2.1 **INTRODUCTION**

This chapter is devoted to provide all the fundamental terminology and notations which are useful for the present work. Basic definitions are given with sufficient illustrations.

2.2 **BASIC DEFINITIONS**

1. **(Graph)**

   A *graph* \( G = (V, E) \) consists of two finite sets: \( V(G) \), the *vertex set* of the graph which is a nonempty set of element called *vertices*, and \( E(G) \), the *edge set* of the graph which is a possibly empty set of element called *edges*, such that each edge \( e \) in \( E(G) \) is assigned an unordered pair of vertices \((u, v)\), called the *end vertices* of \( e \).

2. **(Order of a Graph)**

   The number of vertices in \( G \) is called the *order of a graph* \( G \). It is denoted by \(|V(G)|\).

3. **(Size of a Graph)**

   The number of edges in \( G \) is called the *size of a graph* \( G \). It is denoted by \(|E(G)|\).
4. (Loop/Self loop) :
An edge of a graph which joins a vertex to itself is called a loop. A loop (self loop) at the vertex \( v_i \) is an edge \( e = (v_i, v_i) \).

5. (Multiple edges) :
If two vertices of a graph are joined by more than one edge then these edge are called multiple edges.

6. (Simple graph) :
A graph which has neither loops nor multiple edges is called a simple graph.

7. (Adjacent vertices) :
If two vertices of a graph are joined by an edge then these vertices are called adjacent vertices.

8. (Degree of vertices) :
The number of edges incident on vertex \( v \) of any graph \( G \) is called degree of \( v \). It is denoted by \( \deg(v) \) or \( d(v) \) or \( d_G(v) \).

9. (Incident edges) :
Two edges that have an end vertex in common are called incident edges.

10. (Endpoint/Pendent vertex) :
A vertex of a graph of degree 1 is called endpoints or pendent vertex. An edge of the graph \( G \) which is incident with a pendent vertex is called a pendent edge

11. (Connected and Disconnected graph) :
A graph is said to be connected if there is a path between every pair of vertices of \( G \). A graph which is not connected is called a disconnected graph.
Illustration—2.2.1 : Let us consider the following graph $G$.

In graph $G$ shown in figure—2.1

- Order of graph $G$ is 8, Size of graph $G$ is 8.
- $e_6$ forms a loop at $v_5$, $e_3$ and $e_4$ are multiple edges.
- $v_1$ and $v_2$ are adjacent vertices by $e_1$ edge.
- $d(v_5) = 3$, $d(v_7) = 2$, $d(v_6) = 1$.
- $e_1$ and $e_2$ are incident edges at $v_2$ vertex.
- $v_1$, $v_6$ and $v_8$ are endpoints.
- $G$ is disconnected graph.
12. (Walk) :

A walk is defined as a finite alternating sequence of vertices and edges of the form $v_0e_1v_1e_2...e_nv_n$ which start and end a vertex and each edge in the sequence is incident on the vertex immediately preceding and succeeding it in the sequence.

- The number of edges in a walk is called the **length of the walk**.
- Beginning and ending vertices are equal then, it is called a **closed walk**.

13. (Path) :

A walk in which no vertex is repeated is called a **path**. A path with $n$ vertices is denoted by $P_n$.

14. (Trail) :

A walk in which no edge is repeated is called a **trail**.

15. (Cycle) :

A closed path in which no vertex is repeated except the terminal vertex is called a **cycle**. A cycle with $n$ vertices is denoted by $C_n$.

16. (Unicyclic graph) :

A graph $G$ with exactly one cycle is called a **unicyclic graph**.

**Illustration—2.2.2** : Consider the following graph $G$ shown in figure—2.2.

![Figure—2.2](image-url)
For this graph we note the followings:

• \( W = v_2e_2v_3e_3v_4e_4v_5e_5v_6e_6v_2 \) is a closed walk.

• \( P_4 = v_2e_2v_3e_3v_4e_4v_5 \) is a path of length 3.

• \( C_5 = v_2e_2v_3e_3v_4e_9v_5e_8v_1e_1v_2 \) is a cycle of length 5.

17. \( \textbf{(Euler graph)} : \)

Let \( G = (V, E) \) be a graph. A closed trail in \( G \) is called an \textit{Euler line} if it contains all the edges of the graph \( G \). A graph \( G \) is called an \textit{Euler graph} if it admits an Euler line.

18. \( \textbf{(Tree)} : \)

A graph \( G \) is called a \textit{acyclic graph}, if it does not contain any cycle. A connected graph \( T \) is called a \textit{tree}, if it is an acyclic graph.

19. \( \textbf{(Caterpillar)} : \)

A \textit{caterpillar} is a tree with the property that the removal of its pendant vertices leaves a path. This path is known as \textit{spine} of the caterpillar.

20. \( \textbf{(Complete graph)} : \)

A \textit{complete graph} is a simple graph such that every pair of vertices is joined by an edge. A complete graph on \( n \) vertices is denoted by \( K_n \).

21. \( \textbf{(Regular graph)} : \)

A \textit{regular graph} is a graph if degree of each vertices are same.

