Chapter 5

Bose-Hubbard model with periodic potentials

5.1 Introduction

In this chapter, we discuss the Bose-Hubbard model with periodic commensurate and incommensurate potentials. Motivation for studying this model, as discussed in the introduction, is to understand how the superfluid is destroyed in the presence of these potential and the formation of the other insulating phases like mass density wave and the Bose glass phases. It is well known that pure Bose-Hubbard model has only two phases: superfluid and Mott insulator and its phase diagram is well known. We had reproduced it in the chapter 2. However, when disorder is introduced, a new phase, the Bose glass phase emerges [26]. The disorder Bose-Hubbard model, though studied for many decades [26, 27] still attract many interest. The question of existence of direct superfluid to Mott insulator in the presence of disorder is
still an open problem. In one dimension, however, it has been reported that a direct transition between superfluid and Mott insulator is not possible due to the existence of rare region [126]. We wanted to validate this prediction by systematically studying formation of Bose glass phase, first in the presence of correlated disorder and later in the following chapter, in the presence of random potential.

The Bose-Hubbard model in the presence of periodic potentials is given by

\[ \mathcal{H} = -t \sum_{<i,j>} (a_i^\dagger a_j + h.c) + \frac{U}{2} \sum_i n_i(n_i - 1) + \sum_i \mu_i n_i \]  

(5.1)

where \( \mu_i = \lambda \cos(Qi) \). We consider two different cases: (i) a periodic commensurate potential, where we take \( Q = \pi \) and (ii) a period incommensurate potential. First we consider the former.

### 5.2 Commensurate potential

When \( \lambda = 0 \), the model shows superfluid to Mott insulator transition for \( \rho = 1 \) with the critical onsite interaction \( U_C \sim 3.4 \) as discussed in the Sec.(2.6). This phase transition get modified in the presence of the finite local chemical potential \( \mu_i \). In the model (5.1), the competition is between \( t, U \) and \( \lambda \). It is well known that when the kinetic energy dominates, the system prefers superfluid phase as its natural ground state and the onsite
interaction localizes the bosons. The superfluidity decreases with increasing $U$ and system become an insulator when $\rho = 1$. However, it is not very clear what role does local chemical potential play in the system. In the pure Bose-Hubbard model, Fig. 2.6, suggest that the density of bosons increases with $\mu$ when the system is in the superfluid phase. However, the density remain pinned to an integer density in the Mott insulator phase. Thus it is interesting to study the effect of inhomogeneous local potential on both the superfluid and the Mott insulator.

First we start from a superfluid phase and increase the strength of local periodic commensurate potential $\lambda$. The finite size scaling of correlation length $L/\zeta_L^{SF}$ is plotted as a function of $\lambda$ for different lengths $L$ for $U = 2.5$ in Fig. 5.1. For $\lambda = 0$, the system with $U = 2.5$ is in the superfluid phase and as we increases $\lambda$ we observe that the plots of $L/\zeta_L^{SF}$ for different lengths coalesce for small values of $\lambda$ which confirm that the system continue to be in the superfluid phase as we increase $\lambda$. However, for $\lambda > \lambda_C \sim 2.6$, coalescence of different curve is lost signaling opening up of gap in the system. Thus there exist a SF to non superfluid transition as a function of $\lambda$.

Interesting scenario occurs when we start from a Mott insulator and increase $\lambda$. For example $L/\zeta_L^{SF}$ as a function of $\lambda$ for $U = 5.0$ is plotted in Fig. 5.2. As we know from Sec.2.6 that the system for $U = 5.0$ is in the Mott insulator phase when $\lambda = 0$. Increasing $\lambda$, the $L/\zeta_L^{SF}$ plots for differ-
Figure 5.1: Plot of finite size scaling of correlation length, $L/\zeta_L$ as function of $\lambda$ for $U = 2.5$. The plots for different lengths, coalesce below $\lambda = \lambda_C \sim 2.6$ which yields opening of gap at $\lambda_C \sim 2.6$.

