4.2.3.** *Let R be the direct sum of the minimal right ideals $K_1$, $K_2$, ... , $K_n$ of R. If R has another decomposition as a direct sum of the minimal right ideals $T_1$, $T_2$, ... , $T_m$ of R, then n = m.*

In view of Proposition 4.2.3, we have the following definition.

**Definition 4.2.2.** If R is a direct sum of n minimal right ideals of R, then the dimension of R denoted by dim R, is defined as dim R = n.

**Proposition 4.2.4.** *Suppose that G is a right R-group and* $G = H_1 \oplus H_2 \oplus \ldots \oplus H_r$*,* $H_i$ *are minimal ideals of G. If* $\{0\} \neq H$ *is an ideal of G, then there is a subset I of* $\{1, 2, \ldots, r\}$ *such that H and* $\sum_{i\in I} H_i$ *are isomorphic as right R-groups and* $G = H \oplus \sum_{j\in J} H_j$*, where* $J = \{1, 2, \ldots, r\}$ *- I.*

*Proof.* Let $\{0\} \neq$ H be an ideal of G. Let $T_s = H + H_1 + H_2 + \ldots + H_s$, $s \in A := \{1, 2, \ldots, r\}$. Now $T_s$ is an ideal of G and $T_r = H + H_1 + H_2 + \ldots + H_r = G$. Since $H_{s+1}$ is a minimal ideal of G, $H_{s+1} \subseteq T_s$ or $H_{s+1} \cap T_s = \{0\}$. So, either $T_s = T_{s+1}$ or $T_{s+1} = T_s \oplus H_{s+1}$. Therefore, we get a subset I of A such that $G = H \oplus \sum_{i\in I} H_i$. So, $G/\sum_{i\in I} H_i$ and H are isomorphic as right R-groups. Let J = A - I. Since $G/\sum_{i\in I} H_i$ and $\sum_{j\in J} H_j$ are isomorphic as right R-groups, we have H and $\sum_{j\in J} H_j$ are isomorphic as right R-groups. □

The following Corollary is immediate from Proposition 4.2.4.

**Corollary 4.2.5.** *Let* $R = K_1 \oplus K_2 \oplus \ldots \oplus K_r$*,* $K_i$ *are minimal right ideals of R. If* $\{0\} \neq K$ *is a right ideal of R, then there is a subset I of* $\{1, 2, \ldots, r\}$ *such that K and* $\sum_{i\in I} K_i$ *are isomorphic as right R-groups and* $R = K \oplus \sum_{j\in J} K_j$*, where* $J = \{1, 2, \ldots, r\}$ *- I.*

**Definition 4.2.3.** Let $\nu \in \{0, 1, 2\}$. A near-ring R is called *$J_\nu^r$-semisimple* if $J_\nu^r(R) = \{0\}$.

The following two Propositions are trivial and we state them without proofs.

**Proposition 4.2.6.** *Let $\nu \in \{0, 1, 2\}$. Then R is $J^r_\nu$-semisimple if and only if R is a subdirect product of right $\nu$-primitive near-rings.*

**Proposition 4.2.7.** *Let $\nu \in \{0, 1, 2\}$. Let R satisfies DCC on (right) ideals of R. Then R is $J^r_\nu$-semisimple if and only if R is a subdirect product of a finite number of right $\nu$-primitive near-rings satisfying DCC on (right) ideals.*

**Proposition 4.2.8.** *Let $\nu \in \{0, 1, 2\}$. If R has DCC on right ideals and $D^r_\nu(R) = \{0\}$, then*

1. *R has a left identity.*
2. *$R = K_1 \oplus K_2 \oplus \ldots \oplus K_n$, $K_i$ is a minimal right ideal of R and is a right R-group of type-$\nu$. In particular, R is right completely reducible.*
3. *for every non-zero right ideal K of R, there is a subset I of $A := \{1, 2, \ldots, n\}$ such that K and $\sum_{i\in I} K_i$ are isomorphic as right R-groups and $R = K \oplus (\sum_{j\in J} K_j)$, $J = A - I$. In particular, a minimal right ideal of R is a right R-group of type-$\nu$.*
4. *a right R-group of type-$\nu$ is R-isomorphic to $K_j$, for some $j \in A$ and hence there are only a finite number of mutually non-R-isomorphic right R-groups of type-$\nu$.*
5. *R is zero-symmetric, for $\nu \in \{1, 2\}$.*

*Proof.* (1) Since $D^r_\nu(R) = \{0\}$ and R has DCC on right ideals, by Proposition 1.2.7, we get that R has a left identity.

(2) Again, since $D^r_\nu(R) = \{0\}$ and R has DCC on right ideals, as seen in the proof of

Proposition 4.2.1, we get a finite number of right $\nu$-modular right ideals $M_1$, $M_2$, ... , $M_n$ with $\cap_{i=1}^{n} M_i = \{0\}$ and minimal right ideals $K_1$, $K_2$, ... , $K_n$ of R such that R $= K_1 \oplus K_2 \oplus ... \oplus K_n$, where $K_j = \cap_{i=1, i\neq j}^{n} M_i$ and $R = K_j \oplus M_j$ for all j = 1, 2, ... , n. Since $R/M_j = (K_j + M_j)/M_j \simeq K_j$ as right R-groups and since $R/M_j$ is a right R-group of type-$\nu$, we have that $K_j$ is a right R-group of type-$\nu$, j = 1, 2, ... , n.

