CHAPTER 6

LOCALLY MOST POWERFUL SIMILAR TEST FOR MIXTURES
CHAPTER 6

LOCALLY MOST POWERFUL SIMILAR TEST FOR MIXTURES

In this chapter, we first give the definition of LMP Similar Test and then investigate the conditions under which the LMP Similar Test exists. We also obtain under certain conditions the general form of LMP Similar Test. These results are then applied to obtain LMP Similar Test for the parameters of mixtures of distributions.

6.1 Definition and Some Basic Results

Consider a r.v. $X$ defined over $\mathbb{R}^n$, the $n$-dimensional Euclidean space. Let the probability distribution of $X$ be $P_{\theta \eta}$ where $\theta \in \varphi \subset \mathbb{R}$, $\eta \in \eta \subset \mathbb{R}^n$.

We are interested in testing $H_0: \theta = \theta_0$ against $H_1: \theta < \theta_0$ where $\eta$ is not specified under $H_0$ and $H_1$.

Let $\phi$ be a test function with corresponding power function $\beta_{\phi}(\theta, \eta) = E_{(\theta, \eta)}\phi(X)$.

Let $D_\alpha$ be the class of all similar level $\alpha$ tests of $H_0$ against $H_1$. Thus we have

$$D_\alpha = \{ \phi | \beta_{\phi}(\theta_0, \eta) = \alpha \ \forall \eta \in \eta \}$$
Under this setup, LMP Similar Test of $H_0: \theta = \theta_0$ against $H_1: \theta < \theta_0$ can be defined as follows.

A test $\theta_0$ in $D_\alpha$ is called LMP Similar Test of $H_0$ against $H_1$, if for every other test $\theta$ in $D_\alpha$, and for every $\eta$ in $\eta$, there exists a $\delta > 0$ (not depending on $\eta$) such that

$$\beta_\theta(\theta, \eta) \geq \beta_{\theta_0}(\theta, \eta) \quad \text{where} \quad \theta_0 - \delta < \theta \leq \theta_0$$

Next we shall show that under certain conditions, the LMP Similar Test as defined above exists. The required conditions are given below:

$C_1: \forall \theta \in D_\alpha, \frac{\partial \beta_\theta(\theta, \eta)}{\partial \theta}$ exists

$C_2: \forall \theta \in D_\alpha$, the family $\left\{ \frac{\partial \beta_\theta(\theta, \eta)}{\partial \theta} \mid \eta \in \eta \right\}$ is equicontinuous at $\theta = \theta_0$.

For the definition of equicontinuity and for the conditions under which a family can be equicontinuous, we refer to Dugundji [1975, pp. 186].

Among all tests in $D_\alpha$, if there exists a test $\theta_0$ in $D_\alpha$ which minimizes $\frac{\partial \beta_\theta(\theta, \eta)}{\partial \theta_0}$ for any $\eta \in \eta$,
then such a test \( t_0 \) is the LMP Similar Test of \( H_0 \) against \( H_1 \) according to the definition given above. This can be stated as a lemma as follows.

**Lemma 1**: Let conditions \( C_1 \) and \( C_2 \) hold. If among all tests \( t \) in \( D_\alpha \), the test \( t_0 \) in \( D_\alpha \) satisfies the inequality

\[
\gamma(t_0, \eta) < 0 \quad \forall \eta \in \pi
\]

where

\[
\gamma(t, \eta) = \frac{\partial}{\partial \theta} [ \beta_{t_0}^{t}(\theta, \eta) - \beta_{t}^{t}(\theta, \eta) ]
\]

then \( t_0 \) is the LMP Similar Test of \( H_0 \) against \( H_1 \).

