CHAPTER 5

IDENTIFICATIONS OF BINARY MATROIDS
5.1. INTRODUCTION

Here we discuss the problem which is the reverse of the problem of finding a splitting of binary matroid. Given a binary matroid $M$, can we find a matroid $M'$ whose splitting by some pair is the matroid $M$? Such a matroid if it exists will be called an identification of $M$ and we denote it by $M^{id}$. A necessary condition that a matroid has an identification is that there is at least one pair $x, y$ of elements which is equivalent. That is, set of circuits containing $x$ is same as the set of circuits containing $y$. We give examples and also provide a procedure to find identifications of a matroid. Among other things we provide a set of sufficient conditions for two identifications to be same. Identifications of Eulerian matroids, bipartite matroids, graphic matroids, and circuit matroids are investigated. Results concerning identifications and minors of a matroid are discussed. We denote by $D$, a circuit in a matroid which does not contain the pair $x, y$ of elements under consideration while $D'$ will denote a circuit in $M$ containing both the elements $x$ and $y$.

5.2. PROCEDURE FOR IDENTIFICATION

Let $M = (S, \mathcal{E})$ be a binary matroid with an equivalent pair $x, y$ of elements. Let $C$ be a circuit containing $x$ and $y$. Partition $C$ into two subsets $C_1$ and $C_2$ which contain $x$ and $y$ respectively. Let $S_0$ be a subset of $S-C$ satisfying the following two conditions.
(5.2.1) \( C_1 \cup S_0 \) and \( C_2 \cup S_0 \) do not contain any of the circuits \( D_s \).

(5.2.2) For every collection \( D_1, D_2, \ldots, D_t \) of disjoint circuits containing none of \( x \) and \( y \) we must have a circuit \( D_n^\cap \)
\[ \subseteq \bigcup_{i=1}^n D_i \] such that \( (C_n \cup S_0) \cap (\bigcup_{i=1}^n D_i) \subseteq (C_n \cup S_0) \cap D_n^\cap \)
for \( n = 1, 2 \).

Let \( \mathcal{E}^{id} \) denote the collection of subsets of \( S \) of the following type:

1. All \( D_s \in \mathcal{E}^{id} \)
2. \( C_1 \cup S_0, C_2 \cup S_0 \)
3. \( (C_n \cup S_0) \cup D_j \) for \( n = 1, 2 \), such that \( (C_n \cup S_0) \cup D_j \) do not contain sets of type (1) or (2)
4. All \( D's \in \mathcal{E}^{id} \) where \( D's \) do not contain any sets of type (2) or (3).

§ PROPOSITION 5.2.3 : \((S, \mathcal{E}^{id})\) is a binary matroid.

PROOF : We prove that \( \mathcal{E}^{id} \) satisfies the circuit axioms for a binary matroid (see Definition 1.19 and Proposition 1.32).

To verify the axiom I we notice that \( D_s \) are circuits in \( M \) containing neither \( x \) nor \( y \). \( C_1 \cup S_0 \) is a subset of \( S \) containing only \( x \), while \( C_2 \cup S_0 \) is a subset of \( S \) containing only \( y \). \( D's \) are circuits in \( M \) containing both \( x \) and \( y \). There are several cases to consider. We consider the following two typical cases. Other cases will follow in the same manner.
(i) Let $X$ be of type (2) and $Y$ be of type (3) of $\mathcal{V}^{id}$.

Then $X \not\subseteq Y$ by assumption in (3). In order to prove that $Y \not\subseteq X$, let

(a) $X = (C_1 \cup S_0) \Delta D_j$ and $Y = (C_1 \cup S_0) \Delta D_j$.

Then $Y \subseteq X \Rightarrow (C_1 \cup S_0) \Delta D_j \subseteq C_1 \cup S_0$

$\Rightarrow D_j \subseteq C_1 \cup S_0$.

which is a contradiction to (5.2.1).

When $X = C_2 \cup S_0$, $Y \not\subseteq X$ since $x \in Y$ but $x \not\in X$.

For other combinations of $X$ and $Y$ we can verify the axiom in a similar way.

(ii) Let $X$ and $Y$ both be of type (3),

$X = (C_1 \cup S_0) \Delta D_j$ and $Y = (C_1 \cup S_0) \Delta D_k$.

Then

$X \not\subseteq Y \Rightarrow (C_1 \cup S_0) \Delta D_j \not\subseteq (C_1 \cup S_0) \Delta D_k$

$\Rightarrow [(C_1 \cup S_0) \Delta D_j] \Delta [(C_1 \cup S_0) \Delta D_k] \not\equiv \phi$

$\Rightarrow D_j \Delta D_k \not\equiv \phi$.