Different lengths first converges slowly and coalesce for $2 < \lambda < 3$. For further increase in $\lambda$ the coalescence is lost. So we have a Mott insulator phase to start with which goes over to superfluid as we increase $\lambda$ and finally into a non-superfluid phase with finite gap. To understand the nature of this non-superfluid phase, we calculate the local density of bosons $\rho_i = \langle n_i \rangle$ and plot it as a function of position $i$ in Fig. 5.3 for $\lambda = 0.2$ and 3.8. The $\rho_i$ shows a clear oscillation typical of a density wave for $\lambda = 3.8$. However, no such oscillations were seen for $\lambda = 0.2$ which is, of course, as we know is a Mott insulator phase and has constant value of $\rho_i$. So the gaped phase seen for larger values of $\lambda$ is in fact a density wave phase. Thus model (5.1), when
the local potential is periodic commensurate, has two phase transitions when $U > U_C \approx 3.4$; Mott insulator to superfluid and followed by superfluid to density wave (DW) phase. The superfluid phase is sandwiched between the MI and the DW phases.

For higher values of $U$, the superfluid phase which has sandwiched between MI and DW get smaller and smaller as shown in Figs. 5.4 and 5.5, respectively, for $U = 7.5$ and 12. Eventually at higher value of $U$, this superfluid phase is completely vanishes and the transition now is directly from Mott insulator to the density wave. For example $L/\xi_L^{SF}$ is plotted for $U = 20$ in Fig. 5.6.
Figure 5.3: Plot of local number density $\rho_i$ against $i$ for $U = 5.0$ and two values of $\lambda$. For $\lambda = 3.8$, the oscillation in $\rho_i$ suggest a DW phase. However, no such oscillations are seen for $\lambda = 0.2$ and $\rho_i$ remains constant.

Figure 5.4: Plot of finite size scaling of correlation length, $L/\xi_L^{SF}$ as function of $\lambda$ for $U = 7.5$. The plots for different lengths coalesce for $3.5 < \lambda < 4.2$, thus showing two phase transitions, MI to SF and SF to DW as $\lambda$ is increased.
Figure 5.5: Plot of finite size scaling of correlation length, $L/\xi_L^{SF}$ as function of $\lambda$ for $U = 12.0$. The region where plots of different length coalesce has reduced compare to Fig. 5.4. However, the superfluid phase is still exist between MI and DW phases.
Studying these figures, the phase diagram is obtained for model (5.1) with period commensurate potential and is given in Fig. 5.7. Phase diagram is consist of three phases: the superfluid, the Mott insulator and the density wave. For small $U$, the transition is from SF to DW. For $U > U_C \sim 3.4$, the transition is from MI to SF and then to DW. And finally for $U > 12$ one sees direct MI-DW transition as there is no SF phase sandwiched between them. The hopping matrix element $t = 1$ is negligible compare to $U$ and $\lambda$. The phase transition MI-DW occurs exactly at $\lambda = \frac{U}{2}$. Fig. 5.6 shows MI-DW transition at $\lambda_C = 10.0$. The phase diagram given here has resemblance to the phase diagram of extended Bose-Hubbard model for $\rho = 1$ [75]. The we conclude the model (5.1) with $Q = \pi$ mimic extended Bose-Hubbard model where the role of the nearest neighbor interaction $V n_i n_j$ is played by the periodic commensurate potential with $Q = \pi$. 
Figure 5.6: Plot of finite size scaling of correlation length, $L/\zeta_L^{SF}$ as function of $\lambda$ for $U = 20.0$. In this case system does not enter into the SF phase as $\lambda$ is increased and the transition is direct from MI to DW.

Figure 5.7: Phase diagram of model (5.1) with periodic commensurate potential taking $Q = \pi$. 
5.3 Incommensurate Potential

As discussed in the previous section, when we introduce a local commensurate potential in the Bose-Hubbard model, both the superfluid and the Mott insulator phases are destroyed and the system goes into a new gapped density wave phase. In this section we extend our study to the case of periodic, but incommensurate potential, where the last term of model 5.1 is given by,

\[ \mu_i = \lambda \cos(Qi). \]