(3) We have $R = K_1 \oplus K_2 \oplus ... \oplus K_n$, $K_j$ is a minimal right ideal and is a right R-group of type-$\nu$. Let K be a non-zero right ideal of R. Now by Corollary 4.2.5, we get a subset I of {1, 2, .... , n} such that K and $\sum_{i\in I} K_i$ are isomorphic as right R-groups and $R = K \oplus (\sum_{j\in J} K_j)$, J = {1, 2, ... , n} - I. If K is a minimal right ideal of R, then K is R-isomorphic to $K_j$ for some j ∈ {1, 2, ... , n}.

(4) Let G be a right R-group of type-$\nu$. We have $R = K_1 \oplus K_2 \oplus ... \oplus K_n$ and $K_j$ is a minimal right ideal of R and is a right R-group of type-$\nu$. Let g be a generator of G. So, gR = G. Now $G = gR = gK_1 + gK_2 + ... + gK_n$. Since $G \neq \{0\}$, for some $1 \leq j \leq n$, $gK_j \neq \{0\}$. Clearly $gK_j$ is an R-subgroup of G. Let y ∈ G and $x \in gK_j$. Now y = gr and x = gk for some r ∈ R and $k \in K_j$. y + x - y = gr + gk -gr = g(r + k -r) $\in gK_j$ as $K_j$ is a right ideal of R. So $gK_j$ is a non-zero ideal of G and hence $gK_j = G$. f : $K_j \rightarrow G$ defined by f(k) = gk is a non-zero R-homomorphism of $K_j$ onto G. Since $K_j$ is a right R-group of type-$\nu$, we have Ker f = {0}. Therefore, G is R-isomorphic to $K_j$. Hence there can be utmost n non-R-isomorphic right R-groups of type-$\nu$.

(5) By Corollary 2.2.16, $R_c$, the constant part of R is contained in $D_\nu^r(R)$, for $\nu \in \{1, 2\}$. Since $D_\nu^r(R) = \{0\}$, we have that $R_c = \{0\}$ and hence R is zero-symmetric. □

**Proposition 4.2.9.** *Let $\nu \in \{0, 1, 2\}$. Suppose that R has DCC on right ideals and $D_\nu^r(R) = \{0\}$. If G is a monogenic right R-group, then*

1. *$G = H_1 \oplus H_2 \oplus ... \oplus H_n$, $H_i$ are minimal ideals of G and are right R-groups of type-$\nu$.*

2. *For every non-zero ideal H of G, there is a subset I of A:=$\{1, 2, \ldots, n\}$ such that H is R-isomorphic to $\oplus\sum_{i=I} H_i$ and $G = H \oplus (\oplus\sum_{j=J} H_j)$, $J = A - I$. In particular, a minimal ideal of G is a right R-group of type-$\nu$.*

*Proof.* Since R satisfies DCC on right ideals and $D^r_\nu(R) = \{0\}$, by Proposition 4.2.8, we have $R = K_1 \oplus K_2 \oplus ... \oplus K_n$, $K_i$ is a minimal right ideal of R and is a right R-group of type-$\nu$. Let g be a generator of G. Now $G = gR = gK_1 + gK_2 + ... + gK_n$. As seen in Proposition 4.2.8, it is clear that $gK_i$ is an ideal of G and if $gK_i \neq \{0\}$, then $gK_i$ is R-isomorphic to $K_i$ and hence $gK_i$ is a right R-group of type-$\nu$. So G is a sum of a finite number of minimal ideals $gK_i$ of G. Hence, G is a direct sum of some of the minimal ideals $gK_i$ and each of which is a right R-group of type-$\nu$. This proves (1).

(2) It follows from (1) and Proposition 4.2.4. □

**Theorem 4.2.10.** *Let R be a near-ring. Then the following statements are equivalent:*

1. *$D^r_1(R) = \{0\}$ and R has DCC on right ideals.*

2. *R has a left identity and is a finite direct sum of right ideals which are right R-groups of type-1.*

3. *R has a left identity and if G is a monogenic right R-group and H is an R-subgroup of G, then there exists an ideal K of G such that $G = H + K$ and $H \cap K = \{0\}$.*

4. *R has a left identity and if L is a right R-subgroup of R, then there exists a right ideal K of R such that $R = K + L$ and $K \cap L = \{0\}$.*

5. *R has a left identity and if L and M are right R-subgroups of R with $L \subseteq M$, then there exists an ideal K of $M_R$ such that $M = K + L$ and $K \cap L = \{0\}$.*

*Proof.* (1) $\Rightarrow$ (2) It follows from Proposition 4.2.8.

