CHAPTER 1

PRELIMINARIES
In this chapter we present the definitions, and results on graphs and matroids which are used in the subsequent chapters. The proofs of the results in this chapter can be found in any standard book on Matroid Theory (see for example [9], [10] or [17]). For results in Graph Theory one may refer to [1] or [6].

**Definition 1.1.** A graph \( G \) is a pair \((V(G), E(G))\), where \( V(G) \) is a non empty finite set of elements known as vertices and \( E(G) \) is family of unordered pairs of elements of \( V(G) \). \( V(G) \) is called the vertex (or point) set of \( G \) and \( E(G) \) is the edge family of \( G \).

An edge which joins a vertex to itself is called a loop. Edges which join the same pair of distinct vertices are called parallel edges.

A graph containing neither loop nor parallel edges is known as a simple graph.

Degree of a vertex \( v \in V(G) \), denoted by \( d(v) \), is defined as the number of edges incident at \( v \). A graph with single vertex and no edges is called a trivial graph.

**Definition 1.2 :** A walk (or trail) \( W \) in a graph is defined as a finite alternating sequence of vertices and edges beginning and ending with vertices such that each edge is incident with the vertices preceding and following it and occur exactly once. A vertex may be repeated in a walk. Whenever the terminal vertices of the walk are same, we call it a closed walk. A walk in which
no vertex and hence no edge is repeated is called a path. A closed path in a graph is known as a cycle or a circuit.

DEFINITION 1.3: A closed walk containing all the edges in a graph is called an Eulerian walk.

DEFINITION 1.4: A graph having at least one Eulerian walk is called an Eulerian graph.

DEFINITION 1.5: A graph is said to be connected if each pair of distinct vertices is joined by a path. A nonseparable graph is connected, nontrivial and has no cutvertices. A block of graph $G$ is a maximal nonseparable subgraph. If $G$ is nonseparable, then $G$ itself is called a block.

DEFINITION 1.6: In a connected graph $G$, a cutset is a set of edges whose removal from $G$ leaves $G$ disconnected, provided removal of no proper subset of these edges disconnects $G$. Thus, cutset in a graph $G$ is a minimal disconnecting set of edges of $G$.

DEFINITION 1.7: A spanning tree $T$ of a connected graph $G$ is a connected acyclic subgraph of $G$ such that $V(T) = V(G)$.

THEOREM 1.8 [3]: A connected graph $G$ is Eulerian if and only if each vertex of $G$ has even degree.

THEOREM 1.9 [15]: A connected graph $G$ is Eulerian if and only if the edge set $G$ can be partitioned into cycles.

THEOREM 1.10 [18], [14]: A connected graph $G$ is Eulerian if and only if every edge of $G$ lies on an odd number of cycles.
THEOREM 1.11 [12]: A connected graph \( G \) is Eulerian if and only if the number of subsets of \( E(G) \) each of which is contained in a spanning tree of \( G \), is odd.

THEOREM 1.12 [3]: A graph \( G \) is Eulerian if and only if every cutset of \( G \) is of even size.

THEOREM 1.13 [2]: A graph is Eulerian if and only if the edge set has an odd number of cycle partitions.

DEFINITION 1.14: SPLITTING PROCEDURE IN GRAPHS ([4], [5]):

Let \( G \) be a connected graph and \( v \in V(G) \) with \( d(v) \geq 3 \). If \( x=vv_1 \) and \( y=vv_2 \) are two edges incident at a vertex \( v \) then splitting away a pair \( x,y \) of edges from a vertex \( v \) results in a new graph \( G_{xy} \) obtained from \( G \) by deleting the edges \( x \) and \( y \) and adding a new vertex \( v_{xy} \) adjacent to \( v_1 \) and \( v_2 \). A graph \( G_{xy} \) is called a splitting of \( G \) by the pair \( x,y \). This construction is illustrated in the following figure.

![Splitting Procedure in Graphs](image)

Fig. 1.1
PROPOSITION 1.15 [4]

(1) The graph $G_{xy}$ is Eulerian if and only if $G$ is.

(2) $G_{xy}$ is connected if and only if $G$ is connected and \{x,y\} does not form a cutset.

(3) If $G$ is connected Eulerian graph in which three edges, $x,y,z$ share a common vertex, then $G_{xy}$ or $G_{xz}$ must be connected.

REMARK 1.16: We observe that $G$ can be retrieved from $G_{xy}$ by identifying $v$ and $v_{xy}$.

THEOREM 1.17: ([5], p. V 6]: A graph $G$ has an Eulerian trail $T$ if and only if $G$ can be transformed into a cycle $C$ through repeated applications of the splitting procedure on vertices of valency exceeding 2. Moreover, the number of Eulerian trails of $G$ equals the number of different labelled cycles into which $G$ can be transformed this way.

