Chapter 3

Kurosh-Amitsur Right Jacobson Radicals of Type-0, 1 and 2

By a near-ring we mean a right near-ring. In chapter 1, we introduced and studied the radical J_0^r, the right Jacobson radical of type-0, for near-rings. In chapter 2, we introduced and studied the radicals J_ν^r, the right Jacobson radicals of type-ν, for near-rings, $\nu \in \{1, 2\}$. Let $\nu \in \{0, 1, 2\}$. In this chapter properties of the radical J_ν^r are studied. It is shown that J_ν^r is a Kurosh-Amitsur radical (KA-radical) in the variety of all near-rings R in which the constant part R_c of R is an ideal of R. Thus, unlike the left Jacobson radicals of type-0 and 1 of near-rings, J_0^r is a KA-radical in the class of all zero-symmetric near-rings. J_ν^r is not s-hereditary and hence not an ideal-hereditary radical in the class of all zero-symmetric near-rings.

3.1 Introduction

Throughout this chapter R denotes a right near-ring and all near-rings considered are right near-rings and not necessarily zero-symmetric.

Let $\nu \in \{0, 1, 2\}$. In this chapter properties of the right Jacobson radical of type-ν are studied. It is known that the left Jacobson radicals of type-0 and 1 are not KA-radicals in the class of all zero-symmetric near-rings and only the left Jacobson radicals of type-2 and 3 are KA-radicals in the class of all zero-symmetric near-rings. Surprisingly, J_0^r, the right Jacobson radical of type-0, is a KA-radical in the class of all zero-symmetric near-rings. It is also shown that J_ν^r is a KA-radical even in a bigger class of near-rings namely in the variety of all near-rings R in which the constant part of R is an ideal of R. Moreover, J_ν^r is not s-hereditary and hence not an ideal-hereditary radical in the class of all zero-symmetric near-rings.

This chapter is divided into three sections. In section 2, the properties of J_0^r, the right Jacobson radical of type-0 are studied. We show that if G is a right R-group of type-0 and S is an invariant subnear-rings of R which is also a right ideal of R, then G is a right S-group of type-0. Using it, we also show that J_0^r is complete Hoehnke radical in the class $\mathcal{F}$ of near-rings R for which the constant part of R is an ideal of R. If S is an invariant subnear-ring and right ideal of R and G is a right S-group of type-0, then G is a right R-group of type-0. From this, we show that J_0^r is an idempotent Hoehnke radical in the class of near-rings $\mathcal{F}$. So, J_0^r is a KA-radical in the class of near-rings $\mathcal{F}$. It is also shown that J_0^r is not s-hereditary and hence not ideal-hereditary in the class of all zero-symmetric near-rings.

Let $\nu \in \{1, 2\}$. In section 3, the properties of J_ν^r, right Jacobson radical of type-ν are studied. We show that if G is a right R-group of type-ν and S is an invariant subnear-ring of R, then G is a right S-group of type-ν. With this, it is shown that J_ν^r is complete Hoehnke radical in the class of near-rings $\mathcal{F}$. If S is an invariant subnear-ring R and G is a right S-group of type-ν, then G is a right R-group of type-ν. It is also shown that J_ν^r is an idempotent Hoehnke radical in the class of near-rings $\mathcal{F}$. So, J_ν^r is a KA-radical in the class of near-rings $\mathcal{F}$. Moreover, J_ν^r is not s-hereditary and hence not ideal-hereditary in the class of all zero-symmetric near-rings.

3.2 Properties of the Radical J_0^r

In this section, the properties of J_0^r, the right Jacobson radical of type-0 are studied. We show that if G is a right R-group of type-0 and S is an invariant subnear-rings of R which is also a right ideal of R, then G is a right S-group of type-0. Using it, we also show that J_0^r is complete Hoehnke radical in the class $\mathcal{F}$ of near-rings R for which the constant part of R is an ideal of R. If S is an invariant subnear-ring and right ideal of R and G is a right S-group of type-0, then G is a right R-group of type-0. From this, we show that J_0^r is an idempotent Hoehnke radical in the class of near-rings $\mathcal{F}$. So, J_0^r is a KA-radical in the class of near-rings $\mathcal{F}$. It is also shown that J_0^r is not s-hereditary and hence not ideal-hereditary in the class of all zero-symmetric near-rings.