(2) $\Rightarrow$ (3) Let $R = K_1 \oplus K_2 \oplus \ldots \oplus K_s$, $K_i$ is a minimal right ideal of R and a right R-group of type-1. Let $L_i = K_1 \oplus \ldots \oplus K_{i-1} \oplus K_{i+1} \oplus \ldots \oplus K_s$, $1 \leq i \leq s$. Now $R/L_i \simeq (K_i + L_i)/L_i \simeq K_i/(L_i \cap K_i) = K_i$, as right R-groups. So $L_i$ is a right 1-modular right ideal of R. Since $\cap_{i=1}^{s} L_i = \{0\}$, we have $D_1^r(R) = \{0\}$. Let G be a monogenic right R-group and H be an R-subgroup of G. If G = H, then there is nothing to prove. Suppose that $G \neq H$. By Proposition 4.2.9, $G = G_1 \oplus G_2 \oplus \ldots \oplus G_t$, $G_i$ are ideals of G and right R-groups of type-1. We get an $1 \leq i \leq t$ such that $G_i \not\subseteq H$. Now $H \cap G_i$ is an R-subgroup of $G_i$. Since $G_i$ is a right R-group of type-1, $H \cap G_i = \{0\}$ or $G_i$. Since $G_i \not\subseteq H$, we have $H \cap G_i = \{0\}$. If $H + G_i = G$, we get the result. Suppose that $H + G_i \neq G$. We get $1 \leq j \leq t$ such that $G_j \not\subseteq H + G_i$. Now $i \neq j$. By a similar argument, we get that $G_j \cap (H + G_i) = \{0\}$. This way we get $G_i, G_j, \ldots, G_k$ such that $H + (G_i \oplus G_j \oplus \ldots \oplus G_k) = G$ and $H \cap (G_i \oplus G_j \oplus \ldots \oplus G_k) = \{0\}$.

(3) $\Rightarrow$ (4) Since R has a left identity, R is a monogenic right R-group. So we get (4).

(4) $\Rightarrow$ (5) Let L and M be right R-subgroups of the right R-group R and $L \subseteq M$. By (4), there is a right ideal T of R such that $R = L + T$ and $L \cap T = \{0\}$. Since $L \subseteq M$, we have $L + (T \cap M) = (L + T) \cap M = R \cap M = M$ and $L \cap (T \cap M) = (L \cap T) \cap M = \{0\}$, $T \cap M$ is an ideal of the right R-group M.

(5) $\Rightarrow$ (2) Let K be a non-zero right ideal of R. We claim that K contains a minimal

right ideal of R. Choosing R for M in (5), we get a right ideal T of R such that R = K + T and K $\cap$ T = $\{0\}$. Now R/T = (K + T)/T $\simeq$ K/(K $\cap$ T) = K as right R-groups. Since R has a left identity, R/T has a generator and hence K is a monogenic right R-group. Let k be a generator of the right R-group K. By Zorn's lemma, we get a right ideal L of R which is maximal for the property that L $\subseteq$ K and k $\notin$ L, that is, L $\subset$ K. By (5), we get a right ideal C of R such that R = C $\oplus$ L. Now K = R $\cap$ K = (C $\oplus$ L) $\cap$ K = (C $\cap$ K) $\oplus$ L. Let $0 \neq$ a $\in$ C $\cap$ K. Let A be the right ideal of R generated by a. Since a $\notin$ L and L is maximal in K, we have L $\oplus$ A = K and hence A = C $\cap$ K. Therefore, C $\cap$ K is a minimal right ideal of R contained in K. Let $\mathcal{A} = \{K_\alpha \mid \alpha \in I\}$ be the collection of all minimal right ideals of R. We get a sub collection $\mathcal{B} = \{K_\beta \mid \beta \in J\}$ of $\mathcal{A}$ such that $\sum_{\alpha\in I} K_\alpha = \oplus\sum_{\beta\in J} K_\beta$. We claim that $\oplus\sum_{\beta\in J} K_\beta$ = R. Suppose that $\oplus\sum_{\beta\in J} K_\beta \neq$ R. By (5), we get a right ideal Q of R such that $(\sum_{\beta\in J} K_\beta) \oplus$ Q = R. As seen above, Q contains a minimal right ideal P of R. Now P $\not\subseteq \sum_{\alpha\in I} K_\alpha = \oplus\sum_{\beta\in J} K_\beta$ and P $\in \mathcal{A}$, a contradiction. Therefore, $\oplus\sum_{\beta\in J} K_\beta$ = R. Since R has a left identity, $\mathcal{B}$ is a finite set. Hence R is a direct sum of a finite number of minimal right ideals $K_1$, $K_2$, ... , $K_s$. By Proposition 0.3.1, each $K_i$ is a right R-group of type-0. Let S be an R-subgroup of $K_i$. By (5), we get an ideal U of $K_i$ such that $K_i$ = U + S and U $\cap$ S = $\{0\}$. Now U is a right ideal of R. Since $K_i$ is minimal, U = $\{0\}$ or U = $K_i$. Therefore, S = $K_i$ or S = $\{0\}$. Hence $K_i$ is a right R-group of type-1.

(2) $\Rightarrow$ (1) As seen in (2) $\Rightarrow$ (3), we get that $D_1^r(R) = \{0\}$. Since the right R-group R is completely reducible, R has both DCC and ACC on right ideals. □

**Theorem 4.2.11.** *If R has DCC on right ideals and $D_2^r(R) = \{0\}$, then R is a direct product of a finite number of simple near-rings $T_i$, each of which is a right 2-primitive* *near-ring with* $D_2^r(T_i) = \{0\}$.

*Proof.* The proof follows from Theorem 4.2.10, Proposition 0.3.1 and Theorem 0.3.13.

□

**Corollary 4.2.12.** *If* $R$ *has DCC on right ideals and* $D_2^r(R) = \{0\}$, *then* $R$ *has an identity.*

## 4.3 Right Semisimpe Right Near-Rings with Matrix Units

Throughout this section R stands for a non-zero right near-ring.

