Chapter 6
Forecasting Volatility using Stochastic Volatility Model
Chapter 6

Forecasting Volatility using SV Model

In this chapter, the empirical performance of GARCH(1,1), GARCH-KF and SV models from both estimation and forecasting perspective is investigated. To start with, it is shown that common discrete-time ARMA models as process for the volatility, can have linear state-space representations. Once this is established, it is a simple matter to derive linear state space gaussian equation for these models.

The main empirical contribution of this thesis is to show that the Kalman Filters are closely related to state space form and can be successfully applied to model GARCH processes. In particular, they imply a methodology with a direct one-step ahead forecasts for the volatility, and are identical to the forecasting functions similar to a standard GARCH and a SV model, respectively.

Given the comparison between GARCH(1,1), GARCH-KF and SV models, we might expect GARCH models to perform well in forecasting volatility. As seen from the previous theories, the most common approach for fitting GARCH models is maximum likelihood estimation. However, the maximum-likelihood estimator of the GARCH parameters is found to be inconsistent [44]. The problem is that the likelihood function is mis-specified because the GARCH model fails to capture the contribution of the hidden component of volatility to the forecast errors. Fortunately, the least-squares estimator such as Kalman filter makes the estimation of GARCH parameters consistent and we can implement it in a straightforward fashion. Thus, our analysis of the kalman filtering properties suggests looking at three sets of competing volatility forecast models. First, is produced by the GARCH models. Second,
is produced by the GARCH models using Kalman filter least-squares estimates, and a third set produced by the SV models using Quasi maximum-likelihood estimates.

The models are confronted with the real data and out of sample forecast is used to evaluate the performance for 30-step ahead forecasts. The results reveal that fitting the GARCH models by Kalman filter is an effective strategy for estimation and forecasting. In fact, the Quasi maximum-likelihood version of the SV volatility model does particularly well. It outperforms all of the other models, producing one-step-ahead forecasts whose mean absolute errors and root mean squared errors are optimal. In contrast, both the GARCH(1,1) and Kalman filter versions of the GARCH model perform poorly when compared to SV model. Although the differences in performance become less pronounced as the forecast horizon gets longer, the overall ranking of the models shows little change.

The remainder of the chapter is organized as follows. Next section, discusses the State Space model estimation, illustrates the approach for constructing state space model, and shows that ARMA have a simple state space interpretation which is compared with SV model. Later sections, presents the model fitting results of the GARCH(1,1), GARCH-KF and SV models, describes our approach for assessing the forecasting performance of the models and uses out-of-sample forecast to evaluate how the models perform under these conditions. The final section, discusses the performance of the models in our volatility forecasting application with some concluding remark.

6.1 State Space Model Estimation

In this thesis, two main classes of discrete-time models are used to describe the dynamics of volatility i.e. Stochastic Volatility and GARCH. Stochastic volatility models assume that volatility follows a autoregressive (AR) process. Therefore, these models are standard models applied to a latent stochastic process. GARCH models, on the other hand, assume that volatility can be expressed as a deterministic function of the lagged squared return innovations. These models provide a parsimonious way to model changes in volatility, but they allow for randomness that is specific to the volatility process in a constrictive way.

In regard to the stochastic volatility model, much of the empirical research is focused based on the log-linear model representation given by Taylor [131]. One of the main rea-
sons for the popularity of the SV model is that we can estimate it using linear state-space
methods. We simply square the returns, take logarithms, and the result is a linear state-
space representation that can be analyzed using the Kalman filter.

In this section, we generalize the Kalman filter approach to cover models in which the
volatility or variance follows an autoregressive process and then we use the state-space
framework to forecast the associated series.

This study uses closing value of the daily SENSEX during the time period of 01 January
2006 to 22 August 2013. All the stock market index data are collected from the official
website of BSE, details are given in Section 4.2.1. The daily returns are calculated for
the series in the similar way mentioned in the previous chapter. Our final working sample
consists of 1900 data points for Sensex. In order to make forecasts, the full sample is
divided into two parts comprising 1870 in-sample observations from 01 January 2006 to 03
July 2013 and 30 out of sample observations from 04 July 2013 to 22 August 2013 which
are used for model performance evaluation.

