Chapter − 5

Cordial and Related Labeling of a Graph

5.1 Introduction:


Definition−5.1.1:

A function \( f : V(G) \rightarrow \{0, 1\} \) is called a binary vertex labeling of a graph \( G \) and \( f(v) \) is called label of the vertex \( v \) of \( G \) under \( f \).

For an edge \( e = (u, v) \), the induced function \( f^* : E(G) \rightarrow \{0, 1\} \) defined as \( f^*(e) = |f(u) - f(v)| \), for every edge \( e = (u, v) \in E \). Let \( v_f(0), v_f(1) \) be number of vertices of \( G \) having labels 0 and 1 respectively under \( f \) and let \( e_f(0), e_f(1) \) be number of edges of \( G \) having labels 0 and 1 respectively under \( f^* \).

A binary vertex labeling \( f \) of a graph \( G \) is called a cordial labeling if \( |v_f(0) - v_f(1)| \leq 1 \) and \( |e_f(0) - e_f(1)| \leq 1 \).

For a graph \( G = (V(G), E(G)) \), a vertex labeling function \( f : V(G) \rightarrow \{0, 1\} \) induces an edge labeling function \( f^* : E(G) \rightarrow \{0, 1\} \) defined as \( f^*(e) = f(u) \ast f(v) \), for every edge \( e = (u, v) \in E \).
Then $f$ is called a *product cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

For a graph $G = (V(G), E(G))$, an edge labeling function $f : E(G) \rightarrow \{0, 1\}$ induces a vertex labeling function $f^* : V(G) \rightarrow \{0, 1\}$ defined as $f^*(v) = \Pi f(e)$, for every edge $e = (u,v) \in E$.

Then $f$ is called a *total edge product cordial labeling* if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Cahit [5] proved that every tree is cordial, $K_n$ is cordial $\iff n \leq 3$, $K_{m,n}$ is cordial, $\forall m,n \in \mathbb{N}$, all fans are cordial, wheel $W_n$ is cordial $\iff n \not\equiv 3 \pmod{4}$ and friendship graph (One point union of $t$ copies of the cycle $C_3$) $C_3^{(t)}$ is cordial $\iff t \not\equiv 2 \pmod{4}$.

### 5.2 Cordiality of Graphs:

We know that if we label $v_f(0)$ by 1 and $v_f(1)$ by 0, then $e_f(0)$, $e_f(1)$ would not change in any graph $G$. Also in $K_n$ if we take $v_f(0) = k$, $v_f(1) = n - k$ or $v_f(1) = k$, $v_f(0) = n - k$ then $e_f(1) = k(n - k)$ and $e_f(1) = \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} = \frac{n(n-1)}{2} - k(n - k)$. Thus, to get cordial labeling for $K_n \cup K_n$, it should be require $e_f(0) = e_f(1)$ in both copies of $K_n$. i.e. in each copy of $K_n$, $e_f(1) = k(n - k) = \frac{n(n-1)}{2} - k(n = k) = e_f(0)$.

i.e. $4k(n - k) = n(n - 1)$

$\Rightarrow n^2 - (1 + 4k)n + 4k^2 = 0$

$\Rightarrow n = \frac{1+4k \pm \sqrt{1+8k}}{2}$

$\Rightarrow \sqrt{1+8k}$ should be an integer.

$\Rightarrow k = 0, 1, 3, 6, 10, \ldots$ and $n = 1, 4, 9, 16, 25, \ldots$
i.e. $n$ must be $t^2$ for some $t \in \mathbb{N}$. This proves the sufficient condition for cordiality of union of two copies of the complete graph $K_n$.

Let $f$ be a binary vertex labeling on $K_n$. It is obvious that if $\{f^{-1}(0) = v_f(0), f^{-1}(1) = v_f(1)\} = \{n - l, l\}$, for some $l < n$ in $K_n$ then $e_f(1) = v_f(0) \cdot v_f(1)$ and $e_f(0) = \{C_2 + n-l, C_2\}$.

Kaneria, Makadia and Meera [13] proved that the maximum difference $d_1 = e_f(1) - e_f(0)$ is $\lfloor \frac{n}{2} \rfloor$ when $|v_f(1) - v_f(0)| \leq 1$. They have also proved that by taking $\{v_f(1), v_f(0)\} = \{\frac{n}{2} - i, \frac{n}{2} + i\}$ in $K_n$, for some $i$ ($1 \leq i \leq \frac{n}{2}$), the edge label difference $d_i = e_f(1) - e_f(0) = \frac{n-1}{2} - 2(i^2 + i)$, when $n$ is odd and it is $\frac{n}{2} - 2i^2$, when $n$ is even. This is a decreasing sequence and it stops at $-|E(K_n)|$ by taking $\{v_f(1), v_f(0)\} = \{0, n\}$. In the same paper they have proved that

1. If $n = 4t^2 + 2r$, for some $t, r \in \mathbb{N}$ and $1 \leq r \leq 4t + 1$ then the first negative difference $d_{-1} = e_f(1) - e_f(0)$ in $K_n$ is $-(4t + 2) + r$.
2. If $n = (2t - 1)^2 + 2r$, for some $t, r \in \mathbb{N}$ and $1 \leq r \leq 4t - 1$ then the first negative difference $d_{-1} = e_f(1) - e_f(0)$ in $K_n$ is $-4t + r$.

From the proof of theorem 2.4, 2.5 of Kaneria, Makadia and Meera [13] following facts are obvious.

3. The least positive difference $d_{lp} = e_f(1) - e_f(0)$ in $K_n$ is $r$, when $n = 4t^2 + 2r$ or $n = (2t - 1)^2 + 2r$, for some $t, r \in \mathbb{N}$.
4. The zero difference in $K_n$ (i.e. $d_0 = e_f(1) - e_f(0)$ is zero) is possible, when $n$ is a perfect square.