In this section, right semisimple near-rings with matrix units are studied. Near-rings satisfying DCC on right ideals are considered in this section. Conditions are developed under which a right semisimple near-ring has matrix units. Some generalizations of the Wedderburn-Artin theorem of rings to near-rings are developed.

**Proposition 4.3.1.** *Let $e_1$ and $e_2$ be distributive idempotents in R. If $e_1R$ and $e_2R$ are isomorphic as right R-groups, then $e_1Re_1$ and $e_2Re_2$ are isomorphic as near-rings.*

*Proof.* Let $e_1R$ and $e_2R$ be isomorphic right R-groups. By Proposition 0.3.3, we get distributive elements $e_{12}$ and $e_{21}$ in R such that $e_1e_{12}e_2 = e_{12}$, $e_2e_{21}e_1 = e_{21}$ and $e_{12}e_{21} = e_1$, $e_{21}e_{12} = e_2$. The mapping f of $e_1Re_1$ into $e_2Re_2$ defined by $f(e_1re_1) = e_{21}e_1re_1e_{12}$ is a near-ring isomorphism. □

**Definition 4.3.1.** Let $\nu \in \{0, 1, 2\}$. A distributive idempotent $e \in R$ is said to be right $\nu$-primitive if eR is a right R-group of type-$\nu$.

**Theorem 4.3.2.** *Let $\nu \in \{0, 1, 2\}$. Suppose that R has DCC on right ideals and $D^r_\nu(R) = \{0\}$. If any two right R-groups of type-$\nu$ are R-isomorphic, then R is simple and it has a subnear-ring which is isomorphic to the matrix near-ring $M_n(eRe)$, where $n = dim\ R$ and e is a right $\nu$-primitive idempotent in R. If, in addition R is d.g., then R is (isomorphic to) the matrix near-ring $M_n(eRe)$.*

*Proof.* Since R has DCC on right ideals and $D^r_\nu(R) = \{0\}$, by Proposition 4.2.8, R $= K_1 \oplus K_2 \oplus ... \oplus K_n$, where $K_i = e_iR$ is a minimal right ideal of R which is also a right R-group of type-$\nu$, $e_i$ is a distributive idempotent in R and $e_1 + e_2 + ... + e_n$ is a left identity of R and n = dim R. Now we claim that R is a simple near-ring. Let I be a non-zero ideal of R. Again by Proposition 4.2.8, I is a sum of a finite number of minimal right ideals of R and each of which is a right R-group of type-$\nu$. Let K be a minimal right ideal of R contained in I. By Proposition 4.2.8, we get a distributive idempotent $f_1$ of R such that $K = f_1R$ and is isomorphic to $e_iR$, for all $1 \leq i \leq n$. Now by Proposition 0.3.3, we get distributive elements c and d in R such that $e_icf_1 = c$, $f_1de_i = d$ and $cd = e_i$, $dc = f_1$. Since $f_1 \in K \subseteq I$, $d = f_1de_i \in I$. Now since $d \in I$ and c is distributive, $e_i = cd \in I$. Therefore, $K_i = e_iR \subseteq I$, for all $1 \leq i \leq n$ and hence $R \subseteq I$. So, I = R. Therefore, R is simple. Since R is simple and R has a left identity, R has an identity. Since $K_i$, $1 \leq i \leq n$ are all isomorphic right R-groups, by Proposition 0.3.4, R has $n^2$ matrix units. Let $C_j$ and S be as defined in the Remark 0.3.1. Also, since R is simple, the left annihilator of $Re_{ij}$ is zero in R and hence by Lemma 0.3.5, the left annihilator of $Re_{ij} = C_j$ in S is zero. Let e be a $\nu$-primitive idempotent in R. Therefore, by Proposition 4.3.1, Lemma 0.3.7 and Theorem 0.3.8, the subnear-ring S of R is isomorphic to $M_n(eRe)$. If, in addition R is distributively generated, then by Corollary 0.3.9, R = S and hence R is (isomorphic to) the matrix near-ring $M_n(eRe)$. □

**Lemma 4.3.3.** *Let $e_1$ and $e_2$ be right 1-primitive and right 0-primitive idempotents respectively in R. Then $e_1R$ and $e_2R$ are isomorphic as right R-groups if and only if $e_1Re_2$ contains a non-zero distributive element.*

*Proof.* Let $e_1se_2$ be a non-zero distributive element in $e_1Re_2$. Define $f : e_2R \to e_1R$

by $f(e_2x) = (e_1se_2)e_2x$. Clearly f is an R-homomorphism of $e_2R$ into $e_1R$. Since $0 \neq e_1se_2$ is a distributive element in $e_1R$ and since $e_1R$ is a right R-group of type-1, we have $e_1se_2R = e_1R$. Therefore, f is onto $e_1R$. Ker f $= \{0\}$ or $e_2R$. Since $f \neq 0$, Ker f $= \{0\}$. Hence $e_2R$ and $e_1R$ are isomorphic as right R-groups. On the other hand, if $e_1R$ and $e_2R$ are isomorphic as right R-groups, then by Proposition 0.3.3, we get that $e_1Re_2$ contains a non-zero distributive element. □

**Corollary 4.3.4.** *Let $e_1$ and $e_2$ be right 1-primitive idempotents in R. Then $e_1Re_2$ contains a non-zero distributive element if and only if $e_1R$ and $e_2R$ are isomorphic as right R-groups.*