Throughout this section $\mathcal{F}$ stands for the class of all near-rings R for which the constant part R_c is an ideal of R. This class was considered in P. Fuchs [4], and shown that it is a variety.

From Theorems 1.2.21 and 1.2.23, we have the following:

Proposition 3.2.1. *J_0^r is the Hoehnke radical corresponding to the class of all right 0-primitive near-rings in the class of all near-rings.*

If (A, +) is a group and T is a subset of A, then the subgroup (normal subgroup) of A generated by T is denoted by $< T >_s$ ($< T >_n$).

Remark 3.2.1. Let G be a right R-group. It is clear that H = {g ∈ G | gR = {0}} is an ideal of G. So, if G is simple and gR = {0}, then g = 0.

Theorem 3.2.2. *Let G be a right R-group of type-0. Suppose that S is an invariant subnear-ring and a right ideal of R. If GS ≠ {0}, then G is also a right S-group of* *type-0.*

Proof. Suppose that GS $\neq \{0\}$. Let g $\in$ G and gS := $\{gs \mid s \in S\} \subseteq$ G. Consider the normal subgroup $< gS >_n$ of (G, +). Let r $\in$ R, h $\in$ $< gS >_n$. Now h = $(x_1 + \delta_1(gs_1) - x_1) + (x_2 + \delta_2(gs_2) - x_2) + ... + (x_k + \delta_k(gs_k) - x_k)$, $s_i \in S$, $x_i \in G$, $\delta_i \in \{1, -1\}$. Since SR $\subseteq$ S, hr = $(x_1r + \delta_1(g(s_1r)) - x_1r) + (x_2r + \delta_2(g(s_2r)) - x_2r) + ... + (x_kr + \delta_k(g(s_kr)) - x_kr) \in < gS >_n$. So, $< gS >_n$ is an ideal of the right R-group G and hence it is also an ideal of the right S-group G. Let $0 \neq h \in G$. Suppose that hS = $\{0\}$. Since hR $\neq \{0\}$, $< hR >_n$ is a non-zero ideal of the right R-group G. Since G is a simple right R-group, $< hR >_n =$ G. So, GS = $< hR >_n S \subseteq < hS >_n$ = $\{0\}$, a contradiction to GS $\neq \{0\}$. Therefore hS $\neq \{0\}$. Let g_0 be a generator of the right R-group G. So, g_0 is a distributive element of the right R-group G and g_0R = G. Clearly g_0 is a distributive element of the right S-group G and hence g_0S is a subgroup of (G, +). We have $(g_0S)R = g_0(SR) \subseteq g_0S$. So g_0S is an R-subgroup of G. Let g $\in$ G and s $\in$ S. Since g_0R = G, g = g_0r for some r $\in$ R. So, g + g_0s - g = g_0r + g_0s - g_0r = $g_0(r + s - r) \in g_0S$, as S is a normal subgroup of (R, +). Therefore g_0S is an ideal of the right R-group G and hence g_0S = G. So g_0 is also a generator of the right S-group G. Let K be a non-zero ideal of the right S-group G. Let $0 \neq y \in K$. As seen above, $< yS >_n$ is a non-zero ideal of the right R-group G and hence $< yS >_n$ = G. Since G = $< yS >_n \subseteq$ K, G = K. Therefore, $\{0\}$ and G are the only ideals of the right S-group G and hence G is a right S-group of type-0. □

Proposition 3.2.3. *Let G be a right R-group of type-0 and let T be a right quasi-regular invariant subnear-ring of R. If T is a right ideal of R, then GT = {0}.*

Proof. Suppose that T is a right ideal of R and g_0 is a generator of G. So $g_0(r + s)$ = $g_0r + g_0s$ for all r, s $\in$ R and g_0R = G. L := $(0 : g_0) = \{r \in R \mid g_0r = 0\}$ is a right 0-modular right ideal of R. Therefore L contains the largest right quasi-regular right ideal of R. Since T is a right quasi-regular right ideal of R, $T \subseteq L$, that is, $g_0T = \{0\}$. Let $g \in G$ and $t \in T$. Now $g = g_0r$ for some $r \in R$. $gt = g_0(rt) = 0$, as $rt \in T$. Therefore $GT = \{0\}$. □

Since R_c is an invariant subnear-ring and right quasi-regular in R, we have the following:

Corollary 3.2.4. *If R_c is a normal subgroup of (R, +), then $GR_c = \{0\}$ for all right R-groups G of type-0.*

Corollary 3.2.5. *Let $R \in \mathcal{F}$. If G is a right R-group of type-0, then $GJ_0^r(R) = \{0\}$.*

Proof. Let G be a right R-group of type-0. We have that $I := J_0^r(R)$ is the largest right quasi-regular ideal of R. Since R_c is a right quasi-regular ideal of R, $R_c \subseteq I$. Let $r \in R$ and $i \in I$. Now $ri = (r(0 + i) - r0) + r0 \in I$, as $R_c \subseteq I$. Therefore, $RI \subseteq I$ and hence I is an invariant ideal of R. Therefore, by Proposition 3.2.3, $GI = \{0\}$. □

Proposition 3.2.6. *Let $R \in \mathcal{F}$. Let I be an ideal of R and let $K := I + R_c$. If G is a right K-group of type-0, then G is a right I-group of type-0.*

Proof. Suppose that G is a right K-group of type-0 and g_0 is a generator of G. So g_0 is distributive over K and $g_0K = G$. Let K_c be the constant part of K. Since $K_c = R_c$ is a normal subgroup of K, by Corollary 3.2.4, $GK_c = \{0\}$, that is, $GR_c = \{0\}$. Clearly G is a right I-group. Now $G = g_0K = g_0(I + R_c) = g_0I$ and hence g_0 is a generator of the right I-group G. Let H be a non-zero ideal of the right I-group G. Let $h \in H$ and $k \in K$. $k = i + r_c$, $i \in I$, $r_c \in R_c$ and $h = g_0t$, $t \in I$. Since R_c is an ideal of R, $t(i + r_c) - ti \in R_c$. So, $hk = g_0t(i + r_c) = g_0((t(i + r_c) - ti) + ti) = g_0(t(i +$

r_c) - ti) + g_0(ti) = 0 + (g_0t)i = hi $\in$ H. Therefore, H is a non-zero ideal of the right K-group G and hence H = G. So, G is a right I-group of type-0. □

We show now that the Hoehnke radical J_0^r is complete in the variety $\mathcal{F}$.

Theorem 3.2.7. *Let $R \in \mathcal{F}$. If I is an ideal of R and $J_0^r(I) = I$, then $I \subseteq J_0^r(R)$.*

Proof. Let I be an ideal of R and $J_0^r(I) = I$. Suppose that $I \not\subseteq J_0^r(R)$. So $K := I + R_c$ is an ideal of R and $K \not\subseteq J_0^r(R)$. We prove that there is a right R-group G of type-0 such that $GK \neq \{0\}$. On the contrary suppose that $GK = \{0\}$ for all right R-groups G of type-0. Let Q be a right 0-primitive ideal of R. We get a right 0-modular right ideal L of R such that Q is the largest ideal of R contained in L. Now R/L is a right R-group of type-0. Since $(R/L)K = \{0\}$, we have that $K \subseteq (L : R)$. By Proposition 1.3.5, Q is the largest ideal of R contained in (L : R), as $R_c \subseteq J_0^r(R) \subseteq L$. So $K \subseteq Q$. Hence $K \subseteq J_0^r(R)$, a contradiction to $K \not\subseteq J_0^r(R)$. Therefore, there is a right R-group G of type-0 such that $GK \neq \{0\}$. Since K is an invariant ideal of R, by Theorem 3.2.2, G is a right K-group of type-0. Therefore, by Proposition 3.2.6, G is a right I-group of type-0. This is a contradiction to the fact that $J_0^r(I) = I$. Therefore, $I \subseteq J_0^r(R)$. □

Theorem 3.2.8. *J_0^r is a complete Hoehnke radical in the variety of near-rings $\mathcal{F}$.*

Theorem 3.2.9. *J_0^r is a complete Hoehnke radical in the class of all zero-symmetric near-rings.*

Theorem 3.2.10. *Suppose that S is an invariant subnear-ring of R and also a right ideal of R. If G is a right S-group of type-0, then G is also a right R-group of type-0.*

Proof. Let G be a right S-group of type-0 and g_0 be a generator. We have that g_0 is distributive over S and $g_0S = G$. For $g \in G$ and $r \in R$, define $gr := g_0(sr)$, if $g = g_0s$,