We consider two values of \( Q = Q_1 = \frac{1+\sqrt{5}}{2} \) and \( Q = Q_2 = \frac{1+\sqrt{5}}{2} \) to demonstrate our results, which then can be easily extended to any other values of \( Q \). The reasons for taking these two values are the following. Taking \( Q \) proportional to the golden mean, we make sure that the local chemical potential is incommensurate to the underlying lattice since commensurability can lead to an insulating phases as discussed in the previous section and we would like to avoid such scenario. We would like to study and identify the new phase into which the system goes as we increase \( \lambda \). As we show below, this new phase is nothing but the Bose glass phase [26]. The second reason for choosing two values of \( Q \) is to study how the superfluid and the Mott insulator phases change in the presence of these local potentials when they vary across the lattice in different fashion. The local chemical potentials \( \mu_i \) are plotted against \( i \) in Fig. 5.8 for example for \( \lambda = 1 \). When the
\( Q = Q_2 = \frac{1 + \sqrt{5}}{2}, \) the variation of \( \mu_i \) is smooth over many sites compare to when \( Q = Q_1 = \frac{1 + \sqrt{5}}{2} \). As we discuss below, many interesting aspects of Bose glass phase can be leaned from these two cases and these results helps us in understanding what happens when \( \mu_i \) is completely random which is discussed in the next chapter, where rare regions [126] play an important role in the MI to BG transition. The remaining chapter is organized in the following manner: First we consider the case where \( Q = \frac{1 + \sqrt{5}}{2} \) in Sec. (5.4) and followed by the case where we take \( Q = \frac{1 + \sqrt{5}}{2} \) in Sec. (5.5). Finally we conclude all our results presented in the chapter in Sec. (5.7).

5.4 \( Q = \frac{1 + \sqrt{5}}{2} \)

When \( \lambda \) is finite, we expect three phases [85]: superfluid, Mott insulator and Bose glass. The nature of superfluid and Mott insulator phases are well understood. However, the same is not true in the case of Bose Glass. The global properties of the Bose glass phase is that it has zero gap, exponential decay of superfluid correlation function, finite compressibility. These properties do not help us in understanding the formation of the Bose glass phase, especially in the context of understanding the optical lattice experiments with the addition local potential. In order to remove this lacuna we analyze local
Figure 5.8: Variation of $\mu_i = \lambda \cos(Qi)$ as function of $i$ for $\lambda = 1$. The solid line is for $Q = \frac{1 + \sqrt{5}}{2}$, the golden mean and dashed line is for $Q = \frac{1}{5} \frac{1 + \sqrt{5}}{2}$, one fifth of the golden mean.
properties, especially the local compressibility defined as

\[ \kappa_i = \frac{\delta \mu_i}{\delta \rho_i} \]  

(5.2)

where \( \rho_i = \langle n_i \rangle \). If the local compressibility of a system is finite (zero) across the lattice, we can easily conclude that the system is in the superfluid (Mott insulator). However, as we show below, there can be a situation where the \( \kappa_i \) is zero in some part of the system while remaining finite everywhere else. This is then a Bose glass phase because, this system can not be called a superfluid because the wave function is not extended throughout the system. Neither we can call this a Mott insulator, because the gap is zero. Thus the local compressibility is very useful in understanding the nature of the ground state of an inhomogeneous system.

We obtain \( \kappa_i \) in the following manner. In Chapter (2), for the case of pure Bose-Hubbard model, i.e., \( \lambda = 0 \), we had discussed the relation between density \( \rho = N/L \) and chemical potential \( \mu \) and the corresponding compressibility for a given value of \( U \) (see Figs. 2.6 and 2.7). When \( \lambda = 0 \), the system is homogeneous, i.e., all the lattice sites are identical, thus the system as a whole has a density \( \rho \), the chemical potential \( \mu \) and the compressibility \( \kappa \). However, when \( \lambda \) is finite, the local chemical potential \( \mu_i \) is site dependent, so are the \( \rho_i \) and \( \kappa_i \).

In the presence of finite \( \lambda \), we can define an effective local chemical po-
tential $\mu_i^{\text{eff}} = \mu - \lambda \cos(Qi)$ where $\mu$ is the chemical potential of the system defined as

$$\mu = \frac{\delta E}{\delta N}$$  \hspace{1cm} (5.3)

where $E$ is the energy of the system and $N = \sum_i \rho_i$ with $\rho_i = \langle n_i \rangle$. From these values of $\rho_i$ and $\mu_i^{\text{eff}}$ we calculate the local compressibility $\kappa_i$ following Eq.5.2. We present below this method explicitly for a small value of $\lambda = 0.6$ taking $U = 2$. The local effective chemical potential $\mu_i^{\text{eff}}$ and local density of bosons $\rho_i$ are plotted in Fig. 5.9 for the system density $\rho = N/L = 1$. When $\lambda = 0$, both $\rho_i$ and $\mu_i^{\text{eff}}$ are constant through out the system. However, when local chemical potential vary, the variation of $\rho_i$ as a function of $i$ follow that of $\mu_i^{\text{eff}}$. As $\mu_i^{\text{eff}}$ increases, the local $\rho_i$ also increases and vice versa. In other words the value of $\rho_i$ depends on the value of $\mu_i^{\text{eff}}$. This encourages us to compare the dependence of $\mu_i^{\text{eff}}$ on $\rho_i$ with that of pure system and is given in Fig. 5.10. The curve for $\lambda = 0$ and 0.6 agree with each other. This implies

$$\rho_i(\lambda, \mu) = \rho(0, \mu_i)$$

$$\kappa_i(\lambda, \mu) = \kappa(0, \mu_i)$$  \hspace{1cm} (5.4)