DEFINITION 1.18: A matroid $M$ is a pair $(S, \mathcal{F})$ where $S$ is a finite set and $\mathcal{F}$ is a collection of subsets of $S$ with the following properties:

(1) $\emptyset \in \mathcal{F}$.

(2) If $x \in \mathcal{F}$ and $Y \subseteq X$ then $Y \in \mathcal{F}$.

(3) If $X \in \mathcal{F}$ and $Y \in \mathcal{F}$ and $|X| > |Y|$ then there exists an element $x \in X-Y$ such that $Y \cup \{x\} \in \mathcal{F}$.

Condition (3) in the above definition can be replaced by the following equivalent condition:
(3) If \( X \subseteq \mathcal{F} \) and \( Y \subseteq \mathcal{F} \) and \( |X| = |Y| + 1 \) then there exists an element \( x \in X - Y \) so that \( Y \cup \{x\} \in \mathcal{F} \).

S is called the underlying set of \( M \) whereas members of \( \mathcal{F} \) are called the independent sets of \( M \). A maximal independent set of \( M \) is a base of \( M \). A subset of \( S \) not belonging to \( \mathcal{F} \) is said to be dependent. A minimal dependent subset of \( S \) is called a circuit of \( M \).

**DEFINITION 1.19:** A matroid \( M \) is also defined as the pair \((S, \mathcal{I})\) where \( S \) is a finite set and \( \mathcal{I} \) is a collection of non-empty subsets of \( S \) satisfying the following axioms:

Axiom I: No member of \( \mathcal{I} \) contains another member properly.
Axiom II: If \( X, Y \in \mathcal{I}, X \neq Y, \) and \( u \in X \cap Y \) then there exists \( Z \in \mathcal{I} \) such that \( Z \subseteq X \cup Y - \{u\} \).

Members of \( \mathcal{I} \) are called circuits of \( M \). We say that \( M \) is simple if every circuit of it has at least three elements.

**EXAMPLE 1.20:** (Fano Matroid): The Fano matroid \( F_7 \) is the matroid defined on the set \( S = \{1, 2, 3, 4, 5, 6, 7\} \) whose bases are all 3-element subsets of \( S \) except \( \{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 7\}, \{2, 5, 6\}, \{3, 4, 6\}, \) and \( \{3, 5, 7\} \). Then \( F_7 \) may be drawn as in Fig. 1.2.

![Fig. 1.2](image-url)
The bases are precisely those 3-element subsets of $S$ which do not lie on a line. Set of circuits of Fano matroid is:

$$
\mathcal{C}(F^2) = \{\{1,2,3\}, \{4,5,6,7\}, \{1,4,5\}, \{2,3,6,7\}, \{1,6,7\}, \\
\{2,3,4,5\}, \{2,4,7\}, \{1,3,5,6\}, \{2,5,6\}, \{1,3,4,7\}, \\
\{3,4,6\}, \{1,2,5,7\}, \{3,5,7\}, \{1,2,4,6\}\}.
$$

**Definition 1.21**: (Circuit matroid of a graph). If $S$ is the edge set $E(G)$ of a graph $G$ and $\mathcal{F}$ is the collection of all circuit free subgraphs of $G$ then the pair $(S, \mathcal{F})$ is a matroid. We denote this matroid by $M(G)$ and call it the circuit matroid of the graph $G$.

**Definition 1.22**: (Cutset matroid of a graph). Let $G$ be a graph and $S$ be edge set $E(G)$ of $G$. Let $\mathcal{F}$ be the collection of all cutset free subgraphs of $G$ then the matroid $(S, \mathcal{F})$ is known as a cutset matroid of the graph $G$.

**Definition 1.23**: (Uniform matroid)

Let $|S| = n$ and let $\mathcal{F}$ contain those subsets of $S$ whose cardinality is at most $k$ (for some $0 \leq k \leq n$). Then the pair $(S, \mathcal{F})$ is a matroid. We denote this matroid by $u_{n,k}$ and call it the uniform matroid of rank $k$ on an $n$-element set.

**Definition 1.24**: (Rank of a matroid)

Let $M = (S, \mathcal{F})$ be a matroid and $X \subseteq S$. Then rank of $X$, denoted by $r(X)$, is the cardinality of a maximal independent subset of $X$. That is,

$$
r(X) = \max \{|A| : A \subseteq X \text{ and } A \in \mathcal{F} \}.
$$
Evidently, \( r \) maps the set \( 2^S \) into the set of non-negative integers. We usually write \( r(M) \) for \( r(S) \), and call it the rank of the matroid \( M \).

