CHAPTER 7

MAXIMUM AUTOCORRELATION FOR A MOVING AVERAGE PROCESS
WITH RANDOM COEFFICIENTS

7.1 Introduction

In this chapter, we investigate the properties of the maximum value of the k-th order autocorrelation coefficient of a moving average process of order q with random coefficients. Satisfying certain conditions this work can be considered as an extension of the earlier work by Davies, Pate and Frost (1974) and Anderson (1974), where they have shown that for a moving average process of order q, with non-random coefficients, the k-th order autocorrelation coefficient $\rho_k$ is such that

$$|\rho_k| \leq \cos \left( \frac{\pi}{([q/k]+2)} \right), \quad (1)$$

where $[q/k]$ denotes the integral part of $q/k$. In Section 7.2, we first study the maximum autocorrelation coefficient $\rho_1$ of a q-th order moving average process with random coefficients and then generalise these results in Section 7.3, to the case of the maximum autocorrelation coefficient $\rho_k$ of order k.

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7.2 Maximum value of $\rho_1$

A general moving average process $\{X_t, t=0, \pm 1, \pm 2, \ldots\}$ of order $q$ with random coefficients is defined by the equation

$$X_t = a_t + (\theta_1 + b_{1,t})a_{t-1} + (\theta_2 + b_{2,t})a_{t-2} + \ldots + (\theta_q + b_{q,t})a_{t-q}, \quad (2)$$

where

(i) $\theta_1, \theta_2, \ldots, \theta_q$ are non-random parameters,

(ii) $\{a_t, t = 0, \pm 1, \pm 2, \ldots\}$ is a sequence of uncorrelated random variables with zero means and common variance $\sigma^2$,

(iii) if $b_t = (b_{1,t}, b_{2,t}, \ldots, b_{q,t})'$, then

$\{b_t, t = 0, \pm 1, \pm 2, \ldots\}$ is a sequence of vectors of $q$ uncorrelated random variables $b_{i,t}$ with $E(b_{i,t}) = 0$ and $E(b_{i,t}^2) = \sigma_i^2$, $i = 1, 2, \ldots, q$. Further, $a_t$ and $b_t$, are uncorrelated for any $t$ and $t'$.

Now, the first order autocorrelation coefficient $\rho_1$ for the time series $\{X_t, t = 0, \pm 1, \pm 2, \ldots\}$ defined by (2) can be expressed as

$$\rho_1 = \frac{\sum_{r=0}^{q} \theta_r \theta_{r+1}}{\sum_{r=0}^{q} \theta_r^2 + C}, \quad (3)$$

where $C = \sum_{i=1}^{q} \sigma_i^2$ and $\theta_0 = 1$, $\theta_{q+1} = 0$.

It is easy to verify from (3) that for all real
values of \( q \) and \( q \geq 1 \), and \( c \geq 0 \)

\[-1 < \rho_1 < 1. \quad (4)\]

Further, we note that if for a given set \( \theta_0 = 1, \theta_1, \ldots, \theta_q \),
\( \rho_1(\theta_0, \theta_1, \ldots, \theta_q) \) takes a negative value, we can always
find another set \( \theta^*_0 = 1, \theta^*_1, \ldots, \theta^*_q \) such that
\( \rho_1(\theta^*_0, \theta^*_1, \ldots, \theta^*_q) = -\rho_1(\theta_0, \theta_1, \ldots, \theta_q) \). This can be
achieved by choosing \( \theta^*_0 = 1, \theta^*_1 = -\theta_1, \theta^*_2 = \theta_2, \theta^*_3 = -\theta_3, \ldots, \)
and \( \theta^*_q = (-1)^q \theta_q \). This shows that the maximum value
of \( \rho_1 \) cannot be negative. The maximum value of \( \rho_1 \) can
not be zero also, for if it is zero, all \( \theta \)'s are zero and we have a trivial situation. Hence, we can write

\[
\text{Max} (\rho_1) = \cos \alpha, \quad 0 < \alpha < \pi/2. \quad (5)
\]