6.1.1 ARMA Models Estimates

The goodness of fit criteria for selection of \((p,q)\)-combination is that minimizes which
minimizes the AIC. ARMA(3, 3) found to be the best model, ARMA(2, 2) is the second
best as shown in Table 6.1.

<table>
<thead>
<tr>
<th>AIC for ARMA(p, q) models</th>
</tr>
</thead>
<tbody>
<tr>
<td>p/q</td>
</tr>
<tr>
<td>-----</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>1</td>
</tr>
</tbody>
</table>

Table 6.1: AIC for ARMA (p,q) model
For Estimation of ARMA State Space Model, SSM Toolbox of Matlab which is an asymptotic solver based on maximum likelihood estimation provided by Aston and Peng(2011) [126] was used. The *estimate* and *kalman* functions of SSMODEL class were used to perform the estimate and 1-step ahead prediction for ARMA Model. The model estimates of ARMA(3,3) model in state space form after estimation are:

\[
T = \begin{bmatrix}
0.004278 & 1 & 0 \\
0.9961 & 0 & 1 \\
-0.0007445 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
R = \begin{bmatrix}
1 & 1.0714 \\
0.07902 & 0.00003086
\end{bmatrix}
\]  

(6.1)

\[
Z = \begin{bmatrix}
1 & 0 & 0 & 0
\end{bmatrix},
H_t = 0, Q = [59261.9252]
\]  

(6.2)

### 6.1.2 Stochastic Volatility Estimates

The QML estimation method was used to estimate the model parameters of SV. This was implemented by combining predefined observation Gaussian noise with constant and autoregressive model using model concatenation in SSMODEL class in the SSM toolbox of Matlab provided by Aston and Peng(2011) [126]. Hence the estimates are obtained by the Kalman filter by treating \( \epsilon_t \) and \( \eta_t \) as though they were normal and by minimizing the prediction-error. To solve the QML estimates, *estimate* and *kalman* function were used. This operation was performed on the input return series by specifying the SV Model parameters to the *estimate* function of the toolbox. The maximum likelihood estimation of state space models was performed using *kalman* function of SSM Toolbox. The *signal* functions of SSMODEL class were used to perform the estimate and 1-step ahead prediction for SV Model.

The parameter estimates of the model converge to the results as shown in Table 6.2 and Table 6.3. Estimates of **Stochastic Volatility** model in state space form can then be represented as follows:

\[
T = \begin{bmatrix}
1 & 0 \\
0 & 0.9887
\end{bmatrix},
R = \begin{bmatrix}
0 & 1
\end{bmatrix},
Z = \begin{bmatrix}
1 & 1
\end{bmatrix},
H_t = [4.834], Q = [0.02138]
\]  

(6.3)
### SV Model Parameters Estimates

<table>
<thead>
<tr>
<th>Variables</th>
<th>SV Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
<td>0.9887</td>
</tr>
<tr>
<td>$\zeta$</td>
<td>0.02138</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>4.834</td>
</tr>
</tbody>
</table>

Table 6.2: SV Model Parameter Estimate

### Parameters Estimates

<table>
<thead>
<tr>
<th>Variables</th>
<th>ARMA(3,3)</th>
<th>SV Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_1$</td>
<td>0.004278</td>
<td>0.9888</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>0.9961</td>
<td></td>
</tr>
<tr>
<td>$\phi_3$</td>
<td>-0.0007445</td>
<td></td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>1.071</td>
<td></td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>0.07902</td>
<td></td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>3.086e-05</td>
<td></td>
</tr>
<tr>
<td>$\zeta$</td>
<td>5.926e+04</td>
<td>0.0215</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td></td>
<td>4.825</td>
</tr>
</tbody>
</table>

Table 6.3: Parameter Statistics (ARMA and SV)

### 6.2 Comparison of Out of Sample forecast ARMA and SV

After obtaining the daily volatility series, 1-day ahead forecasts are chosen for the forecasting horizon of 30 days. Furthermore, a period has to be chosen for estimating parameters and a period for predicting volatility. The 1/1/2006 to 3/7/2013 of data are used to estimate the models. Thus, the first day for which an out-of-sample forecast is obtained is 04/07/2013.