This shows that $D_j \Delta D_k$ is a disjoint union $D_1 \cup D_2 \cup \ldots \cup D_\ell$ of circuits of $M$. These $D_i$'s are members of $\mathcal{V}^{id}$ of type (1) and will be contained in $(C_1 \cup S_0) \Delta D_k$ which is a contradiction to our hypothesis. Hence $X \not\subseteq Y$. By similar arguments it follows that $Y \not\subseteq X$. For other possibilities of $X$ and $Y$ from type (3) the axiom can be verified in a similar way.
Axiom II: Let $X, Y \in \mathcal{E}^{id}$, $X \neq Y$. Then $X \Delta Y$ is either a circuit or contains a circuit of $\mathcal{E}^{id}$.

We consider the following typical cases:

(i) Let $X$ be of type (1) and $Y$ be of type (2).

Let $X = D_j$ and $Y = C_i \cup S_o$. If $(C_i \cup S_o) \Delta D_j$ contains some circuit of type (1), then we are through. Otherwise $(C_i \cup S_o) \Delta D_j$ is a circuit of type (3).

(ii) $X$ is of type (1) and $Y$ is of type (3).

Let $X = D_j$ and $Y = (C_i \cup S_o) \Delta D_k$.

Then $X \Delta Y = [(C_i \cup S_o) \Delta D_k] \Delta D_j$

$= (C_i \cup S_o) \Delta (D_k \Delta D_j)$

$= (C_i \cup S_o) \Delta (D_1 \cup D_2 \cup D_3 \cup \ldots \cup D_r)$

where $D_i$s are mutually disjoint. So by (5.2.2) there is a circuit $D_j^i \subseteq D_1 \cup D_2 \cup \ldots \cup D_r$ such that

$(C_i \cap S_o) \cap (D_1 \cup D_2 \cup \ldots \cup D_r) \subseteq (C_i \cup S_o) \cap D_j^i$.

We show that

$(C_i \cup S_o) \Delta D_j^i \subseteq (C_i \cup S_o) \Delta (D_1 \cup D_2 \cup \ldots \cup D_r)$ \hspace{1cm} (I)

For this let $z \in (C_i \cup S_o) \Delta D_j^i$. Then there are two possibilities

(a) $z \in (C_i \cup S_o)$ but $z \not\in D_j^i$.

Then $z \not\in D_1 \cup D_2 \cup \ldots \cup D_r$ because if $z \in D_1 \cup D_2 \cup \ldots \cup D_r$ then $z \in (C_i \cup S_o) \cap D_j^i$ which in turn implies that $z \in D_j^i$; contradiction to the fact that $z \not\in D_j^i$.
Thus in this case \( z \in (C_1 \cap S_o) \Delta (D_1 \cup D_2 \cup \ldots \cup D') \) and (I) is proved.

(b) \( z \in D' \) but \( z \not\in C_1 \cup S_o \).

Then \( z \in D_1 \cup D_2 \cup \ldots \cup D' \) only and hence \( z \in (C_1 \cap S_o) \Delta (D_1 \cup D_2 \cup \ldots \cup D') \). Thus (I) is proved in this case also.

Now if \((C_1 \cup S_o) \Delta D'_j \) contains a circuit of type (1) or (2) then this circuit of \( \varepsilon^{id} \) is contained in \( X \Delta Y \); otherwise \((C_1 \cup S_o) \Delta D'_j \) is a circuit of \( \varepsilon^{id} \) contained in \( X \Delta Y \).

(iii) \( X \) is of type (2) and \( Y \) is of type (3). Let \( X = (C_1 \cup S_o) \) and \( Y = (C_1 \cup S_o) \Delta D'_j \).

Then \( D'_j \) is a circuit in \( \varepsilon^{id} \) of type (1) contained in \( X \Delta Y \).

And if \( X = C_1 \cup S_o, Y = (C_2 \cup S_o) \Delta D'_j \),

then \( X \Delta Y = (C_1 \cup S_o) \Delta (C_2 \cup S_o) \Delta D'_j \)

\[ = C_1 \Delta D'_j \]

\[ = D' \cup D_1 \cup D_2 \ldots \cup D_m. \]

Thus, \( D' \) is a circuit of type (4) and \( D_j \) are circuits of type (1) in \( \varepsilon^{id} \), each of which is contained in the set \( X \Delta Y \).

(iv) \( X \) is of type (2) and \( Y \) is of type (4).