$s \in S$. We show now that this operation is well-defined. Suppose that $g = g_0 s = g_0 t$, $s, t \in S$. Let $r \in R$ and $h := g_0(sr) - g_0(tr)$. Now $hk = (g_0(sr) - g_0(tr))k = g_0((sr)k) - g_0((tr)k) = g_0(s(rk)) - g_0(t(rk)) = g(rk) - g(rk) = 0$, for all $k \in S$. Therefore $hS = \{0\}$ and hence $h = 0$, that is, $g_0(sr) = g_0(tr)$. We show now that G is a right R-group of type-0. It is clear that G is a right R-group. $g_0 = g_0 e$ for some $e \in S$. Now $G \supseteq g_0 R = g_0(eR) \supseteq g_0(eS) = g_0 S = G$. So, $g_0 R = G$. Let $p, q \in R$ and $x = g_0(p + q) - (g_0 p + g_0 q)$. $xs = (g_0(p + q) - (g_0 p + g_0 q))s = (g_0(p + q))s - ((g_0 p + g_0 q))s = g_0(ps + qs) - (g_0 ps + g_0 qs) = (g_0(ps) + g_0(qs)) - (g_0(ps) + g_0(qs)) = 0$, for all $s \in S$. Therefore $x = 0$ and hence g_0 is a generator of the right R-group G. It can be easily verified that the action of R on G is an extension of the action of S on G. So, an ideal of the right R-group G is also an ideal of the right S-group G. Since the right S-group G has no non-trivial ideals, the right R-group G also has no non-trivial ideals. Therefore, G is also a right R-group of type-0. □

We show now that the Hoehnke radical J_0^r is idempotent in the variety $\mathcal{F}$.

Theorem 3.2.11. *Let $R \in \mathcal{F}$. Then $J_0^r(J_0^r(R)) = J_0^r(R)$.*

Proof. Let $I := J_0^r(R)$. I is the largest right quasi-regular ideal of R. Since R_c is a right quasi-regular ideal of R, $R_c \subseteq I$. So I is an invariant ideal of R. Suppose that $J_0^r(I) \neq I$. So there is a right I-group G of type-0. By Theorem 3.2.10, G is a right R-group of type-0. Now by Corollary 3.2.5, $GI = GJ_0^r(R) = \{0\}$. This is a contradiction to the fact that G is a right I-group of type-0. Therefore $J_0^r(I) = I$, that is, $J_0^r(J_0^r(R)) = J_0^r(R)$. □

Theorem 3.2.12. *J_0^r is an idempotent Hoehnke radical in the variety of near-rings $\mathcal{F}$.*

From Theorems 3.2.8 and 3.2.12, we have the following:

Theorem 3.2.13. *J_0^r is a Kurosh-Amitsur radical in the variety of near-rings $\mathcal{F}$.*

Theorem 3.2.14. *J_0^r is a Kurosh-Amitsur radical in the class of all zero-symmetric near-rings.*

Theorem 3.2.15. *J_0^r is not s-hereditary in the class of all zero-symmetric near-rings.*

Proof. Consider G := Z_8, the group of integers under addition modulo 8. Now T : G $\rightarrow$ G defined by T(g) = 5g, for all g $\in$ G is an automorphism of G. T fixes 0, 2, 4, 6 and maps 1 to 5 and 3 to 7. A := {I, T} is an automorphism group of G. {0}, {2}, {4}, {6}, {1, 5} and {3, 7} are the orbits. Let R be the centralizer near-ring $M_A(G)$, the near-ring of all self maps of G which fix 0 and commute with T. An element of R is completely determined by its action on {1, 2, 3, 4, 6}. For each element f of R, f(2), f(4), f(6) are arbitrary in 2G and f(1) and f(3) are arbitrary in G. This example was considered by Kaarli in [8] and shown that I := (0 : 2G) = {f $\in$ R | f(h) = 0, for all h $\in$ 2G} is the only non-trivial ideal of R. Let f_0 be the element of I which fixes all the elements in G - 2G. Clearly f - f_0f $\in$ (2G : G) = {t $\in$ R | t(G) $\subseteq$ 2G} for all f $\in$ R. Since (2G : G) is a proper right ideal of R, f_0 is not right quasi-regular in R. So, I is not a right quasi-regular ideal of R. Since R is a near-ring with identity, it is not right quasi-regular. Therefore, {0} is the largest right quasi-regular ideal of R and hence $J_0^r(R)$ = {0}. So R is J_0^r-semisimple. It is shown in [8] that K := $(4G : G)_I$ = {f $\in$ I | f(G) $\subseteq$ 4G} is a non-zero ideal of I and K^2 = {0}. Since a nil ideal is right quasi-regular, K is a right quasi-regular ideal of I. Therefore {0} $\neq$ K $\subseteq$ $J_0^r(I)$ and hence I is not J_0^r-semisimple. So, J_0^r is not s-hereditary in the class of all zero-symmetric near-rings. □