Using the estimated models, sequential 1-day ahead forecasts are made. Hence, in total 30 daily volatilities are forecasted. With this setup, the models are required to predict volatility for the above mentioned period. The out of sample forecast for ARMA and
Stochastic Volatility models are shown in Figure 6.2 and Figure 6.3 respectively. This shows that the volatility forecast varies a lot between both the models. Thus, it is more relevant to define a confidence interval of the forecast and plot it along with original observations to give a better idea of the risk in the price index for both the methods.

![Estimated volatility for the SENSEX series (SV Model)](image)

Figure 6.1: Stochastic Volatility estimate

In this context, the model which has minimum forecast error terms as MSE, RMSE, MAE and MAPE, is the best volatility forecasting model. In table 6.4 it clearly shows that the SV model has less forecast error values by using all four evaluation measures.

We know that to forecast using the ARMA model, one has to use the price time series. When a price series has been transformed to return series, the conditional distribution of the log return series, will no longer be normal. If logarithms have been taken, the mean and variance of the conditional distribution becomes lognormal[107]. Therefore, to forecast the volatility by ARMA model, we have to convert the price into returns distribution using the following equation (6.4).
Forecast Error Statistics

<table>
<thead>
<tr>
<th></th>
<th>ARMA(3,3)</th>
<th>SV Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSE</td>
<td>2.5452e-6</td>
<td>2.9666e-8</td>
</tr>
<tr>
<td>RMSE</td>
<td>0.0016</td>
<td>1.7224e-4</td>
</tr>
<tr>
<td>MAE</td>
<td>0.0013</td>
<td>1.4227e-4</td>
</tr>
<tr>
<td>MAPE</td>
<td>0.7760</td>
<td>0.1589</td>
</tr>
</tbody>
</table>

Table 6.4: Forecast Error Statistics (ARMA Vs SV)

\[ Var(r_t) = \log \left( 1 + \frac{m^2}{(1+m1)^2} \right) \]  \hspace{1cm} (6.4)

Where, \( m1 \) and \( m2 \) are mean and variance of forecasted price series of ARMA model respectively.
Figure 6.3: Stochastic Volatility Out of Sample Forecast

Figure 6.4: Volatility Forecast Comparison (ARMA and SV)
The results of volatility forecasting of SV model is shown in figure 6.1. The Figure 6.4 shows the 30-day ahead point forecast of both the models plotted along with the actual volatility, which is used as benchmark calculated using equation (5.7). It seems that stochastic volatility has more appropriate forecast as it has lesser residual errors when both models are estimated using MLE/QML based technique.

### 6.3 Comparison of Out of Sample forecast GARCH(1,1), GARCH-KF and SV

The volatility as the results strongly indicate is not stationary, and the time-varying behavior of the volatility depends on the chosen modeling technique. In order to determine the relatively best model of volatility forecast, the quality or the goodness of fit of the estimated volatility is evaluated based on the diagnostic statistics as discussed in Section 5.3.

If we look at the way volatilities are derived, the GARCH based variances are constructed indirectly, and conditional volatility derived by a state space model are calculated directly. The results based on in-sample forecasts refer to model fitting aspects of the respective models, they therefore cannot be employed for comparative purposes for predictive performance of the models. To avoid these problems, the different modeling techniques are formally ranked based on their out-of-sample forecast performance. Following this, the results of the main criteria used to evaluate and compare the respective out-of-sample forecast are shown in Table 6.5. The table also shows the respective performances of MSE, RMSE, MAPE and MAE for each approach under consideration.

The α and β estimates for GARCH based models indicates a high level of persistence, regardless of the estimation model used. One observation is that the Kalman Filter based estimation tend to suggest lesser persistence than the standard GARCH maximum-likelihood estimates for the volatility model. In other words, the mean square errors of the estimates of the GARCH model tend to be larger than the SV based estimates. This implies that the GARCH based MLE forecasts will exhibit a larger response to a given return shock than the Stochastic Volatility forecasts.