Let \( X = (C_1 \cap S_o) \) and \( Y = D'_k \). Then

\[ X \Delta Y = (C_1 \cup S_o) \Delta D'_k \]

\[ = C_1 \Delta S_o \Delta D'_k \Delta C_2 \Delta C_2 \]

\[ = (C_2 \cup S_o) \Delta (C_1 \Delta C_2) \Delta D'_k \]

\[ = (C_2 \cup S_o) \Delta (D_1 \cup D_2 \cup \ldots \cup D_p). \]
As in (ii) there is a circuit

\[ D_j^2 \subseteq D_1 \cup D_2 \cup \ldots \cup D_p \] such that \((C_2 \cup S_o) \Delta D_j^2 \subseteq X \Delta Y.\]

If \((C_2 \cup S_o) \Delta D_j^2\) contains a circuit of type (1) or (2) we are through. Otherwise \((C_2 \cup S_o) \Delta D_j^2\) is a circuit of type (3) in \(E^{id}\) contained in \(X \Delta Y.\)

(v) X is of type (3) and Y of type (4)

Let \(X = (C_1 \cup S_o) \Delta D_r, Y = D_p^r.\) Then

\[ X \Delta Y = (C_1 \cup S_o) \Delta D_r \Delta D_p^r. \]

\[ = C_1 \Delta S_o \Delta D_r \Delta D_p^r \Delta C_2 \Delta C_2 \]

\[ = (C_2 \cup S_o) \Delta C \Delta D_r \Delta D_p \]

\[ = (C_2 \cup S_o) \Delta (D_1 \cup D_2 \cup \ldots \cup D_q). \]

As in case (ii) we can find \(D_j^2 \subseteq D_1 \cup D_2 \cup \ldots \cup D_q\) such that \((C_2 \cup S_o) \Delta D_j^2 \subseteq X \Delta Y\) and we are through.

We conclude from above that \((S, E^{id})\) is a binary matroid and denote it as \(M^{id} = (S, E^{id}).\)

The circuit set of the matroid \((M^{id})_{xy}\) will be denoted by \((E^{id})^{xy}\). Thus,

\[ (E^{id})^{xy} = (E^{id})_o^{xy} \cup (E^{id})_1^{xy} \]

where

\[ (E^{id})_o^{xy} = \{ C \in E^{id} | x, y \in C, \text{ or } x, y \notin C \} \]

and \((E^{id})_1^{xy} = \{ C_1 \cup C_2 | C_1, C_2 \in E^{id}, x \in C_1, y \in C_2 \text{ and there is no member } C' \in (E^{id})_o^{xy} \text{ such that } C' \subseteq C_1 \cup C_2 \} \).
PROPOSITION 5.2.4: \((M^d)_{xy} = M\). That is, splitting of \(M^d\) by the pair \(x, y\) is \(M\).

PROOF: We have to prove that \((\mathcal{C}^d)_{xy} = \mathcal{C}\).

Let \(X \in (\mathcal{C}^d)_{xy}\). We have two cases:

(a) \(X \in (\mathcal{C}^d)_{xy}\) in which case \(X \in \mathcal{C}\) clearly.

(b) \(X \in (\mathcal{C}^d)_{xy}\), so that \(X = Y \cup Z, Y, Z \in \mathcal{C}^d\) are disjoint circuit such that \(x \in Y, y \in Z\) and \(Y \cup Z\) does not contain any circuit in \((\mathcal{C}^d)^o_{xy}\). Then

\[ Y = (C_1 \cup S_0) \Delta D_j, \quad Z = (C_2 \cup S_0) \Delta D_k, \]

\(Y\) and \(Z\) are disjoint therefore

\[ Y \cup Z = Y \Delta Z = C \Delta D_j \Delta D_k. \]

But \(C \Delta D_j \Delta D_k\) does not contain any circuit of type \(D\) or \(D^*\) of \(M\) since \(Y \cup Z\) does not. Consequently \(C \Delta D_j \Delta D_k\) must be a circuit in \(M\), say \(D_k^*\) containing both \(x\) and \(y\). Thus, \(X\) is a circuit in \(\mathcal{C}\). This shows that \((\mathcal{C}^d)_{xy} \subseteq \mathcal{C}\).

To prove the reverse inclusion, we note that every circuit in \(\mathcal{C}\) contains both \(x\) and \(y\), or neither of \(x\) and \(y\). So if \(X \in \mathcal{C}\), and \(X\) is of type \(D\) then \(X \in \mathcal{C}^d\). On the other hand if \(X \in \mathcal{C}\) and \(X\) is of type \(D^*\) then we have again two cases to consider:

(i) \(X = D_k^*\) does not contain any set of type (2) or (3) then

\[ X = D_k^* \in (\mathcal{C}^d)_{0} \]

and we are through.