These relations are known as Local Density Approximation (LDA) [127] which has been used recently in the context of Bose-Hubbard model in the presence of trap potential [96]. As we discuss later in this chapter, LDA is a good approximation only when the system is away from the critical region.
and the local chemical potential vary smoothly over many lattice as the case here.

The $\kappa_i$ calculated from the relation of $\rho_i$ and $\mu_i^{eff}$ using the Eq. (5.2) is given in Fig. 5.11 as a function of $\mu_i^{eff}$. Since we know that $\mu_i^{eff} = \mu - \lambda \cos(Qi)$ we can obtain the value of $\kappa_i$ as a function of $i$ and the same is given in Fig. 5.12. Since the local compressibility $\kappa_i$ is finite throughout the lattice, we conclude that system for $U = 2$ and $\lambda = 0.6$ is in the superfluid phase.

![Graph showing variation of $\mu_i^{eff}$ and $\rho_i$ with $i$.](image)

**Figure 5.9:** Variation of $\mu_i^{eff} = \mu - \lambda \cos(Qi)$ and $\rho_i$ as function of $i$ for $\lambda = 0.6$, $U = 2$ and $\rho = 1$.

Having described the method of obtaining the local compressibility $\kappa_i$, we now extend our discussion on various regions of the phase diagram. We broadly divide the phase diagram into two regions: (i) $U < U_C$ and (ii)
Figure 5.10: Variation of $\rho_i$ as function of $\mu_{\text{eff}}^{i} = \mu - \lambda \cos(Q_i)$ for $\lambda = 0.6$, $U = 2$ and $\rho = 1$.

Figure 5.11: Variation of $\kappa_i$ as function of $\mu_{\text{eff}}^{i} = \mu - \lambda \cos(Q_i)$ for $\lambda = 0.6$, $U = 2$ and $\rho = 1$. 
Figure 5.12: Variation of $\kappa_i$ as function of $i$ for $\lambda = 0.6$, $U = 2$ and $\rho = 1$.

$U > U_C$, where $U_C \sim 3.4$ is the critical onsite interaction for the superfluid to Mott insulator transition in the pure Bose-Hubbard model for density $\rho = 1$.

5.4.1 $U < U_C$

We first discuss the case $U < U_C$. Taking $U = 2$ which is in the superfluid phase when $\lambda = 0$, we increase the value of $\lambda$ and discuss below how this superfluid phase is destroyed. Figs. 5.13 and 5.14 are for $\lambda = 1.5$ where we plot, respectively, $\rho_i$ and $\mu_i^{\text{eff}}$ as a function of $i$ and $\rho_i$ as a function of $\mu_i^{\text{eff}}$. When $\lambda$ is increased, the $\mu_i^{\text{eff}}$ vary over a wider range of values and that lead to a wider spread in the values of $\rho_i$. The system is still in the superfluid because the compressibility, slope of the plot $\rho_i$ versus $\mu_i$ as in Fig. 5.14,
is finite everywhere in the lattice. As we increase $\lambda$ further, the system eventually goes to Bose glass phase. To demonstrate this we plot $\rho_i$ versus $\mu_i^{eff}$ for different values of $\lambda$ in Fig. 5.15. As $\lambda$ increases, the $\rho_i$ decreases with decrease in $\mu_i^{eff}$ and eventually become zero, forming a plateau for smaller values of $\mu_i^{eff}$. The formation of plateau yields zero compressibility and suggests that there are regions in the lattice where the local compressibility is zero. The other regions still have finite compressibility. For example the variation of $\kappa_i$ with respect to $\mu_i^{eff}$ is given in Fig. 5.16 for $\lambda = 4$. For smaller values of $\mu_i^{eff}$ (i.e., for higher values of $\mu_i$, since $\mu_i^{eff} = \mu - \mu_i$ and $\mu_i = \lambda \cos(Qi)$), we observe that $\kappa_i$ goes to zero. From the relation between $\mu_i^{eff}$ and $i$ we calculate $\kappa_i$ as function of $i$ and is given in Fig. 5.17. There are regions in the lattice where the $\kappa_i$ vanishes. The entire lattice are divided into regions of zero and finite local compressibility, giving us the Bose glass phase. To distinguish this Bose glass from the one we obtain when $U > U_C$, we call this phase as BG-I. Though there are regions which are superfluids, the lattice as a whole is no more superfluid, because these superfluid regions are separated by incompressible regions. The variation of $\rho_i$ in the lattice is given in Fig. 5.18 for the same set of $\lambda$'s as in Fig. 5.15. It is very clear from these figures that, there are regions in the lattice, where the number of bosons are very small or zero. This is the incompressible pockets which separates superfluid regions.
Figure 5.13: Variation of $\mu_{i}^{\text{eff}} = \mu - \lambda \cos(Qi)$ and $\rho_i$ as function of $i$ for $\lambda = 1.5$, $U = 2$ and $\rho = 1$.