**Theorem 4.3.5.** *Let $\nu \in \{1, 2\}$. Suppose that R has DCC on right ideals and $D^r_\nu(R) = \{0\}$. If $e_1Re_2$ contains a non-zero distributive element for all right $\nu$-primitive idempotents $e_1$ and $e_2$ in R, then R is simple and has a subnear-ring which is isomorphic to the matrix near-ring $M_n(eRe)$, where n = dim R and e is a right $\nu$-primitive idempotent in R.*

*Proof.* Suppose that $e_1Re_2$ contains a non-zero distributive element, for all right $\nu$-primitive idempotents $e_1$ and $e_2$ in R. By Corollary 4.3.4, any two right R-groups of type-$\nu$ are isomorphic. Let dim R = n. Therefore, by Theorem 4.3.2, R is simple and has a subnear-ring which is isomorphic to $M_n(T)$, where $T \simeq eRe$ and e is a right $\nu$-primitive idempotent in R. □

**Proposition 4.3.6.** *Let R be a near-ring with identity. Suppose that a non-zero distributive element in R has no right zero-divisors and has a right inverse in R. Then, a non-zero distributive element in R is a unit.*

*Proof.* Let a be a non-zero distributive element in R. We get an element $b \in R$ such that $ab = 1$. Let $x, y \in R$. $a(b(x + y) - (bx + by)) = ab(x + y) - (abx + aby) = (x + y) - (x + y) = 0$. Since a has no non-zero right zero-divisors, $b(x + y) - (bx + by) = 0$, that is, $b(x + y) = bx + by$. So, b is a distributive element in R. Now $(ba)^2 = b(ab)a = ba$. So, $ba(ba - 1) = 0$. Since ba has no non-zero right zero-divisors, $ba - 1 = 0$, that is, $ba = 1$. Therefore, a is a unit in R. □

**Lemma 4.3.7.** *Let R be a near-ring with identity and let n be a positive integer. Suppose that $f_{11}^1 M_n(R)$ is a right $M_n(R)$-group of type-0. Then, R is a right R-group of type-0. Hence, if a is a non-zero distributive element in R, then R and aR are isomorphic as right R-groups. Also, if in addition R is finite, then every non-zero distributive element in R is a unit.*

*Proof.* R is a right R-group, monogenic by the identity of R. Let K be a non-zero right ideal of R. Now $(K : R) = \{x \in R \mid xR \subseteq K\} = K$. Let $K_1 = \{(r_1, r_2, \ldots, r_n) \in R^n \mid r_1 \in K, r_2 = r_3 = \ldots = r_n = 0\}$. It is easy to see that $(K_1 : R^n) = \{A \in M_n(R) \mid A(r_1, r_2, \ldots, r_n) \in K_1$, for all $(r_1, r_2, \ldots, r_n) \in R^n\}$ is a right ideal of $M_n(R)$ and is contained in $f_{11}^1 M_n(R)$. Since $f_{11}^1 M_n(R)$ is a right $M_n(R)$-subgroup of $M_n(R)$, $(K_1 : R^n)$ is either $\{0\}$ or $f_{11}^1 M_n(R)$. For each $0 \neq k \in K$, we have $0 \neq f_{11}^k \in (K_1 : R^n)$. Therefore, $(K_1 : R^n) = f_{11}^1 M_n(R)$. So, $f_{11}^1 \in (K_1 : R^n)$ and hence $R \subseteq K$. Therefore, $K = R$ and hence R is a right R-group of type-0. Let a be a non-zero distributive element in R. It is clear that $f : R \to aR$ defined by $f(r) = ar$ is an R-homomorphism of R onto aR. Since $f \neq 0$ and $\text{Ker } f = \{r \in R \mid f(r) = 0\}$ is a right ideal of R, we get that $\text{Ker } f = \{0\}$. Therefore, R and aR are isomorphic as right R-groups. If R is finite, then $aR = R$. So, by Proposition 4.3.6, a is a unit in R. □

**Lemma 4.3.8.** *Let R be a near-ring with identity and let n be a positive integer. Suppose that $f_{11}^1 M_n(R)$ is a right $M_n(R)$-group of type-1. Then R is a right R-group of type-1. Hence, a non-zero distributive element in R is a unit.*

*Proof.* R is a right R-group, monogenic by the identity in R. Let K be a non-zero right R-subgroup of R. Now $(K : R) = \{x \in R \mid xR \subseteq K\} = K$. Let $K_1 = \{(r_1, r_2, \ldots, r_n) \in R^n \mid r_1 \in K, r_2 = r_3 = \ldots = r_n = 0\}$. It is easy to see that $(K_1 : R^n) = \{A \in M_n(R) \mid A(r_1, r_2, \ldots, r_n) \in K_1$, for all $(r_1, r_2, \ldots, r_n) \in R^n\}$ is a right $M_n(R)$-subgroup of $M_n(R)$ and contained in $f_{11}^1 M_n(R)$. Since $f_{11}^1 M_n(R)$ is a right $M_n(R)$-subgroup of $M_n(R)$, $(K_1 : R^n)$ is either $\{0\}$ or $f_{11}^1 M_n(R)$. For each $0 \neq k \in K$, we have $0 \neq f_{11}^k \in (K_1 : R^n)$. Therefore, $(K_1 : R^n) = f_{11}^1 M_n(R)$. So, $f_{11}^1 \in (K_1 : R^n)$ and hence $R \subseteq K$. Therefore, $K = R$ and hence R is a right R-group of type-1. Let a be a non-zero distributive element in R. Now $aR = R$, as aR is a non-zero right R-subgroup of R. So, by Proposition 4.3.6, a is a unit in R. □