(ii) \(X = D_k^*\) contains a circuit of type (2), or (3).

Suppose \(D_k^*\) contains a circuit of type (2), say \(C_1 \cup S_0\). .

81
Define $Y = C \cup S$, $Z = C \cup S = C \cup S$ i.e. $Z = C \cup S$. Then $x \in Y$, $y \in Z$, $X \cap Z = \emptyset$, $Y$ is of type (2). We prove that $Z$ is of type (3).

Firstly,

$$Z = \Delta(C \cup S) = (C \cup S) \setminus \Delta(C \cup S)$$

$$= (C \cup S) \setminus \Delta(C \cup S) \cup C \cup C$$

$$= (C \cup S) \setminus (C \cup S)$$

By (5.2.2) there is a circuit $D^2 \subseteq \cup_{i=1}^{\ell} D_i$ such that

$$(C \cup S) \cap (\cup_{i=1}^{\ell} D_i) = (C \cup S) \cap D^2. \text{ If } (C \cup S) \setminus \Delta D^2 \subseteq (C \cup S) \Delta (D_1 \cup D_2 \cup \ldots \cup D_{\ell}) \text{ then } (C \cup S) \Delta (D_1 \cup D_2 \cup \ldots \cup D_{\ell})$$

This implies that $C \Delta D^2 \subseteq Z$. Since $C \Delta D_k \Delta D^2$ is disjoint union of circuits of type $D$ containing neither $x$ nor $y$, each $D$ is a circuit of $M$ contained in $D_k$ (as $Z = D_k \setminus Y$), which is a contradiction.

Therefore, $Z = (C \cup S) \Delta D^2$

In order to prove that $Z$ is a circuit of type (3) we show that $Z$ does not contain sets of type (1) or (2).
On the contrary assume that $Z$ contains a set $D_j$ of type (1). Then,

$$D_j \subseteq (C_2 \cup S_o) \Delta D_j^2$$

$\Rightarrow D_j \subseteq Z = D_k^c - (C_2 \cup S_o)$

$\Rightarrow D_j \nsubseteq D_k^c$, a contradiction. So $Z$ cannot contain a circuit of type (1).

Next suppose $Z$ contains a circuit of type (2) viz. $C_1 \cup S_o$, or $C_2 \cup S_o$. Then

$$C_2 \cup S_o \nsubseteq Z$$ implies that $(C_2 \cup S_o) \nsubseteq (C_2 \cup S_o) \Delta D_j^2 \Rightarrow D_j^2 \nsubseteq (C_2 \cup S_o)$$

which is a contradiction since by condition (5.2.1) no $D$ is contained in $C_2 \cup S_o$. Also $(C_2 \cup S_o) \nsubseteq Z = (C_2 \cup S_o) \Delta D_j^2$ because $x \in C_2 \cup S_o$ but $x$ does not belong to $(C_2 \cup S_o) \Delta D_j^2$. Finally we see that $Y \cup Z = D_k^c$ does not contain any member of $(\varepsilon^i)^{xy}$ viz circuits of the type $D_s$ or $D_s'$. But this is clear because $D_k^c$, $D_s$ and $D_s'$ are circuits in the set $\varepsilon$. If $D_k^c$ contains a circuit

$$(C_1 \cup S_0) \Delta D_p$$ of type (2) then it leads to a contradiction. Thus,

$$Y \cup Z = D_k^c$$ is a circuit in $(\varepsilon^i)^{xy}$. This completes the proof.

DEFINITION 5.2.5 : The matroid $M^d = (S, \varepsilon^d)$ is called an identification of the matroid $M = (S, \varepsilon)$.

Let us illustrate the above procedure with the help of following examples.

Example 5.2.6 : Consider the matroid $M = (S, \varepsilon)$ where

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$.  

83
and \( \mathcal{E} = \{\{1,2,3\}, \{4,5,6\}, \{8,9,10\}, \{3,6,7,8\}, \{3,4,5,7,8\}, \{1,2,6,7,8\}, \{3,6,7,9,10\}, \{1,2,4,5,7,8\}, \{1,2,6,7,9,10\}, \{3,4,5,7,9,10\}, \{1,2,4,5,7,9,10\}\). 