Now, equating the first partial derivative of \( \rho_1 \) with
respect to \( \theta_r \) \((r = 1, 2, \ldots, q)\) to zero, we find that the
point \((\theta_1, \theta_2, \ldots, \theta_q)\) at which \( \rho_1 \) has an extreme
satisfies the equations

\[
\theta_{r+1} = 2 \cos \alpha \quad \theta_r + \theta_{r-1} = 0, \quad r = 1, 2, \ldots, q. \quad (6)
\]

The general solution of (6) with the boundary condition
\( \theta_0 = 1 \) and \( \theta_{q+1} = 0 \) is

\[
\theta_r = \frac{\sin (q+1-r)\alpha}{\sin (q+1)\alpha}, \quad r = 1, 2, \ldots, q. \quad (7)
\]
If we substitute this value of \( \theta_r \) in \( \rho_1 \), we find that
\( \alpha \) satisfies the following equation:

\[
(1+C) \sin (q+2)\alpha + C \sin q \alpha = 0. \tag{8}
\]

Thus, we see that the maximum value of \( \rho_1 \) is \( \cos \alpha \) where
\( \alpha \) satisfies (8). This statement is true provided the matrix

\[
\begin{pmatrix}
\frac{2}{\theta_r \theta_s} \rho_1 \\
\frac{2}{\theta_r \theta_s}
\end{pmatrix}, r, s = 1, 2, \ldots, q
\]

is negative definite. In fact it is so and the proof is given at the end of this section. We only note at this stage that an explicit expression for \( \cos \alpha \) does not seem to be possible in general case.

We have the following remarks in this context.

**Remark 1**: When \( C = 0 \), we have from (8) \( \alpha = m\pi/(q+2) \),
\( m = 1, 2, \ldots, (q+1) \) and hence the maximum possible value of \( \rho_1 \) is \( \cos (\pi/(q+2)) \) which coincides with the result given by (1).

**Remark 2**: As \( c \to \infty \), the equation (8) reduces to
\( \sin (q+2)\alpha + \sin (q\alpha) = 0 \) which gives \( \alpha = m\pi/(q+1) \),
\( m = 1, 2, \ldots, q \). Thus, the maximum possible value of \( \rho_1 \) is \( \cos (\pi/(q+1)) \) which is interestingly the maximum first order autocorrelation of an ordinary moving average process of order \( q-1 \) with non-random coefficients.

**Remark 3**: From (8), we can express \( C \) as
\[ C = \frac{\sin (q+2)\alpha}{\sin (q+2)\alpha + \sin q\alpha} \]. As \( C \) is a continuous function of \( \alpha \), it is not difficult to show that for a given value of \( q \), \( C \) is a monotonically increasing function of \( \alpha \) in the interval \( 0 < \alpha < \pi/2 \). Hence it follows immediately from Remarks 1 and 2 that

\[ \cos \left( \frac{\pi}{q+1} \right) \leq \text{Max} (\rho_1) \leq \cos \left( \frac{\pi}{q+2} \right) \quad (9) \]

**Remark 4**: The matrix \( D_q = (\frac{\partial^2 \rho_1}{\partial \theta_r \partial \theta_s}), r,s = 1,2, \ldots q \) is negative definite. This is shown as follows.

From (3), we get

\[ \frac{\partial^2 \rho_1}{\partial \theta_r \partial \theta_s} = -\frac{2\rho_1}{q \sum \theta_r^2 + C} \quad \text{if} \quad r = s, \quad (10) \]

\[ = \frac{1}{q \sum \theta_r^2 + C} \quad \text{if} \quad |r-s| = 1, \]

\[ = 0 \quad \text{if} \quad |r-s| > 1. \]

Noting \( \rho_1 = \cos \alpha \), the matrix \( D_q \) can be written as

\[ D_q = \frac{1}{q \sum \theta_r^2 + C} \]

\[
\begin{bmatrix}
-2\cos \alpha & 1 & 0 & 0 & \ldots & 0 \\
1 & -2\cos \alpha & 1 & 0 & \ldots & 0 \\
0 & 1 & -2\cos \alpha & 1 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & -2\cos \alpha \\
\end{bmatrix}
\quad (11)
\]
If we denote by $|D_q|$ the determinant value of $D_q$, we get from (11),

$$|D_q| + 2 \cos \alpha |D_{q-1}| + |D_{q-2}| = 0.$$  

(12)