**Lemma 4.3.9.** *Let R be a near-ring with identity and let n be a positive integer. Suppose that $f_{11}^1 M_n(R)$ is a right $M_n(R)$-group of type-2. Then R is a right R-group of type-2 and hence R is a near-field.*

*Proof.* R is a right R-group, monogenic by the identity in R. Let a be a non-zero element in R. Now $(aR : R) = \{x \in R \mid xR \subseteq aR\} = aR$. Let $(aR)_1 = \{(r_1, r_2, \ldots, r_n) \in R^n \mid r_1 \in aR, r_2 = r_3 = \ldots = r_n = 0\}$. It is easy to see that $f_{11}^a M_n(R) \subseteq ((aR)_1 : R^n) = \{A \in M_n(R) \mid A(r_1, r_2, \ldots, r_n) \in (aR)_1$, for all $(r_1, r_2, \ldots, r_n) \in R^n\} \subseteq f_{11}^1 M_n(R)$. Since $f_{11}^1 M_n(R)$ is a right $M_n(R)$-subgroup of $M_n(R)$ of type-2, $f_{11}^a M_n(R) \subseteq f_{11}^1 M_n(R)$. Therefore, $((aR)_1 : R^n) = f_{11}^1 M_n(R)$. So, $f_{11}^1 \in ((aR)_1 : R^n)$. Therefore, $R \subseteq aR$. Hence, $aR = R$. So, R is a near-field. □

Let S be as defined in Remark 0.3.1.

**Theorem 4.3.10.** *Suppose that R has DCC on right ideals and $D_0^r(R) = \{0\}$. If any two right R-groups of type-0 are R-isomorphic and S = R, then R is simple and is (isomorphic to) $M_n(eRe)$, where n = dim R and e is a right 0-primitive idempotent in R. Moreover, eRe is a right eRe-group of type-0 and for every non-zero distributive element b in eRe, beRe and eRe are isomorphic as right R-groups. If, in addition R is finite, then every non-zero distributive element in eRe is a unit.*

*Proof.* Suppose that any two right R-groups of type-0 are R-isomorphic and S = R. By Theorem 4.3.2, R is simple and is (isomorphic to) $M_n$(eRe), where n = dim R and e is a right 0-primitive idempotent in R. Now eR is a right R-group of type-0. Therefore, $f_{ii}^e M_n$(eRe) is a right $M_n$(eRe)-group of type-0. Now by Lemma 4.3.7, eRe is a right eRe-group of type-0 and for every non-zero distributive element b in eRe, beRe and eRe are isomorphic as right R-groups. If, in addition R is finite, then every non-zero distributive element in eRe is a unit in eRe. □

**Corollary 4.3.11.** *Suppose that R is a d.g. near-ring satisfying DCC on right ideals and $D_0^r(R) = \{0\}$. If any two right R-groups of type-0 are R-isomorphic, then R is simple and is (isomorphic to) $M_n(eRe)$, where n = dim R and e is a right 0-primitive idempotent in R. Moreover, eRe is a d.g. near-ring and is also a right eRe-group of type-0 and for every non-zero distributive element b in eRe, beRe and eRe are isomorphic as right R-groups. If, in addition R is finite, then every non-zero distributive element in eRe is a unit.*

*Proof.* Suppose that any two right R-groups of type-0 are R-isomorphic. Since R is a d.g. near-ring, by Corollary 0.3.6, R = S. We get the Corollary from Theorem 4.3.10, as eRe is d.g. follows from the fact that $M_n$(eRe) is d.g.. □

**Theorem 4.3.12.** *Let $\nu \in \{1, 2\}$. Suppose that R has DCC on right ideals of R and $D^r_\nu(R) = \{0\}$. If any two right R-groups of type-$\nu$ are R-isomorphic and $S = R$, then R is simple and is (isomorphic to) $M_n(eRe)$, where $n = \dim R$ and e is a right $\nu$-primitive idempotent in R. Moreover, eRe is a right eRe-group of type-$\nu$ and every non-zero distributive element b in eRe is a unit. Hence, for $\nu = 2$, eRe is a near-field.*

*Proof.* As seen in Theorem 4.3.10, the proof follows from Theorem 4.3.2 and Lemma 4.3.8. □

**Theorem 4.3.13.** *Let $\nu \in \{1, 2\}$. Suppose that R is a d.g. right $\nu$-primitive near-ring satisfying DCC on right ideals of R. Then, R is simple and is (isomorphic to) $M_n(eRe)$, where $n = \dim R$ and e is a right $\nu$-primitive idempotent in R. Moreover, eRe is a d.g. near-ring and is a right eRe-group of type-$\nu$. Hence for $\nu = 2$, eRe is a division ring and R is a primitive ring.*