After obtaining the daily volatility series, 1-day ahead forecasts are chosen for the fore-
casting horizon of 30 days. Furthermore, a period has to be chosen for estimating parameters and a period for predicting volatility. The 1/1/2006 to 7/7/2013 of data are used to estimate the models. Thus the first day for which an out-of-sample forecast is obtained is 08/07/2013. The reason why the out-of-sample period is chosen to be limited to 30 observations is to ensure proper convergence of the conditional volatility models. The current thesis deals with the volatility forecasting at a more short term horizon, therefore, an estimation period corresponding to seven years of data is justified. It contains enough data to generate stable parameter estimates and it is short enough to reflect current market conditions.

The results of volatility estimated value by using GARCH(1,1), GARCH(1,1)-KF and SV model is shown in Figure 6.5.

![Estimated Volatility for SENSEX](image)

**Figure 6.5: In Sample Volatility estimate (GARCH(1,1), GARH-KF and SV)**

As we will see from this analysis that the Kalman Filter based models prove to be superior in an out-of-sample context. The GARCH or stochastic volatility model require lesser observations when modeled using the Kalman Filter approach. Important thing to note is that the Kalman Filter naturally puts more weight on the most recent observations, the out-of-sample results for the Kalman Filter models do not critically depend on the length of the estimation period.

The goal of this analysis is to establish, that the Kalman Filter based model generates
the best out-of-sample forecasts of the volatility using the real data. To identify overall best approach, the models are estimated, sequential 1-day ahead forecasts are made. Hence, in total 30 daily volatility values are forecasted. With this setup, the models are required to predict volatility for the above mentioned period. The out of sample forecast for GARCH(1,1), GARCH(1,1)-KF and Stochastic Volatility models are shown in figure 6.6.

![Out of Sample Volatility Forecast for SENSEX](image)

Figure 6.6: Out of Sample Forecast (GARCH(1,1), GARH-KF and SV)

A comparison of the results confirms the conjecture that the forecast performance of standard GARCH(1,1) is worse among the volatility forecasting techniques. As compared to the GARCH-KF and the Stochastic Volatility approaches, the degree of GARCH(1,1) inferiority is remarkably low.

The MSE of GARCH(1,1)-KF equals 0.0790, which is only slightly higher than the MSE for the GARCH(1,1) model. For the investigated sample, the Stochastic Volatility techniques clearly outperform their competitors. The average mean squared error of 0.0340 which is nearly 50% lower than in case of GARCH(1,1)-KF. With respect to all four error measures, the SV model ranks first. The GARCH(1,1)-KF model ranks behind the SV model. As the proposed modification to standard GARCH was intended to capture time varying variance and to yield better estimate of the volatility than the standard GARCH model, the comparably weak in-sample performance of GARCH(1,1) is also not surprising.
Within the class of volatility models, the SV approach seems to be better qualified to model the time-varying volatility than the well-established GARCH model. On average, the MAE (MAPE) for the SV model is 0.1552 (0.1619) lower than the error measures for the GARCH-based models, and thus is the best overall model.

In Table 6.5 it clearly shows that the SV model has less forecast error values model by using all four evaluation measures.

<table>
<thead>
<tr>
<th>Forecast Error Statistics</th>
</tr>
</thead>
<tbody>
<tr>
<td>GARCH(1,1)</td>
</tr>
<tr>
<td>MSE</td>
</tr>
<tr>
<td>RMSE</td>
</tr>
<tr>
<td>MAE</td>
</tr>
<tr>
<td>MAPE</td>
</tr>
</tbody>
</table>

Table 6.5: Forecast Error Statistics (GARCH(1,1), GARCH-KF and SV)

Overall, the analysis of out-of-sample estimates suggests that Sensex volatility as modeled by one of the proposed Kalman filter approaches are superior to the considered al-
ternatives. This is in line with previous findings presented by Taufiq [6] for the daily UK market.

The Figure 6.7 shows the 30-day ahead point forecast of both the models plotted along with the actual volatility, which is used as benchmark calculated using equation (5.7). It seems that stochastic volatility has more appropriate forecast as it has lesser residual errors then the other competing models.