Corollary 3.2.16. *J_0^r is not s-hereditary in the class of all near-rings.*

Theorem 3.2.17. *J_0^r is not an ideal-hereditary radical in the class of all zero-symmetric near-rings.*

It is not known whether J_0^r is a KA-radical in the class of all near-rings. J_0^r may fails to be idempotent and Kurosh-Amitsur in the class of all near-rings.

3.3 Properties of the Radicals J_1^r and J_2^r

Let $\nu \in \{1, 2\}$. In this section, the properties of J_ν^r, right Jacobson radical of type-ν are studied. We show that if G is a right R-group of type-ν and S is an invariant subnear-ring of R, then G is a right S-group of type-ν. With this, it is shown that J_ν^r is complete Hoehnke radical in the class of near-rings $\mathcal{F}$. If S is an invariant subnear-ring R and G is a right S-group of type-ν, then G is a right R-group of type-ν. It is also shown that J_ν^r is an idempotent Hoehnke radical in the class of near-rings $\mathcal{F}$. So, J_ν^r is a KA-radical in the class of near-rings $\mathcal{F}$. Moreover, J_ν^r is not s-hereditary and hence not ideal-hereditary in the class of all zero-symmetric near-rings.

From Theorem 2.2.22, we have the following:

Proposition 3.3.1. *Let $\nu \in \{1, 2\}$. J_ν^r is the Hoehnke radical corresponding to the class of all right ν-primitive near-rings in the class of all near-rings.*

Theorem 3.3.2. *Let G be a right R-group of type-1. If S is an invariant subnear-ring of R and $GS \neq \{0\}$, then G is also a right S-group of type-1.*

Proof. Let g ∈ G. Let gS := {gs | s ∈ S} ⊆ G and $< gS >_s$ be the subgroup of G generated by gS. Let h ∈ $< gS >_s$. Now h = $\delta_1(gs_1) + \delta_2(gs_2) + ... + \delta_k(gs_k)$, $s_i \in S$ and $\delta_i \in \{1, -1\}$. Since SR ⊆ S, hr = $(\delta_1(gs_1) + \delta_2(gs_2) + ... + \delta_k(gs_k))r = \delta_1(g(s_1r)) + \delta_2(g(s_2r)) + ... + \delta_k(g(s_kr)) \in < gS >_s$. So $< gS >_s$ is an R-subgroup of G and hence an S-subgroup of G. Let $0 \neq g_0 \in G$. Suppose that $g_0S = \{0\}$. Since G is a simple right R-group and {h ∈ G | hR = {0}} is an ideal of G, we get that $g_0R \neq \{0\}$. Clearly $< g_0R >_s$, the subgroup of G generated by g_0R, is an R-subgroup of G. Since G has no non-trivial R-subgroups, $< g_0R >_s \neq \{0\}$. Therefore $< g_0R >_s = G$.

So, GS = $< g_0R >_s$S $\subseteq$ $< g_0S >_s$ = {0}, a contradiction to GS $\neq$ {0}. Therefore g_0S $\neq$ {0}. Let h be a generator of the right R-group G. Clearly h is a distributive element of the right S-group G and hence hS is an S-subgroup of G. Since (hS)R = h(SR) $\subseteq$ hS, hS is an R-subgroup of G. As seen above, hS $\neq$ {0} and hence hS = G. So h is also a generator of the right S-group G. Let {0} $\neq$ T be a right S-group of G. Let 0 $\neq$ g_1 $\in$ T. As seen above, $< g_1S >_s$ is a non-zero R-subgroup of G and hence $< g_1S >_s$ = G. Since g_1S $\subseteq$ T, G = $< g_1S >_s$ $\subseteq$ T and hence G = T. Therefore {0} and G are the only S-subgroups of G and hence G is a right S-group of type-1. □