*Proof.* Since R is d.g., by Proposition 2.3.2, $J^r_\nu(R) = D^r_\nu(R)$. Since $J^r_\nu(R) = \{0\}$, we have $D^r_\nu(R) = \{0\}$. Since R has DCC on right ideals and $D^r_\nu(R) = \{0\}$, by Proposition 4.2.8, $R = K_1 \oplus K_2 \oplus ... \oplus K_n$, where $K_i = e_iR$ is a minimal right ideal of R which is also a right R-group of type-$\nu$, $e_i \in K_i$ is a distributive idempotent in R and $e := e_1 + e_2 + ... + e_n$ is a left identity of R and $n = \dim R$. Since R is $\nu$-primitive, there is a right R-group G of type-$\nu$ such that $\{0\}$ is the largest ideal of R contained in $(0 : G) = \{r \in R \mid Gr = \{0\}\}$. By Proposition 1.3.8, $(0 : G)$ is an ideal of R. So, $(0 : G) = \{0\}$. Now G is R-isomorphic to $e_jR$, for some $1 \leq j \leq n$. Therefore, $(0 : e_jR) = \{0\}$ and hence $e_jRe_k \neq \{0\}$ for all $1 \leq k \leq n$. So, $e_jDe_k \neq \{0\}$, where D is the set of all distributive elements in R. This shows that $e_jRe_k$ has a non-zero distributive element. Now by Corollary 4.3.4, all $e_iR$ are pairwise R-isomorphic. As seen in Corollary 4.3.11, the proof follows from Theorem 4.3.12. □

**Theorem 4.3.14.** *Let $\nu \in \{1, 2\}$. Suppose that R is a d.g. $J^r_\nu$-semisimple near-ring satisfying DCC on right ideals of R. Then R is a direct sum of a finite number of minimal ideals which are right $\nu$-primitive d.g. near-rings satisfying DCC on right ideals.*

*Proof.* We have $J^r_\nu(R) = \{0\}$, where R is a d.g. near-ring satisfying DCC on right ideals of R. We get a finite number of right $\nu$-primitive ideals $P_i$, i = 1, 2, ... , k of R which are minimal for the property that $\cap_{i=1}^{k} P_i = \{0\}$. Now $R/P_i$ is a right $\nu$-primitive ideal near-ring. Now by Theorem 4.3.13, $R/P_i$ is a simple near-ring. So, $P_i$ is a maximal ideal of R. Hence, R is a direct sum of the ideals $I_j := \cap_{i=1, i\neq j}^{k} P_i$, j = 1, 2, ... , k. Now $R/P_i \simeq (P_i + I_i)/P_i \simeq I_i/P_i \cap I_i = I_i$. So, $I_i$ is a simple near-ring as $P_i$ is a maximal ideal of R. Therefore, $I_i$ are minimal ideals of R. Since R is d.g., $R/P_i$ is a d.g. near-ring. So, $I_i$ is also a d.g. near-ring. Moreover, $I_i$ are right $\nu$-primitive near-rings satisfying DCC on right ideals as $R/P_i$ is a right $\nu$-primitive near-ring satisfying DCC on right ideals. □

Clearly, a right R-group of type-2 is a strictly irreducible in the sense of [17]. So, in view of Propositions 0.3.1, 4.2.8 and Theorem 0.3.14, we have the following:

**Theorem 4.3.15.** *Suppose that R is a finite near-ring and $D^r_2(R) = \{0\}$. If any two right R-groups of type-2 are R-isomorphic and eRe is a non-ring for some right 2-primitive idempotent e in R, then R is (isomorphic to) the matrix near-ring $M_n(eRe)$, where n = dim R. Moreover, eRe is a near-field.*

We give some right completely reducible near-rings which are also matrix near-rings.

**Example 4.3.16.** *By Corollary 0.3.11, if (G, +) is a finite group of order greater than 2, then $M_0(G^n)$ is isomorphic to the matrix near-ring $M_n(M_0(G))$. Let (G, + ) be a finite group, $R = M_0(G^n)$ and $T = M_0(G)$. For $1 \leq i \leq n$, let $H_i := G_1 \times G_2 \times \ldots \times G_{i-1} \times \{0\} \times G_{i+1} \times \ldots \times G_n$, where $G_j = G$ for all $1 \leq j \leq n$. $H_i$ is a normal subgroup of G. Let $K_i := \{f \in R \mid f(a) \in H_i$, for all $a \in G\}$, $1 \leq i \leq n$. $K_i$ is a right ideal of R. Since $\cap_{i=1}^{n} H_i = \{0\}$, we have that $\cap_{i=1}^{n} K_i = \{0\}$. Let $T_i := \cap_{j=1, j \neq i}^{n} K_j$, i = 1, 2, ... , n. If G is a simple group, then each $H_i$ is a maximal normal subgroup of $G^n$ and hence each $K_i$ is a maximal right modular right ideal of R. So, $R = T_1 \oplus T_2 \oplus \ldots \oplus T_n$ and $T_i$ are right R-groups of type-0. If G is a group of prime order, then each $H_i$ is a maximal subgroup of $G^n$ and hence each $K_i$ is a right 1-modular right ideal of R. So, $R = T_1 \oplus T_2 \oplus \ldots \oplus T_n$ and $T_i$ are right R-groups of type-1.*

**Example 4.3.17.** *Let F be a near-field. Consider the matrix near-ring $M_n(F)$. By Theorem 0.3.15, $M_n(F) = f_{11}^1 M_n(F) \oplus f_{22}^1 M_n(F) \oplus \ldots \oplus f_{nn}^1 M_n(F)$, $f_{ii}^1 M_n(F)$ are right ideals of $M_n(F)$ and are right $M_n(F)$-groups of type-2.*

Now we give an example of a finite simple near-ring with identity which is a direct sum of minimal right ideals which are not isomorphic as right R-groups.