**Theorem**–5.2.1 : The index of cordiality for $K_n(n \leq 105)$ is at most 6.

**Proof** : Here we have proved that $\bigcup_{i=1}^lK_n$ is cordial, where $l \leq 6$ and $n \leq 105$. 

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*Table 2*
Kaneria, Makadia and Meera [13] have proved that the maximum edge label difference $d_1 = e_f(1) - e_f(0)$ in $K_n$ is $\lfloor \frac{n}{2} \rfloor$, when $|v_f(1) - v_f(0)| \leq 1$. They have also proved that by taking $\{v_f(1), v_f(0)\} = \{\frac{n}{2} - i, \frac{n}{2} + i\}$ in $K_n$, for some $i$ ($1 \leq i \leq \frac{n}{2}$), then the edge label difference

$$d_i = \frac{n}{2} - 2i^2,$$

when $n$ is even and

$$d_i = \frac{n-1}{2} - 2i(i + 1),$$

when $n$ is odd.

Using these formulas table–1, 2 have computed and they contain type of vertex labels to be require to get conditions $|v_f(1) - v_f(0)|, |e_f(1) - e_f(0)| \leq 1$ in union of some copies of $K_n$. These tables–1, 2 contain maximum index of cordiality for $K_n(n \leq 105)$, which shows that the index of cordiality for $K_n(n \leq 105)$ is atmost 6.

**Illustration–5.2.2** : Index of cordiality for $K_{60}$ is atmost 6. i.e. $\cup_{i=1}^{6} K_{60}$ is a cordial graph. This follows from the table–3, 4.

Using table–3, $2a + 2d + 2g$ type vertex labeling will give require vertex and edge labeling conditions to the graph $\cup_{i=1}^{6} K_{60}$ and so, it is a cordial graph according to table–4.

<table>
<thead>
<tr>
<th>Row</th>
<th>diff</th>
<th>${v_f(1), v_f(0)}$</th>
<th>$e_f(1)$</th>
<th>$e_f(0)$</th>
<th>$d_k = e_f(1) - e_f(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$d_1$</td>
<td>[30,30]</td>
<td>900</td>
<td>870</td>
<td>30</td>
</tr>
<tr>
<td>b</td>
<td>$d_2$</td>
<td>[31,29]</td>
<td>899</td>
<td>871</td>
<td>28</td>
</tr>
<tr>
<td>c</td>
<td>$d_3$</td>
<td>[32,28]</td>
<td>896</td>
<td>874</td>
<td>22</td>
</tr>
<tr>
<td>d</td>
<td>$d_4$</td>
<td>[33,27]</td>
<td>891</td>
<td>879</td>
<td>12</td>
</tr>
<tr>
<td>e</td>
<td>$d_5$</td>
<td>[34,26]</td>
<td>884</td>
<td>886</td>
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<tr>
<td>f</td>
<td>$d_6$</td>
<td>[35,25]</td>
<td>875</td>
<td>895</td>
<td>-20</td>
</tr>
<tr>
<td>g</td>
<td>$d_7$</td>
<td>[36,24]</td>
<td>864</td>
<td>906</td>
<td>-42</td>
</tr>
<tr>
<td>h</td>
<td>$d_8$</td>
<td>[37,23]</td>
<td>851</td>
<td>919</td>
<td>-68</td>
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</table>

*Table–3*
Theorem 5.2.3: The index of cordiality for $K_n$ is atmost 4, when $n = i^2 + j^2$, for some $i, j \in N$.

Proof: Here we prove that union of four copies of $K_n$ is cordial. It is enough to show that $d_t + d_l = 0$, for some $t, l \in N$ and $1 \leq t, l \leq \max\{i, j\} + 1$. We assume here $n = i^2 + j^2$, for some $i, j \in N$ and $i \geq j$. Here following cases to be consider.

Case I: $i = j$. In this case $n$ is even.
Moreover $d_1 = \frac{i^2 + j^2}{2} = i^2$, $d_2 = i^2 - 2$, $d_3 = i^2 - 8$, \ldots, $d_k = i^2 - 2k + 4k - 2$.

If $d_1 + d_k = 0$ then $d_k = i^2 - 2k^2 + 4k - 2 = -i^2$
$\Rightarrow 2i^2 = 2k^2 - 4k + 2$
$\Rightarrow i^2 = (k - 1)^2$
$\Rightarrow k = i + 1$. In this case we get $d_1 + d_{i+1} = 0$.

Case II: $i = j + 1$. In this case $n$ is odd.
Moreover $d_1 = \frac{1}{2}(i^2 + j^2 - 1) = i(i - 1)$, $d_2 = i(i - 1) - 4$, $d_3 = i(i - 1) - 10$, \ldots, $d_k = i(i - 1) - 2k(k - 1)$.

If $d_1 + d_k = 0$ then $i(i - 1) - 2k(k - 1) = -i(i - 1)$
$\Rightarrow k = i$. So, we get $d_1 + d_i = 0$. 

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<th>order of copy in $K_{60}$</th>
<th>$v_f(0)$</th>
<th>$v_f(1)$</th>
<th>$e_f(1)$</th>
<th>$e_f(0)$</th>
<th>diff</th>
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<td>30</td>
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<td>870</td>
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<tr>
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<td>30</td>
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<td>870</td>
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<td>879</td>
<td>12</td>
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<td>24</td>
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<td>180</td>
<td>5310</td>
<td>5310</td>
<td>0</td>
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</table>

Table 4

50
Case–III : \( i = j + 2 \). Here \( n \) is even and
\[
d_1 = \frac{1}{2}(i^2 + (i-2)^2) = i^2 - 2i + 2, d_2 = i^2 - 2i, \ldots, d_k = i^2 - 2(i+2) - 2k^2 + 4k - 2.
\]
If \( d_2 + d_k = 0 \) then \( i^2 - 2i - 2k^2 + 4k = -i^2 + 2i \)
\( \Rightarrow k = i \) and so, we get \( d_2 + d_i = 0 \).