Theorem 3.3.3. *Let G be a right R-group of type-2. If S is an invariant subnear-ring of R and GS $\neq$ {0}, then G is a right S-group of type-2.*

Proof. Since G is also a right R-group of type-1, by Theorem 3.3.2, G is a right S-group of type-1. Let $0 \neq g_0 \in G$. Now g_0S $\neq$ {0} as {h $\in$ G | hS = {0}} is an ideal of the right S-group G. Since G is a right R-group of type-2, we have $gR = G$, for all $0 \neq g \in G$. So, $(g_0S)R = G$. Now $G = (g_0S)R = g_0(SR) \subseteq g_0S$. Therefore $g_0S = G$ and hence G is a right S-group of type-2. □

Proposition 3.3.4. *Let $\nu \in \{1, 2\}$. Let G be a right R-group of type-ν. Suppose that T is a right quasi-regular right ideal of R of type-ν. If T is an invariant subnear-ring of R, then $GT = \{0\}$.*

Proof. Suppose that T is an invariant subnear-ring of R and g_0 is a generator of G. So g_0(r + s) = g_0r + g_0s for all r, s $\in$ R and g_0R = G. L := (0 : g_0) = {r $\in$ R | g_0r = 0} is a right 1-modular right ideal of R. By Theorem 2.4.1, L contains all the right quasi-regular right ideals of R of type-ν. Since T is a right quasi-regular right ideal of R of type-ν, T $\subseteq$ L, that is, g_0T = {0}. Let g $\in$ G and t $\in$ T. Now g = g_0r for some r $\in$ R. gt = g_0(rt) = 0, as rt $\in$ T. Therefore, GT = {0}. □

Corollary 3.3.5. *Let $\nu \in \{1, 2\}$. If G is a right R-group of type-ν, then $GR_c = \{0\}$.*

Proof. Let G be a right R-group of type-ν and g_0 be a generator of G. Now $(0 : g_0)$ is a right ν-modular right ideal of R. By Corollary 2.2.16, $R_c \subseteq D^r_\nu(R)$. So, $R_c \subseteq (0 : g_0)$. Let $g \in G$. Now $g = g_0 r$ for some $r \in R$ as $g_0 R = G$. For $r_c \in R_c$, $gr_c = (g_0 r)r_c = g_0(rr_c) = 0$. Therefore, $GR_c = \{0\}$. □

Corollary 3.3.6. *Let $R \in \mathcal{F}$ and $\nu \in \{1, 2\}$. If G is a right R-group of type-ν, then $GJ^r_\nu(R) = \{0\}$.*

Proof. Let G be a right R-group of type-ν and $I := J^r_\nu(R)$. By Theorem 2.4.2, I is a right quasi-regular ideal of R of type-ν. Since R_c is a right quasi-regular ideal of R, $R_c \subseteq I$. So I is an invariant ideal of R. Therefore, by Proposition 3.3.4, we get that $GI = \{0\}$. □

Proposition 3.3.7. *Let $R \in \mathcal{F}$ and $\nu \in \{1, 2\}$. Let I be an ideal of R and $K := I + R_c$. If G is a right K-group of type-ν, then G is a right I-group of type-ν.*

Proof. Suppose that G is a right K-group of type-ν and g_0 is a generator of G. So g_0 is distributive over K and $g_0 K = G$. Let K_c be the constant part of K. Since $K_c = R_c$, by Corollary 3.3.5, $GR_c = \{0\}$. Clearly G is a right I-group. Now $G = g_0 K = g_0(I + R_c) = g_0 I$ and hence g_0 is a generator of the right I-group G. Let $h \in G$ and $k \in K$. $k = i + r_c$, $i \in I$, $r_c \in R_c$ and $h = g_0 t$, $t \in I$. Now $hk = g_0 t(i + r_c) = g_0((t(i + r_c) - ti) + ti) = g_0(t(i + r_c) - ti) + g_0(ti) = 0 + (g_0 t)i = hi$. So G is a right I-group of type-ν. □

Theorem 3.3.8. *Let $R \in \mathcal{F}$ and $\nu \in \{1, 2\}$. If I is an ideal of R and $J^r_\nu(I) = I$, then $I \subseteq J^r_\nu(R)$.*