Figure 5.14: Variation of $\rho_i$ as function of $\mu_{i}^{\text{eff}} = \mu - \lambda \cos(Qi)$ for $\lambda = 1.5$, $U = 2$ and $\rho = 1$. 
Figure 5.15: Variation of $\rho_t$ as function of $\mu_t^{\text{eff}}$ for different values of $\lambda$, but keeping $U = 2$ and $\rho = 1$. 
Figure 5.16: Variation of $\kappa_i$ as function of $\mu_i^{eff}$ for $\lambda = 4$, $U = 2$ and $\rho = 1$.

Figure 5.17: Variation of $\kappa_i$ as function of $i$ for $\lambda = 4$, $U = 2$ and $\rho = 1$. 
Figure 5.18: Variation of $\rho_i$ as function of $i$ for the same set of $\lambda$ as Fig. 5.15, but keeping $U = 2$ and $\rho = 1$. 
Thus for $U < U_C$, the formation of Bose glass phase (BG-I) is due to the fact that the entire lattice divides into pockets of compressible superfluids separated by incompressible regions where the number of bosons is zero due to large $\mu_i$. Because of these incompressible pockets, the superfluidity vanishes in these regions. The critical $\lambda_C$ for the superfluid to Bose glass transition, however, depends upon the overall density of the bosons in the lattice. It may be noted that, for larger density, the corresponding chemical potential $\mu$ is also larger (see Fig. 2.6) and a bigger value of $\lambda$ is required to bring the values of $\mu_i^{eff}$ which will have zero $\rho_i$. In other words, the $\lambda_C$ for SF to BG transition increases with $\rho$.

5.4.2 $U > U_C$

The way superfluidity is lost when we increase the incommensurate potential is different when $U > U_C$ compare to $U < U_C$, because of the existence of Mott insulator phase for $\rho = 1$. The Mott insulator has a finite gap in its energy spectrum and is given by

$$G(U, \lambda) = \mu^+(U, \lambda) - \mu^-(U, \lambda)$$  \hspace{1cm} (5.5)$$

where $\mu^+(U, \lambda)$ and $\mu^-(U, \lambda)$ are, respectively, the values of $\mu$ at upper and lower knee of the Mott insulator plateau, see for example, Fig. 2.7 for $U = 8$, $\lambda = 0$. 
To demonstrate the formation Bose glass phase when $U > U_C$, we take two values of $U = 6$ and 12. First we consider $U = 12$ and begin from a superfluid phase, i.e., $\rho < 1$, and increase $\lambda$. We consider $\rho = 0.84$, however, the results discussed here is applicable for other densities as well. The variation of $\rho_i$ and $\mu_i^{\text{eff}}$ across the lattice are given in Fig. 5.19 for $\lambda = 0.6$. We observe that the value of $\rho_i$ follows that of $\mu_i^{\text{eff}}$ as we discussed in the previous section for the case of $U = 2$. The variation of $\rho_i$ against $\mu_i^{\text{eff}}$ is given in Fig. 5.20 for the same set of parameter along with that for $\lambda = 0$ for comparison. The agreement between $\lambda = 0$ and 0.6 again suggest validity of LDA [127] method as discussed in the case of $U = 2$ (see Eqs. 5.4). The local compressibility calculated in the same manner as we described in the previous section is given in the Figs. 5.21 and 5.22, respectively, as a function of $\mu_i^{\text{eff}}$ and $i$. Since the compressibility is finite everywhere in the lattice, we conclude that for $\lambda = 0.6$, the system is a superfluid.