Case–IV : \( i = j + 3 \). Here \( n \) is odd and
\[
d_1 = i^2 - 3i + 4, d_2 = i^2 - 3i, \ldots, d_k = i^2 - 3i - 4 - 2k(k - 1).
\]
If \( d_2 + d_k = 0 \) then \( i^2 - 3i + 4 - 2k(k - 1) = -i^2 + 3i \)
\( \Rightarrow i^2 - 3i + 2 = k(k - 1) \)
\( \Rightarrow (i - 1)(i - 2) = k(k - 1) \)
\( \Rightarrow k = i - 1 \) and so, we get \( d_2 + d_{i-1} = 0 \).

General Case : Take \( i = j + s \) i.e. \( s = i - j \).

Subcase–I : \( s \) is even. In this case either \( i, j \) both are even or both are odd
and so, \( n = i^2 + j^2 \) is even.

We would prove here \( d_t + d_l \) and \( t = i - \frac{s}{2} - 1, l = \frac{s}{2} + 1 \).

Since \( n \) is even, \( d_1 = \frac{1}{2}(i^2 + j^2) \), \( d_2 = \frac{1}{2}(i^2 + j^2) - 2, \ldots, d_k = \frac{1}{2}(i^2 + j^2) - 2k^2 + 4k - 2 \).
\( \Rightarrow d_t = \frac{1}{2}(i^2 + j^2) - 2(i - \frac{s}{2} - 1)^2 - 4(i - \frac{s}{2} - 1) - 2 \)
\( = \frac{1}{2}(i^2 + j^2) - 2(i^2 + 1 + \frac{s^2}{4} - is - 2i + s) - 4i + 2s + 2 \)
\( = \frac{1}{2}(j^2 - s^2 - 3i^2) + 2is \)
and \( d_l = \frac{1}{2}(i^2 + j^2) - 2(\frac{s}{2} + 1)^2 + 4(\frac{s}{2} + 1) - 2 \)
\( = \frac{1}{2}(i^2 + j^2) - 2(\frac{s^2}{4} + s + 1) + 2s + 2 \)
\( = \frac{1}{2}(i^2 + j^2 - s^2) \)
\( \Rightarrow d_t + d_l = j^2 - s^2 - i^2 + 2is \)
\( = j^2 - (i - j)^2 - i^2 + 2i(i - j) \)
\( = j^2 - i^2 - j^2 + 2ij - i^2 + 2i^2 - 2ij = 0 \)

51
**Subcase—II :** $s$ is odd.

In this case $n = i^2 + j^2$ is odd.

We would prove that $d_t + d_l = 0$ and $t = i - \frac{s-1}{2}$, $l = \frac{s+1}{2}$.

Since $n$ is odd, $d_1 = \frac{1}{2}(i^2 + j^2 - 1)$, $d_2 = \frac{1}{2}(i^2 + j^2 - 1) - 4$, ..., $d_k = \frac{1}{2}(i^2 + j^2 - 1) - 2k(k - 1)$.

\[
\Rightarrow d_t = \frac{1}{2}(i^2 + j^2 - 1) - 2(i - \frac{s-1}{2})(i - \frac{s-1}{2} - 1)
\]
\[
= \frac{1}{2}(i^2 + j^2 - 1) - 2[i^2 - \frac{(s-1)^2}{4} - i(s - 1) - i + \frac{s-1}{2}]
\]
\[
= \frac{1}{2}(i^2 + j^2 - 1) - 2i^2 - \frac{1}{2}(s^2 - 2s + 1) + 2is - s + 1
\]
\[
= \frac{1}{2}(i^2 + j^2 - 1) - 2i^2 - \frac{s^2}{2} + s + 2is - s + \frac{1}{2}
\]
\[
= \frac{1}{2}(i^2 + j^2) - 2i^2 - \frac{s^2}{2} + 2is
\]

and $d_l = \frac{1}{2}(i^2 + j^2 - 1) - 2(\frac{s+1}{2})(\frac{s-1}{2})$
\[
= \frac{1}{2}(i^2 + j^2 - 1) - \frac{1}{2}(s^2 - 1)
\]
\[
= \frac{1}{2}(i^2 + j^2 - s^2)
\]

\[
\Rightarrow d_t + d_l = i^2 + j^2 - s^2 - 2i^2 + 2is = j^2 - s^2 - i^2 + 2is = 0
\]

Table—5 computed according to above cases and it shows values of $t$ and $l$ to get $d_t + d_l = 0$. Thus, $\cup_{i=1}^{4} K_n (n = i^2 + j^2)$ is a cordial graph and so, the index of cordiality for $K_n (n = i^2 + j^2)$ is atmost four.

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<th>Status of $n$</th>
<th>Value of $t$</th>
<th>Value of $l$</th>
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<td>$i+1$</td>
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<td>$i-1$</td>
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<td>$i$</td>
<td>1</td>
</tr>
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<td>3</td>
<td>$i-2$</td>
<td>even</td>
<td>$i$</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>$i-3$</td>
<td>odd</td>
<td>$i-1$</td>
<td>2</td>
</tr>
<tr>
<td>General</td>
<td>$i-s \ (i.e. \ s=i-j)$</td>
<td>odd/even</td>
<td>$i-[(s-1)/2]$</td>
<td>$i-[(s+1)/2]$</td>
</tr>
</tbody>
</table>

Table—5
Illustration−5.2.4 : Index of cordiality for $K_{148}$ is atmost 4.