Proof. Let I be an ideal of R and $J^r_\nu(I) = I$. Suppose that $I \not\subseteq J^r_\nu(R)$. So $K := I + R_c$ is an ideal of R and $K \not\subseteq J^r_\nu(R)$. We get a right R-group G of type-ν such that GK $\neq \{0\}$. Since K is an invariant ideal of R, by Theorems 3.3.2 and 3.3.3, G is a right K-group of type-ν. Therefore, by Proposition 3.3.7, G is a right I-group of type-ν. This is a contradiction to the fact that $J^r_\nu(I) = I$. Therefore, $I \subseteq J^r_\nu(R)$. □

Theorem 3.3.9. *Let $\nu \in \{1, 2\}$. Then J^r_ν is a complete Hoehnke radical in the variety of near-rings $\mathcal{F}$.*

Theorem 3.3.10. *Let S be an invariant subnear-ring of R. If G is a right S-group of type-1, then G is a right R-group of type-1.*

Proof. Let G be a right S-group of type-1 and g_0 be a generator. So g_0 is distributive over S and $g_0S = G$. For $g \in G$ and $r \in R$, define $gr := g_0(sr)$, if $g = g_0s$, $s \in S$. We show now that this operation is well-defined. Suppose that $g = g_0s = g_0t$, s, t $\in S$. Let $r \in R$ and $h := g_0(sr) - g_0(tr)$. Now $hk = (g_0(sr) - g_0(tr))k = g_0((sr)k) - g_0((tr)k) = g_0(s(rk)) - g_0(t(rk)) = g(rk) - g(rk) = 0$, for all $k \in S$. Therefore $hS = \{0\}$ and hence $h = 0$, that is, $g_0(sr) = g_0(tr)$. We show that G is a right R-group of type-1. It is clear that G is a right R-group. $g_0 = g_0e$ for some $e \in S$. Now $G \supseteq g_0R = g_0(eR) \supseteq g_0(eS) = g_0S = G$. So $g_0R = G$. Let p, q $\in$ R and $x = g_0(p + q) - (g_0p + g_0q)$. $xs = (g_0(p + q) - (g_0p + g_0q))s = (g_0(p + q))s - ((g_0p + g_0q))s = g_0(ps + qs) - (g_0ps + g_0qs) = (g_0(ps) + g_0(qs)) - (g_0(ps) + g_0(qs)) = 0$, for all s $\in$ S. Therefore $x = 0$ and hence g_0 is a generator of the right R-group G. It can be easily verified that the action of R on G is an extension of the action of S on G. So an R-subgroup of the right R-group G is also an S-subgroup of the right S-group G. Since the right S-group G has no non-trivial S-subgroups, the right R-group G also has no non-trivial R-subgroups. Therefore G is also a right R-group of type-1. □

Corollary 3.3.11. *Let S be an invariant subnear-ring of R and let K be a right 1-modular right ideal of S. Then K is an ideal of the right R-group S and S/K is a right R-group of type-1.*

Proof. We have that S/K is a right S-group of type-1. Suppose that K is right modular by $s_0 \in S$. Now $s_0 + K$ is a generator of the right S-group S/K. Let $s + K \in S/K$ and $s + K = (s_0 + K)t$, $t \in S$. Define $(s + K)r := s_0(tr) + K$. As seen in Theorem 3.3.10, S/K is a right R-group of type-1 under the above operation. Now $(s + K)r = (s_0s + K)r = (s_0 + K)sr = s_0(sr) + K$. But from the above operation, $s_0(sr) + K = sr + K$, as $sr \in S$. Therefore, the operation defined above coincides with the natural operation. Hence, K is an ideal of the right R-group S and S/K is a right R-group of type-1. □

Theorem 3.3.12. *Let S be an invariant subnear-ring of R. If G is a right S-group of type-2, then G is a right R-group of type-2.*

Proof. Let G be a right S-group of type-2 and g_0 be a generator. Since G is also a right S-group of type-1, by Theorem 3.3.10, G is a right R-group of type-1, where $gr := g_0(sr)$, if $g = g_0s$, $s \in S$. Moreover, this action of R on G is an extension of the action of S on G. Let $0 \neq g \in G$. Since $gS = G$, we have $G = gS \subseteq gR$. So $gR = G$ and hence G is a right R-group of type-2. □