22. \( \textbf{(Bipartite graph)} : \)

A graph \( G \) is said to be \textit{bipartite} if the vertices can be partitioned into two disjoint subsets \( V_1 \) and \( V_2 \) such that for every edge \( e_i = (v_j, v_k) \in E(G) \), \( v_j \in V_1 \) and \( v_k \in V_2 \).
23. **(Complete bipartite graph)**:

A simple bipartite graph, whose two vertices are adjacent if and only if they are in different partite sets is called a *complete bipartite graph*.

If partite sets $V_1$ and $V_2$ are having $m$ and $n$ vertices respectively then the related complete bipartite graph is denoted by $K_{m,n}$ and $V_1$ is called $m$-vertices part and $V_2$ is called $n$-vertices part of $K_{m,n}$.

24. **($k$-regular graph)**:

A regular graph with vertices of degree $k$ is called a *$k$-regular graph*.

25. **(Chord)**:

A *chord* of a cycle $C_n$ is an edge joining two non adjacent vertices of $C_n$ and such cycle $C_n$ is known as *cycle with one chord*.

26. **(Twin chords)**:

Two chords of a cycle $C_n (n \geq 5)$ are said to be *twin chords* if they form a triangle with an edge of the cycle $C_n$.

27. **(Star graph)**:

A complete bipartite graph $K_{1,n}$ is known as *star graph*.

28. **(Banana tree)**:

A *banana tree* is a tree which is obtained from a family of stars by joining one end vertex of each star to a new vertex.

29. **(Arbitrary super subdivision)**:

Let $G$ be a graph. A graph $H$ is called a *arbitrary super subdivision* of $G$ if $H$ is obtained from $G$ by replacing every edge $e_i$ of $G$ by a complete bipartite graph $K_{2,m_i}$ for some $m_i \in N$, $1 \leq i \leq q$ in such a way that the ends of each $e_i$ are merged with the two vertices part of $K_{2,m_i}$ after removing the edge $e_i$ from the graph $G$. 
30. \textbf{(t-super subdivision)}:

Let $G = (V, E)$ be a graph with $p$ vertices and $q$ edges. A graph $H$ is said to be a \textit{t-super subdivision} of $G$ if $H$ is obtained from $G$ by replacing every edge $e$ of $G$ by a complete bipartite graph $K_{2,t}$ for some $t \in \mathbb{N}$.

31. \textbf{(Cartesian product)}:

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ two graphs. Then the \textit{cartesian product} of $G_1$ and $G_2$ which is denoted by $G_1 \times G_2$ is the graph with vertex set $V = V_1 \times V_2$ consisting of vertices $u = (u_1, u_2)$, $v = (v_1, v_2)$ such that $u$ and $v$ are adjacent in $G_1 \times G_2$ whenever $(u_1 = v_1$ and $u_2$ adjacent to $v_2)$ or $(u_2 = v_2$ and $u_1$ adjacent to $v_1)$.

32. \textbf{(Tensor product)}:

The \textit{tensor product} $G \times H$ of graphs $G$ and $H$ is a graph such that the vertex set of $G \times H$ is the cartesian product $V(G) \times V(H)$ and any two vertices $(u, u')$ and $(v, v')$ are adjacent in $G \times H$ if and only if $u'$ is adjacent with $v'$ and $u$ is adjacent with $v$.

33. \textbf{(Grid graph)}:

The cartesian product of two paths $P_n, P_m$ is known as grid graph, which is denoted by $P_n \times P_m$.

34. \textbf{(Wheel)}:

The \textit{wheel graph} $W_n$ is defined to be the join $K_1 + C_n$. Apex vertex is the vertex which is corresponding to $K_1$ and vertices corresponding to cycle are called rim vertices while the edges corresponding to cycle are called rim edges. We continue to recognize apex of wheel as the apex of respective graphs obtained from wheel.
35. (Web graph) :

A Web graph is the graph obtained by joining all the pendant vertices of a helm to form a cycle and then adding a single pendant edge to each vertex of the outer cycle.

36. (Petersen graph) :

The Petersen graph is an undirected graph with 10 vertices and 15 edges, as shown in figure 2.3.

![Petersen graph](image)

37. (Cycle of graph) :

Let $G$ be a graph and $G^{(1)}, G^{(2)}, \ldots, G^{(n)}$, $n \geq 2$ be $n$ copies of $G$. Let $v \in V(G)$. The graph obtained by joining vertex $v$ of $G^{(i)}$ with same vertex of $G^{(i+1)}$ by an edge, $\forall i = 1, 2, \ldots, n - 1$ and the same vertex $v$ of $G^{(n)}$ with the vertex $v$ of $G^{(1)}$ by an edge is called cycle of graph. It is denoted by $C(n \cdot G)$. If we replace $G$ by $C(n \cdot G)$, such graph becomes $C(n \cdot C(n \cdot G))$, we denote it by $C^2(n \cdot G)$. In general for any $t \geq 2$, $C^t(n \cdot G) = C(n \cdot C^{t-1}(n \cdot G))$. It is obvious that, $C(n \cdot K_1) = C_n$.