**Example 4.3.18.** *Let $Z_n$ be the additive cyclic group of order n. Let p and q be distinct prime numbers and $G = Z_p \times Z_q$. Consider the near-ring $R = M_0(G)$. $K_1 = \{f \in R \mid f(a) \in Z_p \times \{0\}$ , for all $a \in G\}$ and $K_2 = \{f \in R \mid f(a) \in \{0\} \times Z_q$ , for all $a \in G\}$ are right ideals of R and R is the direct sum of $K_1$ and $K_2$ as $h : G \to G$ defined by $h((x,y)) = (x,0)$, $g : G \to G$ defined by $g((x,y)) = (0,y)$ are endomorphisms of G with $1 = h + g$, $h \in K_1$ and $g \in K_2$. Since $Z_p \times \{0\}$ , $Z_q \times \{0\}$ are maximal normal subgroups of G we get that $K_1$, $K_2$ are maximal right ideals of R and hence they are* *also minimal right ideals of R. Now $R = K_1 \oplus K_2$ and is a simple near-ring. Since $K_1$ and $K_2$ don't have the same number of elements, they are not isomorphic as right R-groups.*

*Remark* 4.3.1. Let R be the near-ring considered in Example 4.3.18 and p = 2. Then, by Example 2.2.27, R is right 2-primitive. It is an easy verification that the right ideal $K_1$ of R is not right 2-modular. So, $D_2^r(R) \neq \{0\}$.

We give an example of a near-ring for which some of the results developed in this section can be applied.

**Example 4.3.19.** *Let G be a finite simple non-abelian additive group. By Corollary 0.3.12, $E(G^2)$ is isomorphic to the matrix near-ring $M_2(E(G))$. As mentioned soon after Corollary 19 of [16], $E(G^2) = M_0(G^2)$. So, $M_0(G^2)$ is a simple d.g. near-ring with DCC on right ideals. Let $i \in \{1, 2\}$. Let $G_1 = G\times\{0\}$ and $G_2 = \{0\}\times G$ . Since $G_i$ is a maximal (minimal) normal subgroup of $G^2$, $K_i = (G_i : G^2) = \{m \in M_0(G^2) \mid m(a) \in G_i, \text{for all } a \in G^2\}$ is a maximal right ideal of $M_0(G^2)$. Moreover, $K_1 \cap K_2 = \{0\}$. Thus $J^r_{1/2}(M_0(G^2)) = \{0\}$. This shows that $M_0(G^2) = K_1 \oplus K_2$, where $K_i$ is a minimal right ideal of $M_0(G^2)$. Define $e_i : G^2 \to G^2$ by $e_i((a_1, a_2)) = (b_1, b_2)$, where $b_j = a_i$ if $j = i$ and 0 if $j \neq i$. Now $e_i$ is a group homomorphism and hence it is a distributive idempotent in $M_0(G^2)$ and $e_iM_0(G^2) \subseteq K_i$. Since $e_1$ and $e_2$ are orthogonal distributive idempotents in $M_0(G^2)$ and $e_1 + e_2 = 1$, by Proposition 0.3.2, we get that $e_iM_0(G^2)$ is a right ideal of $M_0(G^2)$. Thus, $K_i = e_iM_0(G^2)$. The mapping $e_{12} : G^2 \to G^2$ defined by $e_{12}((a_1, a_2)) = (a_2, 0)$ is an endomorphism of $G^2$. So, $e_{12}$ is a distributive element in $M_0(G^2)$. It is an easy verification that the mapping $h : e_2M_0(G^2) \to e_1M_0(G^2)$ defined by $h(e_2m) = e_{12}(e_2m)$ is an isomorphism of the right $M_0(G^2)$-groups. So, $K_1$ and $K_2$ are isomorphic as right $M_0(G^2)$-groups. Since* *a minimal right ideal $K$ of $M_0(G^2)$ is isomorphic to $K_j$ for some $j \in \{1, 2\}$ as right $M_0(G^2)$-groups, we get that any two minimal right ideals of $M_0(G^2)$ are isomorphic as right $M_0(G^2)$-groups.*

The following example shows that the left Jacobson radicals are not relevant for the extension of Wedderburn-Artin theorem of rings to near-rings.

**Example 4.3.20.** *Let $G$ be a finite simple non-abelian additive group. Now by Theorem 0.1.4, $E(G) = M_0(G)$. So, $M_0(G)$ is a finite simple d.g. near-ring with identity. Moreover, $J_2(M_0(G)) = \{0\}$ and each minimal left ideal of $M_0(G)$ is isomorphic to $G$ as left $M_0(G)$-groups. So, $M_0(G)$ is a direct sum of a finite number of minimal left ideals which are pair wise isomorphic as left $M_0(G)$-groups. Since each distributive element of $M_0(G)$ is an endomorphism of $(G, +)$, 0 and the automorphisms of $(G, +)$ are the only distributive elements of $M_0(G)$. Therefore, $M_0(G)$ has no nontrivial matrix units. Hence, $M_0(G)$ is not isomorphic to a matrix near-ring $M_n(S)$, where $n > 1$.*