Corollary 3.3.13. *Let S be an invariant subnear-ring of R. If K is a right 2-modular right ideal of S, then K is an ideal of the right R-group S and S/K is a right R-group of type-2.*

Theorem 3.3.14. *Let $R \in \mathcal{F}$ and $\nu \in \{1, 2\}$. Then $\mathcal{J}_\nu(\mathcal{J}_\nu(R)) = \mathcal{J}_\nu(R)$.*

Proof. Let I := $J^r_\nu(R)$. Since R_c is a right quasi-regular ideal of R, $R_c \subseteq I$. So I is an invariant ideal of R. Suppose that $J^r_\nu(I) \neq I$. So there is a right I-group G of type-ν. By Theorems 3.3.10 and 3.3.12, G is a right R-group of type-ν. Now by Corollary 3.3.6, $GI = GJ^r_\nu(R) = \{0\}$. This is a contradiction to the fact that G is a right I-group of type-ν. Therefore $J^r_\nu(I) = I$, that is, $J^r_\nu(J^r_\nu(R)) = J^r_\nu(R)$. □

Theorem 3.3.15. *Let $\nu \in \{1, 2\}$. Then J^r_ν is an idempotent Hoehnke radical in the variety of near-rings $\mathcal{F}$.*

Theorem 3.3.16. *Let $\nu \in \{1, 2\}$. Then J^r_ν is a Kurosh-Amitsur radical in the variety of near-rings $\mathcal{F}$.*

Theorem 3.3.17. *Let $\nu \in \{1, 2\}$. Then J^r_ν is a Kurosh-Amitsur radical in the class of all zero-symmetric near-rings.*

Theorem 3.3.18. *Let $\nu \in \{1, 2\}$. Then J^r_ν is not s-hereditary in the class of all zero-symmetric near-rings.*

Proof. Let R be the near-ring $M_A(G)$ considered in Theorem 3.2.15. We have that I := (0 : 2G) = $\{f \in R \mid f(h) = 0$, for all $h \in 2G\}$ is the only non-trivial ideal of R. Let f_0 be the element of I which fixes all the elements in G - 2G. Let K := (2G : G) = $\{t \in R \mid t(G) \subseteq 2G\} \neq R$. Clearly f - $f_0f \in K$ for all $f \in R$. So $f_0 + K \in R/K$ is distributive over R. Let $h \in R - K$. We show now that $(h + K)R = R/K$. Since $h \notin K$, there is an element $a \in G$ such that $b := h(a) \notin 2G$. Since each $f \in R$ maps 2G to 2G, we have that $a \notin 2G$. We construct an element $s \in R$, such that s = 0 on 2G and s(1) = s(3) = a, so that s(5) = s(7) = a + 4. Since hs maps 2G to $\{0\}$ and maps G - 2G to G - 2G, we get that t_0 - hs $\in$ K and hence $(h + K)s = t_0 + K$, where t_0 is the identity element of R. So $(h + K)R = R/K$. Therefore, R/K is a right R-group of type-2. Moreover, $(R/K)I \neq \{0\}$. Therefore $J_2^r(R) = \{0\}$. It is shown in [8] that $K := (4G : G)_I$ is a non-zero ideal of I and $K^2 = \{0\}$. Since a nil ideal is right quasi-regular, K is a right quasi-regular ideal of I. Therefore $\{0\} \neq K \subseteq J_\nu^r(I)$ and hence I is not J_ν^r-semisimple. So J_ν^r is not s-hereditary in the class of all zero-symmetric near-rings. □

Corollary 3.3.19. *Let $\nu \in \{1, 2\}$. Then J_ν^r is not s-hereditary in the class of all near-rings.*

Theorem 3.3.20. *Let $\nu \in \{1, 2\}$. Then J_ν^r is not an ideal-hereditary radical in the class of all zero-symmetric near-rings.*

Corollary 3.3.21. *Let $\nu \in \{1, 2\}$. Then J_ν^r is not an ideal-hereditary radical in the class of all near-rings.*

It is not known whether J_ν^r is a KA-radical in the class of all near-rings. But it looks that the radical J_ν^r may fails to be idempotent and Kurosh-Amitsur in the class of all near-rings.