The edge label difference sequence $K_{148}$ is $d_1 = 74$, $d_2 = 72$, $d_3 = 66$, $d_4 = 56$, $d_5 = 42$, $d_6 = 24$, $d_7 = 2$ and $d_8 = -24$. According to table−5, $148 = 12^2 + 2^2$ and so, value of $t = 12 - \lfloor \frac{10}{2} \rfloor = 8$, $l = \lceil \frac{10+1}{2} \rceil = 6$. Moreover $d_6 + d_8 = 0$. Table−6 shows that the union of four copies of $K_{148}$ is cordial and so, the index of cordiality for $K_{148}$ is atmost four.

<table>
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<td>79</td>
<td>5451</td>
<td>5427</td>
<td>24</td>
</tr>
<tr>
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<td>69</td>
<td>5451</td>
<td>5427</td>
<td>24</td>
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<tr>
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<td>67</td>
<td>81</td>
<td>5427</td>
<td>5451</td>
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</tr>
<tr>
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<td>81</td>
<td>67</td>
<td>5451</td>
<td>5427</td>
<td>-24</td>
</tr>
<tr>
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<td>296</td>
<td>21756</td>
<td>21756</td>
<td>0</td>
</tr>
</tbody>
</table>

Table−6

5.3 INDEX OF CORDIALITY OF COMPLETE GRAPHS :

Theorem−5.3.1 : If $d_i = 4$ in $K_n$ then $d_{i-2} + 2d_i + d_{i+2} = 0$ and in this case the index of cordiality for $K_n$ is atmost 4.

Proof : According to discussion of cordiality of $K_n$,

$$d_i = \frac{n-1}{2} - 2(i^2 + 1), \quad \text{when } n \text{ is odd.}$$

$$= \frac{n}{2} - 2i^2, \quad \text{when } n \text{ is even}$$

$$\Rightarrow d_{i-1} - d_{i-2} = d_i - d_{i-1} + 4 = d_{i+1} - d_i + 8$$

$$= d_{i+2} - d_{i+1} + 12$$

$$\Rightarrow d_{i-2} + d_i = 2d_{i-1} - 4,$$

$$d_{i+2} + d_i = 2d_{i+1} - 4 \text{ and}$$

$$d_{i+1} + d_{i-1} = 2d_i - 4.$$

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\[ d_{i+2} + 2d_i = 2(d_{i+1} + d_{i-1}) - 8 \]

\[ = 2(2d_i - 4) - 8 \]

\[ = 2(2 \cdot 4 - 4) - 8 \]

\[ = 0 \]

Since vertex difference for \( d_i, d_{i-2} \) and \( d_{i+2}, d_i \) is precisely two, we shall get \( v_f(0) = v_f(1) \) and \( e_f(1) - e_f(0) = d_{i-2} + 2d_i + d_{i+2} = 0 \) in \( \bigcup_{i=1}^4 K_n \). Thus, \( \bigcup_{i=1}^4 K_n \) is cordial and the index of cordiality for \( K_n \) is atmost 4, when \( d_i = 4 \), for some \( i \).

**Illustration-5.3.2**: The index of cordiality for \( K_{89} \) is atmost 4. Because according to table-1, \( d_5 = e = 4 \) and \( d_3 + 2d_5 + d_7 = c + 2e + g = 0 \).

**Theorem-5.3.3**: If \( d_i = 5 \) in \( K_n \) then \( d_{i-2} + d_{i-1} + d_{i+1} + d_{i+2} = 0 \) and in this case the index of cordiality for \( K_n \) is atmost 4.

**Proof**: By Theorem-5.3.1

\[ d_{i+2} + d_{i-2} + 2d_i = 2(d_{i+1} + d_{i-1}) - 8 \]

\[ \Rightarrow d_{i+2} + d_{i-2} + d_{i+1} + d_{i-1} + 4 = 2(2d_i - 4) - 8 \]

\[ \Rightarrow d_{i+2} + d_{i+1} + d_{i-1} + d_{i-2} = 2(2 \cdot 5 - 4) - 8 - 4 \]

\[ = 0 \]

Since vertex difference for \( d_{i+2}, d_{i+1} \) and \( d_{i-1}, d_{i-2} \) is precisely one, we shall get \( v_f(0) = v_f(1) \) and \( e_f(1) - e_f(0) = d_{i+2} + d_{i+1} + d_{i-1} + d_{i-2} = 0 \) in the union of four copies of \( K_n \) and so, it is cordial. Thus, the index of cordiality for \( K_n \) is atmost 4, when \( d_i = 5 \), for some \( i \).

**Theorem-5.3.4**: If \( d_i = 1 \) in \( K_n \) then \( d_{i-1} + 2d_i + d_{i+1} = 0 \) and in this case the index of cordiality for \( K_n \) is atmost 4.
Proof: According to proof of Theorem 5.3.1
\[ d_i - d_{i-1} = d_{i+1} - d_i + 4 \]
\[ \Rightarrow d_{i+1} + d_{i-1} = 2d_i - 4 \]
\[ \Rightarrow d_{i+1} + 2d_i + d_{i-1} = 4 \cdot d_i - 4 = 4 \cdot 1 - 4 = 0 \]

Since vertex difference for \( d_{i+1}, d_i \) and \( d_{i-1} \) is precisely one, we shall get \( v_f(0) = v_f(1) \) and \( e_f(1) - e_f(0) = d_{i+1} + 2d_i + d_{i-1} = 0 \) in the union of four copies of \( K_n \) and so, it is cordial graph. Therefore, the index of cordiality for \( K_n \) is atmost 4, when \( d_i = 1 \), for some \( i \).

Theorem 5.3.5: If \( d_i = 9 \) in \( K_n \) then \( d_{i-3} + 2d_i + d_{i+3} = 0 \) and in this case the index of cordiality for \( K_n \) is atmost 4.