**Proof**: From condition \( C_2 \), it follows that the family \( \{ \gamma(t, \eta) \mid \eta \in \pi \} \) is equicontinuous at \( \theta = \theta_0 \), and therefore the inequality (1) implies that \( \forall \eta \in \pi \), there exists \( \delta > 0 \) (not depending on \( \eta \)), such that for \( \theta_0 - \delta < \theta \leq \theta_0 \), \( \gamma(t, \eta) < 0 \) which, in turn, implies that \( \forall \eta \in \pi \), \( \beta_{t_0}^{t}(\theta, \eta) - \beta_{t}^{t}(\theta, \eta) \) is decreasing in \( \theta \in (\theta_0 - \delta, \theta_0] \). Hence for any \( \theta \in (\theta_0 - \delta, \theta_0] \), we have

\[
\beta_{t_0}^{t}(\theta, \eta) - \beta_{t}^{t}(\theta, \eta) \geq \beta_{t_0}^{t}(\theta_0, \eta) - \beta_{t}^{t}(\theta_0, \eta) = \alpha - \alpha = 0
\]
which completes the proof.

**Remark 1**: In Statistical literature, we find that the condition $C_2$ is not assumed in connection with LMP Similar Tests [Refer, for example, Ahmed (1961) and Spjotvoll (1968)]. However, we feel that the condition $C_2$ is required which is clear from the above Lemma. If $C_2$ were not assumed in the above Lemma, we can only show that $\forall \eta \in \mathbb{R}$ there exists $\delta(\eta)$ such that

$$\beta_0(\theta, \eta) \geq \beta_0(\theta, \eta)$$

when $\theta_0 - \delta(\eta) \leq \theta \leq \theta_0$

But the infimum $\{\delta(\eta) \mid \eta \in \mathbb{R}\}$ may become zero. So, to guarantee a positive $\delta$, we require the condition $C_2$.

**Lemma 1** proves the existence of LMP Similar Test of $H_0$ against $H_1$ under certain conditions. However, to get the form of the LMP Similar Test we need some more conditions which are stated below:

$C_2$: The probability distribution $P_{\theta, \eta}$ of $X$ has a p.d.f. $f(x \mid \theta, \eta)$ w.r.t. some $\sigma$-finite measure $\nu$ on $\mathbb{R}^n$ and the support

$$X = \{x \mid f(x \mid \theta, \eta) > 0\}$$

does not depend on $(\theta, \eta)$.
There exists a boundedly complete sufficient statistic \( T = T(X) \) for the family of distributions under \( H_0 \), given by
\[
\{ f(x \mid \theta_0, \eta) \mid \eta \in \eta \}.
\]

The derivative of every power function w.r.t. \( \theta \) evaluated at \( \theta = \theta_0 \) can be taken inside the integral so that we have:
\[
\frac{\partial \beta_0(\theta, \eta)}{\partial \theta_0} = \int \frac{\partial f(x \mid \theta, \eta)}{\partial \theta_0} \frac{\partial g(x)}{\partial \theta_0} d \nu.
\]

With these conditions, we can obtain the form of LMP Similar Test which is given in Theorem 1 below. This theorem is also stated without proof in Spjotvoll (1968).

**Theorem 1**: Assume that all the five conditions, \( C_1 \) to \( C_5 \), hold good. Then we have

1. Among all tests in \( D_\alpha \), the following test \( \theta_0 \) minimizes
\[
\frac{\partial \beta_0(\theta, \eta)}{\partial \theta} \bigg|_{(\theta_0, \eta_0)}.
\]
where \( c(t) \) and \( \gamma(t) \) are determined by \( \mathbb{E}_{X|T=t} \theta_0(X) = \alpha \) \( \forall t \); \( \mathbb{E}_{X|T=t} \) denotes the expectation taken w.r.t. the conditional distribution of \( X \) given \( T = t \) under \( \theta = \theta_0 \).

(ii) If \( \theta_0 \) does not depend on \( \eta_0 \), then it is the LMP Similar Test of \( H_0 : \theta = \theta_0 \) against \( H_1 : \theta < \theta_0 \) when \( \eta \) is unspecified.