There are three pairs of equivalent elements viz., 1 and 2, 4 and 5, and 9 and 10. We consider the pair 1,2 and a circuit \( C = \{1,2,6,7,8\} \) containing the pair. Partition \( C \) into the sets \( C_1 \) and \( C_2 \) as

\[
C_1 = \{1,6,7\}, \quad C_2 = \{2,8\}.
\]

Now, \( S - C = \{3,4,5,9,10\} \) and \( S_0 = \{3\} \subseteq S - C \) satisfies the conditions (5.2.1) and (5.2.2). Then the collection \( \mathcal{E}_{id} \) is

\[
\mathcal{E}_{id} = \{\{3,4,5,7,8\}, \{3,6,7,9,10\}, \{3,6,7,8\}, \{4,5,6\}, \{8,9,10\}, \{1,8\}, \{2,4,5,7\}, \{2,6,7\}, \{2,3,9,10\}, \{1,2,3\}\}.
\]

We can verify that \( M_{id} = (S, \mathcal{E}_{id}) \) is a binary matroid and \( (M_{id})_{12} = M \) where \( (M_{id})_{12} \) mean the splitting of \( M_{id} \) by the pair 1,2.

Example 5.2.7 : Let \( S = \{1,2,3,4,5,6,7\} \)

and \( \mathcal{E} = \{\{1,2,3\}, \{2,4,7\}, \{2,5,6\}, \{4,5,6\}, \{1,3,5,6\}, \{1,3,4,7\}\} \).

In the matroid \( M = (S, \mathcal{E}) \) we consider the pair 1,3 of equivalent elements and the circuit \( C = \{1,3,4,7\} \) containing this pair. Consider \( C_1 = \{1,4\}, C_2 = \{3,4\} \). \( C_1 \) and \( C_2 \) form partition of \( C \) containing 1 and 3 respectively. \( S - C = \{2,5,6\} \). Here no nontrival subset \( S_0 \) of \( S - C \) satisfies (5.2.1) and (5.2.2). Thus we take \( S_0 = \emptyset \), then
Thus, $M^id = (S, \varepsilon^id)$ is an identification of $M$.

In the following proposition we give a set of sufficient conditions under which two identifications are identical.

**Proposition 5.2.8**: Let $M = (S, \varepsilon)$ be a binary matroid and $x, y$ be a pair of equivalent elements in $M$. Suppose $M^id_1 = (S, \varepsilon^id_1)$ and $M^id_2 = (S, \varepsilon^id_2)$ are two identifications of $M$ generated respectively by $C_1 \cup S_o, C_2 \cup S_o$, and by $C'_1 \cup S'_o, C'_2 \cup S'_o$. If there is a circuit $D'_k$ in $M$ such that

$$C'_1 = D'_k \cap (C_1 \cup S_o), C'_2 = D'_k - C' \text{ and } S'_o = S_o \cap (S - D'_k)$$

then $M^id_1 = M^id_2$.

**Proof**: Suppose there is a circuit $D'_k$ in $M$ such that the given conditions are satisfied. We prove that the circuit sets of two matroids are same. For this we prove that

(i) $C'_1 \cup S'_o = C_1 \cup S_o$

(ii) $C'_2 \cup S'_o = (C_2 \cup S_o) \Delta D'_r$ where $D'_r$ is some circuit in $M$ containing neither $x$ nor $y$.

(i) We have $C'_1 = D'_k \cap (C_1 \cup S_o) = C_1 \cup (D'_k \cap S_o)$ as $C_1 \cap S_o = S'o$ so that $C'_1 \cup S'_o = C_1 \cup (D'_k \cap S_o) \cup (S_o \cap (S-D'_k)) = C_1 \cup S_o$.

85
(ii) \( C_2' \cup S_o' = C_2' \Delta S_o' = D_k' \Delta C_1' \Delta S_o' \)
\[ = D_k' \Delta (C_1 \Delta (D_k' \cap S_o')) \Delta S_o' \]
\[ = D_k' \Delta C_2 \Delta C_1 \Delta D_k' \Delta S_o' \Delta S_o' \]
\[ = C_2 \Delta D_k' \Delta C \Delta (D_k' \cap S_o') \Delta S_o' \]
\[ = C_2 \Delta D_k' \Delta C \Delta D_k' \Delta S_o' \Delta (D_k' \cap S_o') \Delta S_o' \]
\[ = C_2 \Delta D_k \Delta S_o \Delta D_r \Delta (D_k' \cap S_o') \Delta S_o' \]
\[ = (C_2 \Delta S_o) \Delta D_r \Delta S_o \Delta (D_k' \cap S_o') \Delta S_o' \]
\[ = (C_2 \Delta S_o) \Delta D_r \Delta S_o' \Delta S_o' \]
\[ = (C_2 \cup S_o) \Delta D_r. \]

Thus, in order to prove that \( \phi_1^{id} = \phi_2^{id} \), it is enough to prove that the circuits containing \( y \) in \( \phi_1^{id} \) and in \( \phi_2^{id} \) are same.