38. (Mean graph) :

A function $f$ is called mean labeling of a graph $G = (V, E)$ if $f : V \rightarrow \{0, 1, \ldots, q\}$ is injective and the induced function $f^* : E \rightarrow \{1, 2, \ldots, q\}$ defined as $f^*(e) = \left\lceil \frac{f(u) + f(v)}{2} \right\rceil$ is bijective for every edge $e = (u, v) \in E$. A graph $G$ is called mean graph if it admits a mean labeling.
39. (Star of G) :

Let $G$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$. Let $G^{(0)}, G^{(1)}, \ldots, G^{(p)}$ be $p+1$ copies of $G$. Join each vertex $v_i$ of $G^{(0)}$ with the corresponding vertex $v_i$ of $G^{(i)}$, $\forall \ i = 1, 2, \ldots, p$. Such graph is known as star of $G$ and it is denoted by $G^*$. We call $G^{(0)}$ as central copy of $G^*$. It is obvious that $K_2^* = P_6$.

40. (Path union) :

Let $G$ be a graph and $G^{(1)}, G^{(2)}, \ldots, G^{(n)}$, $n \geq 2$ be $n$ copies of $G$. The graph obtained by joining vertex $v$ of $G^{(i)}$ with same vertex of $G^{(i+1)}$ by an edge, $\forall \ i = 1, 2, \ldots, n-1$ is called the path union of graph $G$, It is denoted by $P(n \cdot G)$. If $G = K_1$ then $P(n \cdot K_1) = P_n$.

41. (Joint sum of graphs) :

Consider $t$ copies of a graph $G_0$. Then graph $G = \langle G^{(1)}_0; G^{(2)}_0; \ldots; G^{(t)}_0 \rangle$ obtained by joining two copies of the graph $G^{(i)}_0$ and $G^{(i+1)}_0$ by a vertex $1 \leq i \leq t-1$ is called joint sum of graphs.

42. (One point union) :

A graph $G$ obtained by replacing each edge of $K_{1,t}$ by a path $P_n$ on $n+1$ vertices is called one point union for $t$ copies of path $P_n$. We denote such graph $G$ by $P^t_n$.

43. (Open star of graphs) :

A graph obtained by replacing each vertex of $K_{1,n}$ except the apex vertex by the connected graphs $G_1, G_2, \ldots, G_n$ is known as open star of graphs. We denote such graph by $S(G_1, G_2, \ldots, G_n)$.

If we replace each vertices of $K_{1,n}$ except the apex vertex by a connected graph $G$. i.e. $G_1 = G$, $G_2 = G$, ..., $G_n = G$, such open star of graph is denoted by $S(n \cdot G)$.
44. (Barycentric subdivision):

If every edge of a graph $G$ is subdivided into two edges by a new vertex, then the resulting graph is called barycentric subdivision of the graph $G$. In other word 1-super subdivision of $G$ is known as barycentric subdivision of the graph $G$.

45. (Index of cordiality):

The index of cordiality for $G$ is $n$ if union of $n$ copies of $G$ is cordial, but union of less than $n$ copies of $G$ do not have cordial labeling.

46. ($\alpha$-labeling):

A function $f$ is called $\alpha$-labeling of a graph $G$ if $f$ is a graceful labeling (See Definition−3.2.1) for $G$ and there exist an integer $k$ ($0 \leq k \leq q−1$) such that for every $e = (x, y) \in E(G)$, either $f(x) \leq k < f(y)$ or $f(y) \leq k < f(x)$. A graph $G$ with an $\alpha$-labeling is necessarily bipartite graph.

47. (Harmonious graph):

A function $f$ is called harmonious labeling of a graph $G$ if $f: V(G) \rightarrow \{0, 1, 2, \ldots, q−1\}$ is injective and the induced function $f^*: E(G) \rightarrow \{0, 1, 2, \ldots, q−1\}$ defined as $f^*(e) = (f(u) + f(v)) \mod q$ is bijective, $\forall e = (u, v) \in E(G)$. A graph which admits harmonious labeling is called harmonious graph.

48. (Super graph):

A subgraph of a graph $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. In such a case, $G$ is known as a super graph of $H$.

49. (Comb tree):

A comb tree means a graph obtained by taking path union of $t$ copies of path $P_n$ at pendent vertex of $P_n$. 
50. (Simple lobster) :

A simple lobster means a path union of $t$ copies of $P_5$ in which join these consecutive copies of $P_5$ by an edge at the middle vertex of $P_5$.

2.3 CONCLUDING REMARKS :

Chapter–2 provides a brief account of basic concept, definitions and some notations which are prerequisites for advancement of the remaining chapters. Common families of graph like cycle, path, wheel, tree, grid graph, complete graph and complete bipartite graph are defined. We have tried our best to prepare platform for the remaining part of thesis.

For other standard terminology and notations we refer Harary [10], West [17], Gross and Yellen [9] and Clark and Holton [6]. The next chapter–3 is aimed to discuss about graceful labeling, some known results about graceful labeling, smooth graceful labeling, semi smooth graceful labeling and their applications to get graceful labeling for some graphs obtained by some graph operations.