Proof:

(1) Consider the size condition

\[
\beta_{\alpha}(\theta_0, \eta) = \alpha \quad \forall \eta
\]

By condition \( C_4 \), this size condition is equivalent to

\[
\mathbb{E}_{X|T=t} \theta_0(X) = \alpha \quad \forall t
\]

(3)

Consider
\[
\beta_\theta(\alpha, \eta) = \int x \cdot \theta(x) \, dP_{\theta, \eta}(x)
\]

\[
= \int x \cdot \frac{\theta(x | \theta, \eta)}{f(x | \theta_0, \eta_0)} \, dP_{\theta_0, \eta_0}(x)
\]

\[
\frac{\partial \beta_\theta(\alpha, \eta)}{\partial \theta} \bigg|_{(\theta_0, \eta_0)}
\]

\[
= \int x \cdot \frac{\partial \log f(x | \theta, \eta)}{\partial \theta} \bigg|_{\theta_0} \, dP_{\theta_0, \eta_0}(x)
\]

\[
= E_{(\theta_0, \eta_0)} \left[ \frac{\partial \log f(x | \theta, \eta)}{\partial \theta} \bigg|_{(\theta_0, \eta_0)} \right]
\]

\[
= \frac{T}{E_{(\theta_0, \eta_0)} \left[ \frac{\partial \log f(x | \theta, \eta)}{\partial \theta} \bigg|_{(\theta_0, \eta_0)} \right]}
\]

where \( E_{(\theta_0, \eta_0)} \) denotes the expectation taken w.r.t. the distribution of \( T \) when \((\theta, \eta) = (\theta_0, \eta_0)\).

Now minimising \( \frac{\partial \beta_\theta(\alpha, \eta)}{\partial \theta} \bigg|_{(\alpha, \eta)} \) is equivalent to maximising
for all $t$. Maximising the latter subject to (3) yields the test in (2), by using the Neyman Pearson Lemma.

(ii) The test $\theta_0$ given in (2) minimizes

$$\frac{\partial \beta_0(\theta, \eta)}{\partial \theta} \bigg|_{(\theta_0, \eta_0)}$$

among all tests in $D_\alpha$. This implies

$$\frac{\partial \beta_0(\theta, \eta)}{\partial \theta} \bigg|_{(\theta_0, \eta_0)} < \frac{\partial \beta_0(\theta, \eta)}{\partial \theta} \bigg|_{(\theta_0, \eta_0)}$$

\forall \theta \in D_\alpha

That is $y(\theta_0, \eta_0) < 0$ where

$$y(\theta, \eta) = \frac{\partial}{\partial \theta} \left[ \beta_0(\theta, \eta) - \beta_0(\theta, \eta) \right]$$

If the test $\theta_0$ does not depend on $\eta_0$, then "$y(\theta_0, \eta_0) < 0"$ implies $y(\theta_0, \eta) < 0 \forall \eta \in \eta$.

Hence by Lemma 1, the test is the LMP Similar Test of $H_0$ against $H_1$. 
Remark 2: Assume that $C_1$ to $C_5$ hold good.

Consider the test in (x) with inequalities reversed.

Further, suppose this test does not depend on $\eta_0$.

Then such a test will maximise $\frac{\partial A(\theta, \eta)}{\partial \eta_0}$ for any $\eta$ in $\mathbb{N}$ among all similar tests and in fact it will be the LMP Similar Test of $H_0 : \theta = \theta_0$ against $H_1' : \theta > \theta_0$.

6.2 LMP Similar Test for the Mixture Proportion:

Let $X_1, \ldots, X_n$ be $n$ i.i.d. r.v.s. each with p.d.f.

$$f(x) = pf_1(x | \eta) + qf_2(x | \eta), \quad -\infty < x < \infty$$

where $0 \leq p \leq 1$, $q = 1 - p$, $\eta \in \mathbb{R}^k$ are unknown parameters; $f_1$ and $f_2$ are two p.d.f.'s w.r.t. a common $\sigma$-finite measure.

Consider the problem of testing $H_0 : p = 1$

against $H_1 : p < 1$ when $\eta$ is not specified.