**DEFINITION 1.25:** (Dual of a matroid)

Let \( M = (S, \mathcal{I}) \) be a matroid and \( \mathcal{I} \) be the collection of bases of \( M \). Define \( \mathcal{I}^* = \{X | S - X \in \mathcal{I}\} \). Then \( M^* = (S, \mathcal{I}^*) \) is a matroid with base set \( \mathcal{I}^* \). \( M^* \) is called dual of a matroid \( M \).

**DEFINITION 1.26:** Two matroids \( M_1 = (S_1, \mathcal{I}_1) \) and \( M_2 = (S_2, \mathcal{I}_2) \) are said to be isomorphic if there exists a bijection between \( S_1 \) and \( S_2 \) so that independent subsets of \( S_1 \) correspond to independent subsets of \( S_2 \) and dependent subsets of \( S_1 \) correspond to dependent subsets of \( S_2 \). We write this as \( M_1 \cong M_2 \).

**DEFINITION 1.27:** A matroid \( M \) is said to be graphic (respectively cographic) if there exists some graph \( G \) such that \( M \) is isomorphic to the circuit (respectively cutset) matroid of \( G \).

**PROPOSITION 1.28:** ([10], page 198) Fano matroid \( F_7 \) is neither graphic nor cographic.

**DEFINITION 1.29:** (Eulerian, bipartite matroids): A matroid \( M = (S, \mathcal{I}) \) is said to be eulerian if \( S \) can be expressed as a disjoint union of circuits of \( M \).

\( M \) is called bipartite if every circuit of \( M \) is of even size.

**DEFINITION 1.30:** (Representable matroid). A matroid \( M \) on a set \( S \) is said to be representable over a field \( F \) if there exists a vector space \( V \) over \( F \) and a map \( \phi : S \rightarrow V \), which preserves the rank.
Such a map \( \phi \), is called a representation of \( M \) and we say that a matroid \( M \) as representable if it is representable over some field.

**Definition 1.31**: (Binary matroids). A matroid \( M \) on a set \( S \) is defined to be binary if it is representable over the Galois field \( \text{GF}(2) \).

Example: (1) Every graphic matroid is binary.

(2) Fano matroid \( F_7 \) is binary.

**Theorem 1.32** ([9], Page 304)

The following statements are equivalent for a matroid \( M \).

(1) \( M \) is binary.

(2) If \( C_1 \) and \( C_2 \) are distinct circuits, then their symmetric difference \( C_1 \Delta C_2 \) contains a circuit.

(3) If \( C_1 \) and \( C_2 \) are distinct circuits, then \( C_1 \Delta C_2 \) is a disjoint union of circuits.

(4) The symmetric difference of any set of circuits is a disjoint union of circuits.

(5) If \( C \) is a circuit and \( D \) is a cutset then \( |C \cap D| \) is even.

(For cutset see definition 1.46).

Throughout this thesis we consider only binary matroids. Matroid will mean binary matroid.

**Definition 1.33**: A matroid \( M = (S,F) \) is said to be connected if for every pair \( x, y \) of elements of \( S \) there exists a circuit of \( M \) containing both \( x \) and \( y \).
DEFINITION 1.34: A matroid $M = (S, \mathcal{I})$ is said to be a circuit matroid if $S$ is a circuit of $M$. $M$ is called discrete matroid if $S$ is a base of $M$. Every subset of $S$ is independent in a discrete matroid.

DEFINITION 1.35: (Deletion of a set from a matroid):

If $M = (S, \mathcal{I})$ is an arbitrary matroid and $X \subseteq S$ then define $\mathcal{I}'$ so that for $Y \subseteq S-X$, $Y \in \mathcal{I}'$ if and only if $Y \in \mathcal{I}$. Then $\mathcal{I}'$ is a collection of independent subsets of a matroid on $S-X$. This matroid is denoted by $M_{\setminus X}$ and is called the deletion of $X$ from $M$ or the restriction of $M$ to $S-X$.

Example: Consider the set $S = \{1, 2, 3, 4\}$ and
$\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{2,3,4\}, \{1,2,4\}\}.$

$M = (S, \mathcal{I})$ is a matroid.

If $X = \{2,4\}$ then $M_{\setminus X} = (S-X, \mathcal{I}')$ is the matroid with $S-X = \{1,3\}$ and $\mathcal{I}' = \{\emptyset, \{1\}, \{3\}, \{1,3\}\}$.

DEFINITION 1.36: (Contraction of an element).

If $x$ is not a loop, then a contraction of $x$ in $M$ denoted by $M_{/x}$ is the matroid $(S-\{x\}, \mathcal{I}')$ where a subset $Y$ of $S-\{x\}$ is in $\mathcal{I}'$ if and only if $Y \cup \{x\}$ was independent before contraction.