We increase the values of $\lambda$ now. We plot $\rho_i$ and $\mu_i^{\text{eff}}$ versus $i$ in Fig. 5.23 and $\rho_i$ versus $\mu_i^{\text{eff}}$ in Fig. 5.24 for $\lambda = 1.0, 1.5, 2.0$ and 2.5. The variation $\rho_i$ as function of $\mu_i^{\text{eff}}$ agree with the pure system for small values of $\lambda$. However, as $\lambda$ increases, this agreement is not so good. The variation of $\rho_i$ versus $\mu_i^{\text{eff}}$ smoothen out near the Mott insulator region. It may be noted that for pure system, the compressibility diverges as MI phase is approached from both $\rho < 1$ and $\rho > 1$ sides. However, this divergence is lost as soon as
Figure 5.19: Variation of \( \rho_i \) and \( \mu_{i\text{eff}} = \mu - \lambda \cos(Q_i) \) for \( \lambda = 0.6 \), \( U = 12 \) and \( \rho = 0.84 \).

Figure 5.20: Variation of \( \rho_i \) as function of \( \mu_{i\text{eff}} \) for \( \lambda = 0.6 \), \( U = 12 \) and \( \rho = 0.84 \).
Figure 5.21: Variation of $\kappa_i$ as function of $\mu_i^{\text{eff}}$ for $\lambda = 0.6$, $U = 12$ and $\rho = 0.84$.

Figure 5.22: Variation of $\kappa_i$ as function of $i$ for $\lambda = 0.6$, $U = 12$ and $\rho = 0.84$. 
inhomogeneity is introduced in the local chemical potential and that lead to the smothering of the $\rho_i$ versus $\mu_i^{\text{eff}}$ plots. In other works the value of chemical potential at the lower knee $\mu^-(U, \lambda) > \mu^-(U, \lambda = 0)$. Similarly $\mu^+(U, \lambda) \leq \mu^-(U, \lambda = 0)$. i.e., the gap $G(U, \lambda) = \mu^+(U, \lambda) - \mu^-(U, \lambda)$ is smaller than that of pure system, $G(U, \lambda = 0) = \mu^+(U, \lambda = 0) - \mu^-(U, \lambda = 0)$.

When values of $\mu_i^{\text{eff}}$ are such that $\mu^-(U, \lambda) < \mu_i^{\text{eff}} < \mu^+(U, \lambda)$, the $\rho_i = 1$, which can be seen from Fig. 5.23 and these regions has zero compressibility, thus a Mott insulator phase. For example $\kappa_i$ is plotted with respect to $\mu_i^{\text{eff}}$ and $i$ respectively in Figs. 5.25 and 5.26 for $\lambda = 2.5$. From these figures we observe that whenever, $\mu_i^{\text{eff}} > \mu^-(U, \lambda)$, $\kappa_i = 0$. However, when $\mu_i^{\text{eff}} < \mu^-(U, \lambda)$ the compressibility is finite. This leads to an interesting scenario. The entire system now breaks into pockets of Mott insulator and superfluid regions, and this is the Bose glass phase for $U > U_C$. We see this phase is gapless because there are superfluid region and it is compressible because we can add any number of bosons in these superfluid regions without causing any energy. We call this phase as BG-II to distinguish it from BG-I. The system which was in superfluid for small values of $\lambda$ goes over to a Bose glass for higher values of $\lambda$. This Bose glass phase, BG-II, is different from the one we saw when $U < U_C \sim 3.4$, i.e, BG-I. Here the entire lattice is divided into pockets of superfluids separated by incompressible Mott insulator regions. The difference between $U < U_C$ and $U > U_C$ is that in the former case,
the incompressible region has no bosons, the effective chemical potentials are small to hold bosons. However, in the latter case, the incompressible regions are Mott insulators.

Figure 5.23: Variation of $\rho_i$ and $\mu_i^{\text{eff}} = \mu - \lambda \cos(Qi)$ for (a) $\lambda = 1.0$, (b) $\lambda = 1.5$, (c) $\lambda = 2.0$ and (d) $\lambda = 2.5$ keeping $U = 12$ and $\rho = 0.84$. 
Figure 5.24: Variation of $\rho_i$ versus $\mu_i^{\text{eff}} = \mu - \lambda \cos(Qi)$ for (a) $\lambda = 1.0$, (b) $\lambda = 1.5$, (c) $\lambda = 2.0$ and (d) $\lambda = 2.5$ keeping $U = 12$ and $\rho = 0.84$.