Let $\phi$ be any test function with corresponding power function $\beta_\phi(p, \eta) = \phi(p, \eta)^*(X_1, \ldots, X_n)$. By Lemma 3 of Appendix II, we note that the conditions $C_1$, $C_2$ and
C₅ (given in the previous section) are always satisfied by every power function \( \beta_i(p, \eta) \). Assume that the conditions \( C_3 \) and \( C_4 \) also hold good. With these assumptions, we prove the following:

**Theorem 2:** If the test \( t_1 \) given below does not depend on \( \eta_0 \), then it is the LMP Similar Test of \( H_0 : p = 1 \) against \( H_1 : p < 1 \).

\[
\hat{t}_1(x_1, \ldots, x_n) = \begin{cases} 
1 & \text{if } \sum_{i=1}^{n} \lambda(x_i | \eta_0) > c(t) \\
\gamma(t) & < \\
0 & < 
\end{cases} \quad (4)
\]

where \( \lambda(x | \eta) = f_{x_i}(x | \eta) / f_{1}(x | \eta) \), \( c(t) \) and \( \gamma(t) \) are determined by \( \prod_{p=1}^{p} t_1(x_1, x_2, \ldots, x_n) = \alpha \).

**Proof:** The joint p.d.f. of \( x_1, \ldots, x_n \) is given by

\[
L(x_1, \ldots, x_n) = \prod_{i=1}^{n} \left[ p_{f_1}(x_i | \eta) + q_{f_2}(x_i | \eta) \right]
\]
\[ \theta \log L(x_1 \ldots x_n) = \frac{\sum_{i=1}^{n} f_1(x_i | \eta) - f_2(x_i | \eta)}{\theta \mu} \]

\[ \theta \log L(x_1 \ldots x_n) \bigg|_{(p, \eta) = (1, \eta_0)} = \sum_{i=1}^{n} \left[ 1 - \lambda(x_i | \eta_0) \right] \]

Now applying Theorem 1, we get the result.

In the next section we shall illustrate the test in (4).

6.3 Examples:

Example 1: Let \( X_1, \ldots, X_n \) be n i.i.d. r.v.s. each with p.d.f.

\[ f(x) = \frac{p e^{-x/\eta}}{\eta} + \frac{q \eta^{k+1} e^{-x/\eta}}{(k+1) \Gamma(k+1)}, \quad x > 0 \]

where \( 0 \leq p \leq 1 \), \( 0 < \eta < \infty \) are unknown but \( k > 1 \) is known.

We shall find the LMP Similar Test of \( H_0 : p = 1 \) against \( H_1 : p < 1 \) when \( \eta \) is not specified.
From Lemma 3 of Appendix II, we get that the conditions $C_1$, $C_2$, and $C_3$ are satisfied.

Since the support of this mixture does not depend on $(p, \eta)$, condition $C_3$ is also satisfied.

Under $H_0$, $T = n^{-1} \sum_{i=1}^{n} x_i$ is a complete sufficient statistic. Hence $C_4$ is satisfied. Thus the test in (4) is applicable.

Consider
\[ \sum_{i=1}^{n} \lambda(x_i | \eta, \zeta) > c(t) \text{ where } c(t) \text{ is some constant depending on } t = n^{-1} \sum_{i=1}^{n} x_i. \]

This inequality reduces to
\[ \sum_{i=1}^{n} x_i^k > c(t) \cdot \eta^k \cdot \Gamma(k+1) \]
i.e.
\[ \sum_{i=1}^{n} (x_i/t)^k > c(t) \cdot \eta^k \cdot \Gamma(k+1) / t^k \]
or
\[ \sum_{i=1}^{n} (x_i/t)^k > \text{constant (t)}. \]

We note that the distribution of $\sum_{i=1}^{n} (x_i/T)^k$
does not depend on \( \eta \). Also \( T \) is a complete sufficient statistic for the distributions under \( H_0 \). Hence under \( H_0 \), \( T \) and \( \sum_{i=1}^{n} (X_i/T) \) are independent.