\( C_2 \cup S_o \) is a circuit of type (2) in \( \phi_1^{id} \) containing \( y \). Now
\[ (C_2 \cup S_o) = (C_2 \cup S_o) \Delta D_r \Delta D_r = [(C_2 \cup S_o) \Delta D_r] \Delta D_r \]
\[ = (C_2' \cup S_o') \Delta D_r. \]

We claim that \( (C_2' \cup S_o') \Delta D_r \) does not contain circuits of type \( (C_2' \cup S_o') \) and \( (C_2' \cup S_o') \Delta D_r \) because
\[ (C_2' \cup S_o') \not\subseteq (C_2' \cup S_o') \Delta D_r \rightarrow (C_2' \cup S_o') \Delta (C_2' \cup S_o') \Delta D_r \]
\[ \not\subseteq (C_2' \cup S_o') \Delta D_r \]
\[ \rightarrow D_r \not\subseteq (C_2' \cup S_o') \Delta D_r \], which is contradiction.

Also, \( (C_2' \cup S_o') \Delta D_r \not\subseteq (C_2' \cup S_o') \Delta D_r \).
\[ D_p \Delta D_r \subseteq (C_2' \cup S_o') \Delta D_r \]

\[ D_1 \cup D_2 \cup \ldots \cup D_m \subseteq (C_2' \cup S_o') \Delta D_r \]

which is again a contradiction to the fact that \((C_2' \cup S_o') \Delta D_r\) does not contain any circuit of type (2). This shows that \((C_2' \cup S_o') \Delta D_r\) i.e. \(C_2 \cup S_o\) is a circuit in \( \mathcal{S} \) containing \( y \). Similarly if \((C_2 \cup S_o) \Delta D_p\) is a circuit of type (3) in \( \mathcal{S}_2^{id} \) containing \( y \) then

\[ (C_2 \cup S_o) \Delta D_p = (C_2 \cup S_o) \Delta D_r \Delta D_r \Delta D_p \]

\[ = [(C_2 \cup S_o) \Delta D_r] \Delta D_r \Delta D_p \]

\[ = (C_2' \cup S_o') \Delta D_q \quad \text{for some } D_q. \]

And \((C_2' \cup S_o') \Delta D_q\) does not contain any circuit of type (1) viz., a \( D \) or a circuit of type (2) viz. \((C_2' \cup S_o') \Delta D_j\) of \( \mathcal{S}_2^{id} \) (this can be proved as above). Therefore \((C_2' \cup S_o') \Delta D_q\) is a circuit in \( \mathcal{S}_2^{id} \) of type (3) containing \( y \), i.e. \((C_2 \cup S_o) \Delta D_p\) is a circuit of \( \mathcal{S}_2^{id} \). Thus, it follows that \( \mathcal{S}_1^{id} \subseteq \mathcal{S}_2^{id} \). Similarly \( \mathcal{S}_2^{id} \subseteq \mathcal{S}_1^{id} \). This completes the proof.

The above conditions are not necessary as can be seen from the following example. Consider the matroid \( M = (S, \mathcal{S}) \) where

\( S = \{1,2,3,4,5,6\} \) and

\( \mathcal{S} = \{\{2,5\}, \{4,6\}, \{1,2,3,4\}, \{1,2,3,6\}, \{1,3,5,6\}, \{1,3,4,5\}\}. \)

The pair 1,3 of elements is equivalent.

The identification generated by

\( C_1 = \{1,2\}, \quad C_2 = \{3,4\}, \quad \text{and} \quad S_o = \emptyset \)
is same as the identification generated by

\[ C'_1 = \{1, 8\}, \quad C'_2 = \{3, 7\}, \quad S'_o = \emptyset. \]

But \( C'_1, C'_2 \notin C'_1 \cup C'_2 \), and \( C'_1, C'_2 \notin C_1 \cup C_2 \).