With the initial conditions $D_{-1} = 0$ and $D_0 = 1/(\sum_{r=0}^{q} \theta_r^2 + C)$, we get the general solution of (12) as

$$D_k = (-1)^k \cdot \frac{1}{\sin(\alpha)} \frac{\sin((k+1)\alpha)}{\sum_{r=0}^{q} \theta_r^2 + C}, \quad k = 1, 2, \ldots, q.$$  

(13)

As from Remark 3, we know that $\pi/(q+2) < \alpha < \pi/(q+1)$, it follows immediately that $|D_k| < 0$ for $k$ odd and $|D_k| > 0$ for $k$ even. Thus the matrix $D_q$ is negative definite. Accordingly, $\cos \alpha$, $0 < \alpha < \pi/2$ where $\alpha$ satisfies (8) is the maximum value of $\rho_1$.

7.3 Maximum value of $\rho_k$

In this section, we generalise the result of the earlier section i.e. we obtain the maximum autocorrelation coefficient of order $k$ ($k \leq q$) of a moving average process defined by (2). For such a process, the $k$-th order autocorrelation coefficient is obtained as

$$\rho_k = \frac{\sum_{r=0}^{q} \theta_r \theta_{r+k}}{\sum_{r=0}^{q} \theta_r^2 + C}, \quad k = 1, 2, \ldots, q.$$  

(14)

where $\theta_{q+1} = \theta_{q+2} = \ldots = \theta_{q+k} = 0$. 
For the maximisation of $\rho_k$, we argue in the same way as we have done for $\rho_1$ in the preceding section and we find that if $\cos \alpha_k, \ 0 < \alpha_k < \pi/2$ is the maximum value for $\rho_k$, then the points $(\theta_0, \theta_1, \ldots, \theta_q)$ at which the maximum occurs satisfies the following difference equation

$$\theta_{r+k} - 2 \cos \alpha_k \theta_r + \theta_{r-k} = 0, \ r = 1, 2, \ldots, q, \ (15)$$

subject to boundary condition $\theta_{-(k-1)} = \theta_{-(k-2)} = \ldots = 0$, $\theta_0 = 1$, $\theta_{q+1} = \theta_{q+2} = \ldots = 0$, $\theta_{q+k} = 0$. The general solution of (15) is

$$\theta_r = A_s \sin \frac{\alpha_k r}{k} + B_s \cos \frac{\alpha_k r}{k}, \ r = 1, 2, \ldots, q, \ (16)$$

where $s = r - \lfloor r/k \rfloor$ and $\lambda = 0, 1, 2, \ldots$. This means that $s$ assumes the $k$ values $0, 1, \ldots, (k-1)$ and hence there are $k$ sets of $\theta$-values which constitute the complete solution. Now, from the boundary conditions, it is easy to show that all $\theta$s except $\theta_0, \theta_{2k}, \theta_{2k'}, \ldots, \theta_Q$ are zero where $Q = \lfloor q/k \rfloor$. Thus, in effect, the problem of maximisation of $\rho_k$ of a general moving average process of order $q$ with random coefficients is equivalent to the problem of maximisation of $\rho_1$ of a moving average process of order $Q$. Hence, we say that the maximum value of $\rho_k$ will be $\cos \alpha_k$ where $\alpha_k$ satisfies the equation

$$(1+C) \sin(Q+2)\alpha_k + C \sin Q \alpha_k = 0. \quad (17)$$
When \( C = 0 \), from the Remark 1, the maximum value of \( \rho_k \) is \( \cos \left( \frac{\pi}{Q+2} \right) \), which tallies with the result given by Davies et al. (1974) and Anderson (1974).

For sake of completeness, we attach the Table 7.1 showing the maximum values of \( \rho_1 \) for a moving average process with random coefficients for the different combinations of \( q \) and \( C \). From the table it is apparent that for \( q > 5 \) and \( C > 5 \) the maximum autocorrelation of order 1 does not seem to change much; and is an agreement with Remark 2 of Section 7.2.
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