From these observations, we get that the following unconditional test is the LMP Similar Test of \( H_0 : p = 1 \) against \( H_1 : p < 1 \).

\[
\delta_1(x_1, \ldots, x_n) = \begin{cases} 
1 & \text{if } v \geq c \\
0 & \text{otherwise}
\end{cases}
\]

(5)

where \( v = n^{-1} \sum_{i=1}^{n} (x_i/t) \), \( c \) is found to satisfy the size condition.

In Sec 5.3, we have shown that the test in (6) is also LMP Invariant Test for the same problem, and that under \( H_0 \), \( \sqrt{n} (V - \Gamma(k+1)) \) is asymptotically normal with mean zero and variance \( \Gamma(k+1) - (k+1) \cdot \Gamma'(k+1) \).

**Example.** Let \( X_1, \ldots, X_n \) be \( n \) i.i.d. r.v.s. each having the p.d.f.

\[
f(x) = \frac{pe^{-(x-\eta)^2/2}}{\sqrt{2\pi}} + \frac{qe^{-(x-\eta-k)^2/2}}{\sqrt{2\pi}}, -\infty < x < \infty
\]
where \( p \) and \( \eta \) are known but \( k \neq 0 \) is known.

We shall obtain the LMP Similar Test of \( H_0: p = 1 \) against \( H_1: p < 1 \) when \( \eta \) is not specified.

As in the previous example, we note that the conditions \( C_1 \) to \( C_5 \) of Sec 6.1 are satisfied.

\[ T = n^{-1} \sum_{i=1}^{n} X_i \] is a complete sufficient statistic under \( H_0 \). Hence using Theorem 2 and noting that

\[ U = n^{-1} \sum_{i=1}^{n} e^{k(X_i - T)} \] is independent of \( T \) under \( H_0 \), we find that the following unconditional test is the LMP Similar Test of \( H_0: p = 1 \) against \( H_1: p < 1 \)

\[ s_1(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } u \geq c \\ 0 & \text{otherwise} \end{cases} \] (6)

where \( c \) is found to satisfy the size condition.

In Sec 6.3, we have shown that the test in (6) is also LMP Invariant Test. In the same section, we have also shown that \( \sqrt{n} (U - e^{k^2/2}) \) is asymptotically normal with mean zero and variance \( e^{k^2} - e^{k^2} \sigma^2(1+k^2) \) under \( H_0 \).
6.4 **LMP Similar Test for the Parameter of the Mixture Component**

Let \( X_1, \ldots, X_n \) be \( n \) i.i.d. r.v.s. each with p.d.f.

\[
f(x) = pg(x \mid \theta, \eta) + qg(x \mid \theta, \eta), -\infty < x < \infty
\]

where \( 0 < p < 1 \) is known but \( \theta \) and \( \eta \) are unknown.

We are interested in obtaining LMP Similar Test

\( H_0 : \theta = \theta_0 \) against \( H_1 : \theta > \theta_0 \) when \( \eta \) is not specified.

Let \( \psi \) be any test function with corresponding power function \( \beta_\psi(\theta, \eta) = \mathbb{P}(\psi(X_1, \ldots, X_n)) \).

Assuming that the conditions \( C_1 \) to \( C_5 \) of Sec 6.1 are satisfied for this set up and using Remark 2 we get the following theorem:

**Theorem 3**: If the test \( \psi \) given below does not depend on \( \eta_0 \), then it is the LMP Similar Test of

\( H_0 : \theta = \theta_0 \) against \( H_1 : \theta > \theta_0 \)


\[
\theta(x_1, \ldots, x_n) = \begin{cases} 
\gamma(t) & \text{if } \sum_{i=1}^{n} \frac{\partial \log g(x_i|\theta, \eta)}{\partial \theta} \left| \left( \theta_0, \eta_0 \right) > c(t) \\
0 & \text{otherwise}
\end{cases}
\]

where \( c(t) \) and \( \gamma(t) \) are determined by the size condition

\[
E_{\theta_0} \left( \frac{X}{t} \right) \theta(x_1, \ldots, x_n) = \alpha.
\]

6.5 An Example:

We shall illustrate the test in (7) for a specific mixture.