5.3. REMARKS

(5.3.1) In Theorem 4.3.1, we proved that a binary matroid is Eulerian if and only if its splitting by any pair is Eulerian. Therefore an identification of an Eulerian matroid is Eulerian and conversely. For, if an identification \( M^d = (S, \varphi^d) \) of a matroid \( M = (S, \varphi) \) is Eulerian then \( (M^d)_{xy} = M \). By Theorem 4.3.1, \( (M^d)_{xy} \) is Eulerian, hence \( M \) is Eulerian. On the other hand if \( M \) is Eulerian then \( M = (M^d)_{xy} \) is Eulerian and so by Theorem 4.3.1, again \( M^d \) must be Eulerian.

(5.3.2) If the identification of a matroid is bipartite then the matroid must be bipartite. But the converse may not be true. For example, consider the matroid \( M = (S, \varphi) \) where \( S = \{1, 2, 3, 4, 5, 6\} \) and \( \varphi = \{\{1, 2, 3, 4\}, \{5, 6\}\} \). Then its identification by the pair 5, 6 is the matroid \( M^d = (S, \varphi^d) \) where \( S = \{1, 2, 3, 4, 5, 6\} \), \( \varphi^d = \{\{5, 6\}, \{1, 2, 5\}, \{3, 4, 6\}, \{1, 2, 3, 4\}\} \), which is not bipartite.

(5.3.3) If \( M^d = (S, \varphi^d) \) is an identification of the matroid \( M = (S, \varphi) \), \( r \) denote the rank of \( M \), \( r^d \) denote the rank of \( M^d \) then from Theorem 4.1.8 (rank theorem), it follows that \( r^d(M^d) = r(M) - 1 \).
Fano matroid \( F_7 \) and its dual \( F_7^* \) have no identifications, since they do not have a pair of equivalent elements. The matroids \( M(K_5) \), \( M^*(K_5) \), \( M(K_3,3) \) and \( M^*(K_3,3) \) also have no identifications for the same reason.

Identification of a graphic matroid need not be graphic. Consider for example the graphic matroid \( M = (S, \mathcal{C}) \) where \( S = \{1,2,3,4,5,6,7\} \), and
\[
\mathcal{C} = \{\{1,2,3\}, \{2,4,7\}, \{2,5,6\}, \{4,5,6\}, \{1,3,5,6\}, \{1,3,4,7\}\}.
\]
The matroid is graphic and corresponds to the graph in the figure.

Now to find the identification, consider the pair 1,3 of equivalent elements. \( C = \{1,2,3\} \) is a circuit containing this pair. \( S - C = \{4,5,6,7\} \). Partition \( C \) into two subsets as \( C_1 = \{1\}, C_2 = \{2,3\} \). Then \( S - C = \{4,5\} \subseteq S - C \) satisfies the two conditions (5.2.1) and (5.2.2) so that
\[
\mathcal{E}^{id} = \{\{2,4,7\}, \{2,5,6\}, \{4,5,6,7\}, \{1,4,5\}, \{2,3,4,5\}, \{1,2,5,7\}, \{1,2,4,6\}, \{1,6,7\}, \{3,5,7\}, \{3,4,6\}, \{2,3,6,7\}, \{1,2,3\}, \{1,3,5,6\}, \{1,3,4,7\}\}.
\]
Then the matroid $M^\text{id} = (S, \mathcal{C}^\text{id})$ which is identification of $M$ is the Fano matroid and hence is non-graphic.

The above example also illustrates the fact that identification in matroids is more general than the identifications of vertices in the graph. The graphs obtained by identifying the vertex $v$ with the other vertices are shown in following fig.

![Fig.](image)

and Fano matroid does not correspond to a circuit matroid of any of these graphs.

\begin{enumerate}
\item \textbf{5.3.6} If $M = (S, \mathcal{C})$ is a circuit matroid with $|S| = n$ then the number of identifications of $M$ is equal to $2^n - 1$ because number of distinct identifications is equal to the number of ways to obtain one subset $C_1$ in the partition of $C$. Every $C_1$ will give us one identification. Thus, problem reduces to finding all the subsets of $n$-set except empty set. Hence the number is $2^n - 1$.

For the discrete matroid the number of identification is $2^{n-1} - 1$.
\end{enumerate}
In Section 4.2, we have proved that the effect of splitting a matroid and then deleting (contracting) a set in it is same as deleting (contracting) a set first and then splitting the resulting matroid. Following examples show that operations of identification followed deletion (contraction) give different result from the operation of deletion (contraction) followed by identification.