Proof: According to proof of Theorem 5.3.1
\[ d_{i-2} - d_{i-3} = d_{i-1} - d_{i-2} + 4 = d_i - d_{i-1} + 8 \]
\[ = d_{i+1} - d_i + 12 = d_{i+2} - d_{i+1} + 16 \]
\[ = d_{i+3} - d_{i+2} + 20 \]
\[ \Rightarrow d_{i+3} - d_i = 3(d_{i-2} - d_{i-3}) - 48 \]
\[ d_i - d_{i-3} = 3(d_{i-2} - d_{i-3}) - 12 \]
\[ \Rightarrow d_{i+3} + d_{i-3} = -48 + 12 + 2d_i \]
\[ \Rightarrow d_{i+3} + 2d_i + d_{i-3} = 4d_i - 36 = 0 \]

Since vertex difference for \( d_{i+3}, d_i \) and \( d_{i-3} \) are precisely 3, we can get \( v_f(0) = v_f(1) \) and \( e_f(1) - e_f(0) = d_{i-3} + 2d_i + d_{i+3} = 0 \) in the union of four copies of \( K_n \) and so, it is cordial. Thus, the index of cordiality for \( K_n \) is atmost 4, when \( d_i = 9 \), for some \( i \).

Theorem 5.3.6: If \( d_i = 13 \) in \( K_n \) then \( d_{i-3} + d_{i-2} + d_{i+2} + d_{i+3} = 0 \) and in this case the index of cordiality for \( K_n \) is atmost 4.
Proof: by Theorem−5.3.5

\[ d_{i+3} + d_{i-3} = -36 + 2d_i \] and
\[ d_{i+2} - d_i = 2(d_{i-2} - d_{i-3}) - 28, \]
\[ d_i - d_{i-2} = 2(d_{i-2} - d_{i-3}) - 12 \]
\[ \Rightarrow d_{i+3} + d_{i-3} = 2d_i - 36 \] and
\[ d_{i+2} + d_{i-2} = 2d_i - 16 \]
\[ \Rightarrow d_{i-3} + d_{i-2} + d_{i+2} + d_{i+3} = 4d_i - 36 - 16 = 4 \cdot 13 - 52 = 0 \]

Since vertex difference for \( d_{i+3}, d_{i+2} \) and \( d_{i-3} \) is precisely one, we can get \( v_f(0) = v_f(1) \) and \( e_f(1) - e_f(0) = d_{i-3} + d_{i-2} + d_{i+2} + d_{i+3} = 0 \) in the union of four copies of \( K_n \). Thus, \( \bigcup_{i=1}^{4} \) is a cordial graph and so, the index of cordiality for \( K_n \) is atmost 4, when \( d_i = 13 \), for some \( i \).

**Theorem−5.3.7:** If \( d_i = 28 \) in \( K_n \) then \( d_{i-4} + d_{i-1} + d_{i+5} = 0 \) and in this case the index of cordiality for \( K_n \) is atmost 6.

Proof: According to proof of Theorem−5.3.1

\[ d_i - d_{i-1} = d_{i+1} - d_i + 4 = d_{i+5} - d_{i+4} + 20 \]
\[ = d_{i-1} - d_{i-2} - 4 = d_{i-2} - d_{i-3} - 8 = d_{i-4} - d_{i-3} - 12 \]
\[ \Rightarrow d_{i+5} - d_i = 5(d_i - d_{i-1}) - 60 \]
\[ d_{i-4} - d_i = -4(d_i - d_{i-1}) - 24 \]
\[ \Rightarrow d_{i+5} + d_{i-4} - 2d_i = d_i - d_{i-1} - 84 \]
\[ \Rightarrow d_{i+5} + d_{i-1} + d_{i-4} = 3d_i - 84 = 0 \]

To get \( v_f(1) - v_f(0) = 0 \) in \( K_n \), we shall take two copies of \( K_n \) which produce the difference \( d_{i-4}, d_{i-1} \) and \( d_{i+5} \). Thus, the union of six copies of \( K_n \) produce \( v_f(1) = v_f(0) \) and \( e_f(1) - e_f(0) = 2(d_{i-4} + d_{i-1} + d_{i+5}) = 0 \) and so, it is cordial.

Thus, the index of cordiality for \( K_n \) is atmost 6, when \( d_i = 28 \), for some \( i \).
Theorem−5.3.8: If \( d_i = 21 \) for some \( i \) in \( K_n \) then \( d_{i-4}+d_{i-1}+d_i+d_{i+5} = 0 \) and in this case the index of cordiality for \( K_n \) is atmost 8.

Proof: By Theorem−5.3.7

\[
\begin{align*}
\d_{i+5} - \d_i &= 5(\d_i - \d_{i-1}) - 60 \\
\d_{i-4} - \d_i &= -4(\d_i - \d_{i-1}) - 24 \\
\Rightarrow \d_{i+5} + \d_{i-4} &= \d_i - \d_{i-1} + 2\d_i - 84 \\
\Rightarrow \d_{i+5} + \d_i + \d_{i-1} + \d_{i-4} &= 4\d_i - 84 = 0
\end{align*}
\]

To get \( v_f(1) - v_f(0) \) in \( K_n \), we shall take two copies of \( K_n \), each produces the difference \( \d_i, \d_{i-1}, \d_{i-4} \) and \( \d_{i+5} \). Thus, the union of eight copies of \( K_n \) produce \( v_f(1) = v_f(0) \) and \( e_f(1) - e_f(0) = 2(d_{i-4} + d_{i-1} + d_i + d_{i+5}) = 0 \) and so, it is a cordial graph. Thus, the index of cordiality for \( K_n \) is atmost 8, when \( \d_i = 21 \), for some \( i \).