Let \( X_1, \ldots, X_n \) be \( n \) i.i.d. r.v.s each having the p.d.f.

\[
f(x) = \begin{cases} 
p \frac{e^{-(x-\eta)^2/2}}{\sqrt{2\pi}} + q \frac{e^{-(x-\eta)^2/2}}{e^{\frac{(x-\eta)^2}{2}}} & , -\infty < x < \infty
\end{cases}
\]

where \( 0 < p < 1 \) is known but \( \theta \) and \( \eta \) are unknown.

We shall obtain LMP Similar Test of \( H_0: \theta = 1 \)

against \( H_1: \theta > 1 \) when \( \eta \) is not specified.
First let us show that the conditions $C_1$ to $C_5$ are satisfied for this problem.

Let $\phi$ be any test function with corresponding power function given by

$$
\beta_n(\phi, \eta) = \int_{\mathbb{R}^n} \left[ p e^{-(x_1-\eta)^2} + \frac{q}{\theta} e^{-(x_1-\eta)^2/\theta} \right] dx_1 \cdots dx_n
$$

where $x = (x_1, \ldots, x_n)$, $dx = (dx_1, \ldots, dx_n)$ and $g(x | \theta, \eta)$

$$
= (2\pi)^{-n/2} \exp \left\{ - (n-r) \log \theta - \sum_{i \in I_\theta} (x_i - \eta)^2 - 1 \sum_{i \in I_\theta} (x_i - \eta)^2 (2\theta)^{-1} \right\}
$$

(Refer the expansion (4) of Appendix II).

$g(y \mid \theta, \eta)$ is the joint p.d.f. of $n$ normal r.v.s., $r$ of which are normal with mean $\eta$ and variance one, $(n-r)$ of which are normal with mean $\eta$ and variance $\theta^2$. Hence it follows that the conditions $C_1$ and $C_5$.
listed in Sec 6.1 are satisfied for this mixture.

Since the support of this mixture does not depend on \((\theta, \eta)\), condition \(C_3\) is also satisfied. Under \(H_0\),
\[
T = n^{-1} \sum_{i=1}^{n} X_i
\]
is a complete sufficient statistic. Thus condition \(C_4\) is also valid.

The only condition to be verified is \(C_5\). Towards this end, consider

\[
\frac{\partial \rho_k(\theta, \eta)}{\partial \theta} = \sum_{r=0}^{n} \frac{p_r q^{n-r} \phi(x) g_r(x | \theta, \eta) \sum_{i=1}^{n} (x_i - \eta) \phi_r(x_i | \theta, \eta) \int g_r(x | \theta, \eta) dx}{g_r(x | \theta, \eta)}
\]
and so

\[
\frac{\partial \rho_k(\theta, \eta)}{\partial \theta^2} = \sum_{r=0}^{n} \frac{p_r q^{n-r} \phi(x) g_r(x | \theta, \eta) h_r(x | \theta, \eta) dx}{g_r(x | \theta, \eta)}
\]

where
\begin{align*}
& h_{r}(x \mid \theta, \eta) \\
& = \left[ \sum_{i \in \mathbb{I}_r} (x_i - \eta)^{q} / \beta^{-i} (n-r)^{\nu} \right] \sum_{i \in \mathbb{I}_r} (x_i - \eta)^{q} / \beta^{-i} (n-r)^{\nu} .
\end{align*}

Therefore

\begin{align*}
\left| \frac{\partial \pi_{\theta_{e}}(\theta, \eta)}{\partial \theta} \right| & \leq \sum_{r=0}^{n} \sum_{r=0}^{n} \beta^{-i} C_{r}^{i} \int_{\mathbb{R}^{n}} \left| h_{r}(x \mid \theta, \eta) \right| dx (8)
\end{align*}

since \( f(x) \leq 1 \).