Example: In the above matroid let $x = 2$ then $M_{/2} = ((1,3,4), \mathcal{I}')$ where $\mathcal{I}' = \{\emptyset, \{1\}, \{3\}, \{4\}, \{1,3\}, \{1,4\}, \{3,4\}\}$.
DEFINITION 1.37: (Contraction of a set in a matroid):

Let \( M = (S, \mathcal{F}) \) be a matroid and \( X \subseteq S \). Let \( X_0 \) be a maximal independent subset of \( X \) in \( M \). Then a contraction of \( X \) in \( M \) denoted by \( M/_{X} \) is the matroid \((S-X, \mathcal{F}')\) where a subset \( Y \) of \( S-X \) is in \( \mathcal{F}' \) if and only if \( Y \cup X_0 \) is in \( \mathcal{F} \).

PROPOSITION 1.38: ([9], Page 107).

Let \( M = (S, \mathcal{F}) \) be a matroid and \( X \subseteq S \). Let \( \mathcal{C}(M) \) denote the set of circuits of \( M \). Then

(i) \( \mathcal{C}(M/_{X}) = \{C \subseteq S-X : C \in \mathcal{C}(M)\} \).

(ii) The circuits of \( M/_{X} \) consists of the minimal non empty members of \( \{C-X : C \in \mathcal{C}(M)\} \).

PROPOSITION 1.39 [9]: If \( X \subseteq S \), then for all \( Y \subseteq S-X \)

(i) \( r_{M/_{X}}(Y) = r_{M}(Y) \).

(ii) \( r_{M/_{X}}(Y) = r_{M}(Y \cup X) - r_{M}(X) \).

PROPOSITION 1.40: Let \( X \subseteq S \) be an arbitrary subset in the matroid \( M = (S, \mathcal{F}) \). Then

\[ (M/_{X})^* = M_{\setminus X}^*. \]

DEFINITION 1.41 (Minors)

Let \( M \) be a matroid on a set \( S \) then by deleting or contracting a subset of \( S \) in \( M \) we get a new matroid from \( M \). Matroids so obtained are called minors of \( M \).
PROPOSITION 1.42 ([10], Page 164 or [9], page 109)

If \( N \) is a minor of \( M = (S, \mathcal{F}) \) then there are two disjoint subsets \( A, B \subset S \) so that \( N = (M_{\setminus A})_{/B} \). (That is, every minor can be obtained by just one suitable deletion and just one suitable contraction).

THEOREM 1.43 ([17], Page 162)

Any minor of a binary matroid is binary.

LEMMA 1.44 ([17], Page 167).

Let \( M \) be a binary matroid on a set \( S \), let \( x \in S \) and let \( C \) be a circuit of \( M \) with \( x \in C \). Then \( C - \{x\} \) is a circuit of \( M_{/\{x\}} \). If \( x \not\in C \) then either \( C \) is a circuit of \( M_{/\{x\}} \) or is the disjoint union of two circuits of \( M_{/\{x\}} \).

PROPORTION 1.45 ([10], Page 162):

Let \( M = (S, \mathcal{F}) \) be a matroid and \( \mathcal{C} \) denote the set of circuits of \( M \). If \( X \in \mathcal{C}, Y \in \mathcal{C} \) and \( X \cap Y \neq \emptyset \) then for every \( x \in X \) and \( y \in Y \) there exists a circuit \( Z \in \mathcal{C} \) so that \( x, y \in Z \) and \( Z \subseteq X \cup Y \).

DEFINITION 1.46: (Cutset of a matroid):

If \( M = (S, \mathcal{F}) \) is a matroid and \( M^* \) is its dual then circuits of \( M^* \) are called the cutsets (or cocircuits) of \( M \).

DEFINITION 1.47: (Bridge of a matroid)

The element \( x \in S \) is a bridge of the matroid \( M = (S, \mathcal{F}) \) if \( \{x\} \) is a cutset of \( M \).
PROPOSITION 1.48 : ([10], Page 190)

If $M$ is a binary bipartite matroid and $Q$ is a cutset of $M$ then $M/Q$ is also bipartite.

PROPOSITION 1.49 : [7] Let $M = (S, \mathcal{F})$ be a binary matroid and $X \subseteq S$. Then $X$ is disjoint union of circuits of $M$ if and only if $X$ intersects each cutset evenly.

THEOREM 1.50 : ([9], P. 441) : A binary matroid is graphic if and only if it has no minor isomorphic to $F_7, F_7^*, M^*(K_5)$ or $M^*(K_{3,3})$.

DEFINITION 1.51 : If $B$ is a base of $M$ and $x \in S - B$ then there exists a unique circuit $C = C(x, B)$ such that $x \in C \subseteq B \cup \{x\}$. This circuit $C(x, B)$ is called the fundamental circuit of $x$ in the base $B$. 