Remark−5.3.9: The edge label difference \( d_5 = 21 \) in \( K_{106} \) as well as in \( K_{123} \), which satisfies the condition \( d_{i-4} + d_{i-1} + d_i + d_{i+5} = 0 \) by taking \( i = 5 \). So, the index of cordiality for them is atmost 8. In fact the index of cordiality for them is atmost 4, because \( 106 = 5^2 + 9^2 \) (sum of square of two integers) and \( d_6 = 1 \) in \( K_{123} \), so, according to the Theorem−5.3.4 it satisfies \( d_5 + 2d_6 + d_7 = 0 \).

5.4 Graceful and Cordial Labeling for Star Related Graphs:

Theorem−5.4.1: The graph \( G \) obtained by joining some stars \( < K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t} > \) is graceful.
Proof : Let \( v_{i,0}, v_{i,1}, \ldots, v_{i,n_i}, i = 1, 2, \ldots, t \) be vertices of star graphs \( K_{1,n_i}, i = 1, 2, \ldots, t \). We shall join these graphs \( K_{1,n_i} \) and \( K_{1,n_i+1} \) by new vertex \( u_i \), where \( 1 \leq i \leq t - 1 \) by their apex vertices. We define a labeling function \( f : V \rightarrow \{0, 1, \ldots, q\} \), where \( q = \sum_{i=1}^{t} n_i + (2t - 2) \) as follows:

\[
\begin{align*}
    f(v_{i,0}) &= i - 1, & \forall i = 1, 2, \ldots, t. \\
    f(v_{1,j}) &= q - (j - 1), & \forall j = 1, 2, \ldots, n_1. \\
    f(u_j) &= q - \sum_{i=1}^{j} n_i - (j - 1) \\
    f(v_{j,i}) &= q - \sum_{l=1}^{j-1} n_l - (j + i - 2), & \forall i = 1, 2, \ldots, n_j; \\
    & \quad \forall j = 2, 3, \ldots, t.
\end{align*}
\]

The above labeling pattern give rise a graceful labeling to the graph \( G \).
Corollary−5.4.3 : The graph \( G = < K_{1,n_1}, K_{1,n_2}, \ldots, K_{1,n_t} > \) is cordial.

Proof : Above graph \( G \) is a tree and so, the number of vertices for \( G \) is \( q + 1 \), where \( q = \) the number of edges for \( G \), which is here \( \Sigma_{i=1}^{t} n_i + 2(t - 1) \).

So, the graceful labeling \( f \) for \( G \) is bijective map. Hence, it satisfies \( |v_f(\text{odd}) - v_f(\text{even})| \leq 1 \). Therefore, by Theorem−5.4.2 \( G \) is a cordial graph.

5.5 Various Labeling for Cycle of Paths :

Theorem−5.5.1 : \( C(t \cdot P_n) \) is graceful, where \( t \equiv 0, 3 \pmod{4} \) and \( n \in \mathbb{N} \).

Proof : Let \( G \) be a cycle of graphs formed by \( t \) copies of path \( P_n \). Let \( u_{i,j} \) \((1 \leq j \leq n)\) be vertices of \( i^{th} \) copy of path \( P_n \) in \( C(t \cdot P_n) \), \( \forall i = 1, 2, \ldots, t \).

We shall join \( u_{i,n} \) last vertex of \( P_i \) with \( u_{i+1,n} \) vertex of \( P_{i+1} \) by an edge, \( \forall i = 1, 2, \ldots, t - 1 \) and also join \( u_{t,n} \) last vertex of \( P_t \) with \( u_{1,n} \) to form the cycle of graphs \( C(t \cdot P_n) \).

We define a labeling function \( f : V(G) \rightarrow \{0, 1, \ldots, q\} \), where \( q = t \cdot n \) as follows.

\[
\begin{align*}
f(u_{1,j}) &= q - \left(\frac{j-1}{2}\right), & \text{when } j \equiv 1 \pmod{2} \\
&= \left(\frac{j-2}{2}\right), & \text{when } j \equiv 0 \pmod{2}, \forall j = 1, 2, \ldots, n;
\end{align*}
\]

\[
\begin{align*}
f(u_{2,j}) &= f(u_{1,j}) + (-1)^j(q + 1 - n)\forall j = 1, 2, \ldots, n;
\end{align*}
\]

\[
\begin{align*}
f(u_{i,j}) &= f(u_{i-2,j}) - (-1)^{i+j}(n), & \forall j = 1, 2, \ldots, n, \forall i = 3, 4, \ldots, \left\lfloor \frac{t}{2} \right\rfloor;
\end{align*}
\]

\[
\begin{align*}
f(u_{\left\lfloor \frac{t}{2} \right\rfloor + 1,j}) &= f(u_{\left\lfloor \frac{t}{2} \right\rfloor - 1,j}) + \frac{1}{2} + (-1)^j(n + \frac{1}{2}), & \forall j = 1, 2, \ldots, n;
\end{align*}
\]

\[
\begin{align*}
f(u_{\left\lfloor \frac{t}{2} \right\rfloor + 2,j}) &= f(u_{\left\lfloor \frac{t}{2} \right\rfloor,j}) + \frac{1}{2} - (-1)^j(n + \frac{1}{2}), & \forall j = 1, 2, \ldots, n;
\end{align*}
\]

\[
\begin{align*}
f(u_{i,j}) &= f(u_{i-2,j}) - (-1)^{i+j}(n), & \forall j = 1, 2, \ldots, n, \forall i = \left\lfloor \frac{t}{2} \right\rfloor + 3, \left\lfloor \frac{t}{2} \right\rfloor + 4, \ldots, t.
\end{align*}
\]
Above labeling pattern give rise a graceful labeling to the given graph $G$.