In the integral on the RHS of (8), put

\begin{align*}
u_{1} = \begin{cases} x_i - \eta & \text{for } i \in \mathbb{I}_r \\ 0 & \text{for } i \notin \mathbb{I}_r \end{cases}
\end{align*}

The Jacobian of the transformation is \( \beta^{n-r} \) and we have

\begin{align*}
\left| \frac{\partial \pi_{\theta_{e}}(\theta, \eta)}{\partial \theta} \right| & \leq k / \beta^{n-r}
\end{align*}

where

\begin{align*}
k = \sum_{r=0}^{n} \sum_{r=0}^{n} \beta^{-i} C_{r}^{i} \int_{\mathbb{R}^{n}} \beta^{i} (u) du
\end{align*}
\[ h_r(x) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) \]

\[ \sum_{i=1}^{n} \frac{e^{u_i}}{(2\pi)^{n/2}} \left| \sum_{r \in \mathcal{R}_r} (n-r)^{n-1} \right| \]

\[ d \mu = (du_1, \ldots, du_n). \]

Note that \( k \) is independent of \( \theta \) and \( \eta \).

For \( \theta \in [\alpha^{-1}, \alpha] \), we have

\[ \left| \frac{\partial^k \beta_\theta}{\partial \theta^k} (\theta, \eta) \right| < 4k \]

which implies that the family

\[ \left\{ \frac{\partial \beta_\theta}{\partial \theta} (\theta, \eta) \mid \eta \in \mathbb{R} \right\} \]

is equi-continuous at \( \theta = 1 \).

Hence the condition \( C_2 \) is also true for this case.

Thus all the conditions \( C_1 \) to \( C_6 \) are satisfied and so Theorem 3 is applicable.

Now consider

\[ g(x \mid \theta, \eta) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{(x-\eta)^2}{2\theta^2} - \log \theta \right). \]

\[ \frac{\partial}{\partial \theta} \log g(x \mid \theta, \eta) \bigg|_{(\theta, \eta) = (1, \eta_0)} = (x - \eta_0)^2 - 1 \]

Let \( c(t) \) be some constant depending on
\[ t = n^{-1} \sum_{i=1}^{n} x_i. \]

Consider
\[ \sum_{i=1}^{n} \frac{3 \log g(x_i | \theta, \eta)}{3 \theta} |(l, \eta_0) > c(t) \]

i.e. \[ \sum_{i=1}^{n} (x_i - \eta_0)^k - 1 > c(t) \]

i.e. \[ \sum_{i=1}^{n} x_i^k - k n \eta_0 t + \eta_0^k - 1 > c(t) \]

i.e. \[ \sum_{i=1}^{n} x_i^k > c(t) + k n \eta_0 t - \eta_0^k + 1 = c'(t) \quad \text{(say)} \]

i.e. \[ \sum_{i=1}^{n} (x_i - t)^k > c''(t) \]

where \( c''(t) \) is some constant depending on \( t \).

We note that the distribution of \( \sum_{i=1}^{n} (X_i - T)^k \) does not depend on \( \eta \). Also \( T \) is a complete sufficient statistic under \( H_0 \). Hence under \( H_0 \), \( T \) and \( \sum_{i=1}^{n} (X_i - T)^k \) are independent.
Using these observations in conjunction with Theorem 3, we get that the following unconditional test is the LMP Similar Test of $H_0: \theta = 1$ against $H_1: \theta > 1$.

$$
\tau_2(x_1, \ldots, x_n) = \begin{cases} 
1 & \text{if } \sum_{i=1}^{n} (x_i - t)^2 > c \\
0 & \text{otherwise}
\end{cases}
$$

where $c$ is chosen to satisfy the size condition.

This test is also LMP Invariant Test as seen in Sec. 5.5 for the same testing problem.