(i) Deletion and identification:

Consider the matroid \( M = (S, \mathcal{E}) \) where \( S = \{1,2,3,4,5,6,7\} \),
\[
\mathcal{E} = \{\{1,2,3\}, \{2,4,7\}, \{2,5,6\}, \{4,5,6,7\}, \{1,3,5,6\}, \{1,3,4,7\}\}.
\]
The pair 1,3 of elements is equivalent and \( X = \{2,5\} \subset S \).
Then \( M_\setminus X = (\{1,3,4,6,7\}, \mathcal{E}') \) where
\[
\mathcal{E}' = \{\{1,3,4,7\}\}.
\]
We find the identification of \( M_\setminus X \) by the pair 1,3. Here
\( C = \{1,3,4,7\} \) is a circuit containing the pair 1,3. Partition \( C \)
into two sets \( C_1 \) and \( C_2 \) as:
\[
C_1 = \{1,4\}, \quad C_2 = \{3,7\}.
\]
Now, \( S-C = \{2,5,6\} \).
Then \( S_\circ = \{2,6\} \) satisfies the conditions (5.2.1) and (5.2.2).
Thus,
\[
(M_\setminus X)^{\text{id}} = (\{1,3,4,7\}, \mathcal{E}^{\text{id}}) \text{ where}
\]
\[
\mathcal{E}^{\text{id}} = \{\{1,2,4,6\}, \{2,3,6,7\}, \{1,3,4,7\}\}.
\]
Next, consider \( M^{\text{id}} = (S, \mathcal{E}^{\text{id}}) \) by the pair 1,3 and the circuit
\( C = \{1,3,4,7\} \) containing the pair. Then
\[
S-C = \{2,5,6\}, \quad S_\circ = \{2,6\}, \quad C_1 = \{1,4\}, \quad C_2 = \{3,7\}.
\]
\[ \mathcal{E}^\text{id} = \{\{2,4,7\}, \{2,5,6\}, \{4,5,6,7\}, \{1,2,4,6\}, \{2,3,6,7\}, \{1,6,7\}, \\
\{1,4,5\}, \{1,2,5,7\}, \{3,4,6\}, \{3,5,7\}, \{2,3,4,5\}, \{1,2,3\}, \\
\{1,3,5,6\}, \{1,3,4,7\}\}. \]

Therefore,

\[(M^\text{id})_\setminus \setminus \setminus = (\{1,3,4,6,7\}, \mathcal{E}^-) \text{ where} \]

\[\mathcal{E}^- = \{\{1,6,7\}, \{3,4,6\}, \{1,3,4,7\}\}.\]

It follows that,

\[(M^\text{id})_\setminus \setminus \setminus \setminus \setminus \text{ id } \neq (M^\text{id})_\setminus \setminus \setminus \setminus \setminus \text{ id }^-\]

(ii) Contraction and Identification:

Following example show that if \( M = (S, \mathcal{E}) \) is a matroid and \( X \subseteq S \) then

\[(M_{/X})^\text{id} \neq (M^\text{id})_{/X} \]

where the identifications correspond to same equivalent pair in \( M \).

Consider, the matroid \( M = (S, \mathcal{E}) \) where \( S = \{1,2,3,4,5,6,7\} \),

and \( \mathcal{E} = \{\{1,2,3\}, \{2,4,7\}, \{2,5,6\}, \{4,5,6,7\}, \{1,3,5,6\}, \\
\{1,3,4,7\}\}. \)

The elements 1 & 3 are equivalent since circuit sets containing 1 & 3 are same. \( C = \{1,3,4,7\} \) is a circuit containing 1 & 3.

\( S\setminus C = \{2,5,6\} \). Partition \( C \) into two sets \( C_1 \) and \( C_2 \) as

\( C_1 = \{1,4\}, C_2 = \{3,7\} \), let \( S_\circ = \{5,6\} \subseteq S \).

Then

\[ M^\text{id} = (S, \mathcal{E}^\text{id}) = F_7, \text{ Fano matroid}, \]

92
and
\[ \varepsilon \left( \mathcal{M}^d \big/ \Sigma \right) = \{4,7\}, \{6\}, \{1,4\}, \{1,7\}, \{3,7\}, \{3,4\}, \{1,3\} \}. \] (I)

On the other hand,
\[ \varepsilon (\mathcal{M} / \Sigma) = \{1,3\}, \{4,7\}, \{6\} \],
and \[ \varepsilon ((\mathcal{M} / \Sigma)^d) = \{1\}, \{3\}, \{4,7\}, \{6\} \]. (II)

From (I) and (II) it follows that
\[ \varepsilon \left( \mathcal{M}^d \big/ \Sigma \right) \not\sim \varepsilon \left( \mathcal{M} / \Sigma \right)^d. \]

Hence \( \mathcal{M}^d \big/ \Sigma \not\sim \mathcal{M} / \Sigma \).