Thus, $G = C(t \cdot P_n)$ is a graceful graph.

**Illustration**–5.5.2: $C(7 \cdot P_5)$ and its graceful labeling shown in *figure–5.1*, where $p = q = 35$.

![Figure 5.1](image)

*Figure–5.1*: Cycle graph $C(7 \cdot P_5)$ and its graceful labeling.

**Theorem**–5.5.3: $C(t \cdot P_n)$ is cordial, where $t \equiv 0, 1, 3 \pmod{4}$ and $n$ is odd or $t \in N$ and $n$ is even.

**Proof**: Let $u_{i,j}$ ($1 \leq i \leq t$, $1 \leq j \leq n$) be vertices of the cycle graph $C(t \cdot P_n)$ like previous *theorem–5.1*.

To define a labeling function $f : V(t \cdot P_n) \rightarrow \{0, 1\}$, we have to consider following two cases.

**Case–I**: $t \equiv 0, 1, 3 \pmod{4}$ and $n \equiv 1 \pmod{2}$

- $f(u_{1,j}) = 0$ when $j \equiv 0, 1 \pmod{4}$
- $= 1$, when $j \equiv 2, 3 \pmod{4}$, $\forall j = 1, 2, \ldots, n$;
- $f(u_{i,j}) = f(u_{1,j})$ when $i \equiv 0, 1 \pmod{4}$
- $= 1 - f(u_{1,j})$, when $i \equiv 2, 3 \pmod{4}$, $\forall j = 1, 2, \ldots, n$, $\forall i = 1, 2, \ldots, t$. 

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Case–II: $t \in \mathbb{N}$ and $n \equiv 0 \pmod{2}$

$$f(u_{i,j}) = 0 \quad \text{when } j \equiv 0, 1 \pmod{4}$$

$$= 1, \quad \text{when } j \equiv 2, 3 \pmod{4}, \forall \ j = 1, 2, \ldots, n, \forall \ i = 1, 2, \ldots, t.$$  

Above labeling pattern give rise a cordial labeling to the graph $C(t \cdot P_n)$ and so, it is a cordial graph.

**Illustration–5.5.4:** $C(9 \cdot P_5)$ and its cordial labeling shown in figure–5.2, where $p = q = 45, v_f(0) = 23, v_f(1) = 22, e_f(0) = 23$ and $e_f(1) = 22.$

![Cycle graph $C(9 \cdot P_5)$ and its cordial labeling.](image)

**Theorem–5.5.5:** $C(t \cdot P_n)$ is product cordial graph, where $t, n \in \mathbb{N}$ and $n \geq 2.$

**Proof:** Let $G$ be a cycle of $t$ copies of path $P_n$. It is obvious that $|V(G)| = p = q = |E(G)| = t \cdot n$. Let $u_{i,j}$ $(1 \leq i \leq t, 1 \leq j \leq n)$ be vertices of the cycle graph $G$ like previous theorem–5.5.1. Let us redefine $V(G) = \{u_{i,j} / i = 1, 2, \ldots, t, j = 1, 2, \ldots, n\}$ by $V(G) = \{v_l / l = 1, 2, \ldots, p - t\} \cup \{w_s / s = 1, 2, \ldots, t\}$, where $u_{i,j} = v_{i(n-1)+1-j}, \forall \ i = 1, 2, \ldots, t, \forall \ j = 1, 2, \ldots, n - 1$ and $u_{i,n} = w_i, \forall \ i = 1, 2, \ldots, t.$
We define a labeling function \( f : V(G) \rightarrow \{0, 1\} \) as follows.

\[
\begin{align*}
  f(w_s) &= 1 \quad \forall s = 1, 2, \ldots, t; \\
  f(v_l) &= 1 \quad \text{when } l \leq \left\lceil \frac{q}{2} \right\rceil - t \\
           &= 0, \quad \text{when } l > \left\lceil \frac{q}{2} \right\rceil - t, \forall l = 1, 2, \ldots, p - t.
\end{align*}
\]

Above labeling pattern give rise a product cordial labeling to the graph \( C(t \cdot P_n) \) and so, it is a product cordial graph.

**Illustration**–5.5.6 : \( C(10 \cdot P_3) \) and its product cordial labeling shown in **figure**–5.3, where \( p = q = 30, \ v_f(0) = 15, \ v_f(1) = 15, \ e_f(0) = 15 \) and \( e_f(1) = 15 \).

\[
\begin{array}{c}
  1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

*Figure*–5.3  Cycle graph \( C(10 \cdot P_3) \) and its product cordial labeling.

**Theorem**–5.5.7 : \( C(t \cdot P_n) \) is total edge product cordial graph, where \( t, n \in N \) and \( n \geq 2 \).

**Proof** : Let \( G \) be a cycle of \( t \) copies of path \( P_n \). It is obvious that \( |V(G)| = p = q = |E(G)| = t \cdot n \). Let \( u_{i,j} \ (1 \leq i \leq t, \ 1 \leq j \leq n) \) be vertices of the cycle graph \( G \) like previous **theorem**–5.5.1. Let us redefine \( V(G) = \{u_{i,j} / i = 1, 2, \ldots, t, j = 1, 2, \ldots, n\} \) by \( V(G) = \{w_k / k = 1, 2, \ldots, p\} \), where \( u_{i,j} = w_{(j-1)t+i}, \forall i = 1, 2, \ldots, t, \forall j = 1, 2, \ldots, n \).

We shall define \( E(G) = \{e_k / k = 1, 2, \ldots, q\} \) by

\[
e_k = (w_k, w_{k+t}), \forall k = 1, 2, \ldots, q - t.
\]
\[ e_k = (w_k, w_{k+1}), \forall k = q - t + 1, q - t + 2, \ldots, q - 1. \]
\[ e_q = (w_q, w_{q-t}). \]

Now we shall define the labeling function \( f : E(G) \rightarrow \{0, 1\} \) as follows.

\[
\begin{align*}
    f(e_k) &= 1 & \text{when } k \leq \left\lceil \frac{q}{2} \right\rceil \\
    &= 0, & \text{when } k > \left\lceil \frac{q}{2} \right\rceil, \forall k = 1, 2, \ldots, q.
\end{align*}
\]

Above labeling pattern give rise a total edge product cordial labeling to the graph \( G \) and so, \( G \) is a total edge product cordial graph.

**Theorem-5.5.8:** \( C(t \cdot P_n) \) is a mean graph, where \( t, n \in N, t \) is even and \( n \geq 2. \)

**Proof:** Let \( u_{i,j} \ (1 \leq j \leq n) \) be vertices of \( i^{th} \) copy of \( P_n \) in \( C(t \cdot P_n), \forall i = 1, 2, \ldots, t. \) We join last vertex \( u_{i,n} \) of \( P_n^{(i)} \) with \( u_{i+1,n} \), vertex of \( P_n^{(i+1)} \) by an edge, \( \forall i = 1, 2, \ldots, t-1 \) and also join \( u_{t,n} \) last vertex of \( P_n^{(t)} \) with \( u_{1,n} \) to form the cycle graph \( C(t \cdot P_n) \).

To define a labeling function \( f : V(G) \rightarrow \{0, 1, \ldots, q\}, \) where \( q = t \cdot n, \) we consider following two cases.

**Case-I:** \( \frac{t}{2} \) is odd.

\[
\begin{align*}
    f(u_{1,j}) &= q + 1 - j, & \forall j = 1, 2, \ldots, n; \\
    f(u_{2,j}) &= f(u_{1,n}) + j - (n + 1), & \forall j = 1, 2, \ldots, n; \\
    f(u_{i,j}) &= f(u_{i-2,j}) - 2n, & \forall j = 1, 2, \ldots, n, \forall i = 3, 4, \ldots, \frac{t}{2}; \\
    f(u_{\frac{t}{2}+1,n}) &= f(u_{\frac{t}{2},n}) - 1; \\
    f(u_{\frac{t}{2}+1,j}) &= f(u_{\frac{t}{2}-1,j}) - (2n + 1), & \forall j = 1, 2, \ldots, n - 1; \\
    f(u_{\frac{t}{2}+2,j}) &= f(u_{\frac{t}{2},j}) - (2n + 1), & \forall j = 1, 2, \ldots, n; \\
    f(u_{\frac{t}{2}+3,n}) &= f(u_{\frac{t}{2}+1,n}) - (2n + 1); \\
    f(u_{\frac{t}{2}+3,j}) &= f(u_{\frac{t}{2}+1,j}) - 2n, & \forall j = 1, 2, \ldots, n - 1; \\
    f(u_{i,j}) &= f(u_{i-2,j}) - 2n, & \forall j = 1, 2, \ldots, n, \forall i = \frac{t}{2} + 4, \frac{t}{2} + 5, \ldots, t.
\end{align*}
\]
Case–II : \( \frac{t}{2} \) is even.

\[
\begin{align*}
    f(u_{1,j}) &= q + j - n, \quad \forall j = 1, 2, \ldots, n; \\
    f(u_{2,j}) &= f(u_{1,1}) - j, \quad \forall j = 1, 2, \ldots, n; \\
    f(u_{i,j}) &= f(u_{i-2,j}) - 2n, \quad \forall j = 1, 2, \ldots, n, \forall i = 3, 4, \ldots, \frac{t}{2}; \\
    f(u_{\frac{t}{2}+1,n}) &= f(u_{\frac{t}{2},n}) - 1; \\
    f(u_{\frac{t}{2}+1,j}) &= f(u_{\frac{t}{2}-1,j}) - (2n + 1), \quad \forall j = 1, 2, \ldots, n - 1; \\
    f(u_{\frac{t}{2}+2,j}) &= f(u_{\frac{t}{2},j}) - (2n + 1), \quad \forall j = 1, 2, \ldots, n; \\
    f(u_{\frac{t}{2}+3,n}) &= f(u_{\frac{t}{2}+1,n}) - (2n + 1); \\
    f(u_{\frac{t}{2}+3,j}) &= f(u_{\frac{t}{2}+1,j}) - 2n, \quad \forall j = 1, 2, \ldots, n - 1; \\
    f(u_{i,j}) &= f(u_{i-2,j}) - 2n, \quad \forall j = 1, 2, \ldots, n, \forall i = \frac{t}{2} + 4, \frac{t}{2} + 5, \ldots, t.
\end{align*}
\]

Above labeling pattern \( f \) give rise a mean labeling to \( C(t \cdot P_n) \) and so, it is a mean graph.

5.6 Concluding Remarks:

We have discussed here graceful labeling and cordial labeling for star related graphs. We have also discussed cordial labeling for union of some copies of the complete graph \( K_n \) (\( n \geq 4 \)) and the index of cordiality for \( K_n \) (\( 4 \leq n \leq 105 \) and for all \( n \) with some conditions). This work wash out the impression of cordial labeling.

Here, we pose conjecture that \( \exists m \in N \) such that the index of cordiality for \( K_n \) is atmost \( m, \forall n \in N \) (Most probable value of \( m \) is 8). In [16] Vaidya, Dani, Kanani and Vihol proved that the graph \( < K_{1,n}^1, K_{1,n}^2, \ldots, K_{1,n}^k > \) obtained by \( k \) copies of \( K_{1,n} \) is cordial, which is particular case of our Corollary–5.4.5 by taking \( t = k \) and \( n_1 = n_2 = \ldots = n_t = n \).