Now we discuss $U = 6$. We plot $\rho_i$ and $\mu_i^{\text{eff}}$ versus $i$ in Fig. 5.27 and $\rho_i$ versus $\mu_i^{\text{eff}}$ in Fig. 5.28 for $\lambda = 0.4$ to 2.0 keeping $\rho = 0.84$. For small values of $\lambda$, the variation $\rho_i$ as function of $\mu_i^{\text{eff}}$ agree with that of the pure system. However, the behavior differ when MI phase is approached as discussed in
Figure 5.25: Variation of $\kappa_i$ as function of $\mu^{\text{eff}}_i$ for $\lambda = 2.5$, $U = 12$ and $\rho = 0.84$.

Figure 5.26: Variation of $\kappa_i$ as function of $i$ for $\lambda = 2.5$, $U = 12$ and $\rho = 0.84$. 
the case of $U = 12$. Comparing Fig. 5.28 with Fig. 5.24, we see that, when
the system is in the superfluid phase, the deviation of $\rho_i$ versus $\mu_i^{eff}$ from the
pure system curve is less for $U = 6$ compare to $U = 12$. The most interesting
result, however, is that there is no MI region for $U = 6$. The system continue
to be in the superfluid phase even for $A = 2.0$. Compressibility, the slope of
the $\rho_i$ versus $\mu_i^{eff}$ curve, is finite everywhere in the lattice. It may be noted
that we have a Bose glass when $U = 12$ for $\lambda = 2$, and a Mott insulator
when $U = 6$, $\lambda = 0$. For $U = 6$, the Bose glass phase, BG-I, however, occur
at a higher values of $\lambda$ in the same manner as we saw for the case of $U = 2
where the system is divided into compressible superfluids and incompressible
regions with nearly zero bosons. So the situation for $U = 6$ is different from
$U = 12$. For $U = 12$ we have a superfluid to Bose Glass-II phase transition
as a function of $\lambda$. The Bose glass phase occur when the $\mu_i^{eff}$ values are
such that $\mu^- < \mu_i^{eff} < \mu^-$ and it consist of superfluid and Mott insulator
pockets. However, for $U = 6$, the MI region get smeared out and SF to BG-I
transition occur at much higher value of $\lambda$ and behave in the same manner
as the case $U < U_C$. 
Figure 5.27: Variation of $\rho_i$ and $\mu_{\text{eff}} = \mu - \lambda \cos(Qi)$ as a function of $i$ for different values of $\lambda$ keeping $U = 6$ and $\rho = 0.84$. 
Figure 5.28: Variation of $\rho_i$ versus $\mu_i^{eff} = \mu - \lambda \cos(Q_i)$ for different values of $\lambda$ keeping $U = 6$ and $\rho = 0.84$. 
In this section we discuss the phases of model 5.1 in the presence of the periodic incommensurate potential \( \mu_i = \lambda \cos(Qi) \) with \( Q = \frac{1+\sqrt{5}}{2} \). The variation of \( \mu_i \) is given in Fig. 5.8 for \( \lambda = 1 \) and comparing it with that for \( Q = \frac{1+\sqrt{5}}{2} \), we see that variation of \( \mu_i \) across the lattice is faster for the former case. We begin with the case \( U < U_C \) and take \( U = 2 \). In Fig. 5.29 we plot \( \rho_i \) versus \( \mu_{i}^{\text{eff}} \) for \( \rho = 1 \), which clearly demonstrate the phase transition from superfluid to Bose glass, BG-I, as \( \lambda \) is increased. Comparing Figs. 5.29 and 5.15 for the cases with \( Q = \frac{1+\sqrt{5}}{2} \) and \( Q = \frac{11+\sqrt{5}}{8} \), respectively, following conclusions are drawn. First the comparison with finite \( \lambda \) case with pure system. In the case of \( Q = \frac{11+\sqrt{5}}{8} \), the LDA (i.e., Eqs. 5.4) is a good approximation, however, that is not the case with \( Q = \frac{1+\sqrt{5}}{2} \). The variation of \( \rho_i \) with respect to \( \mu_{i}^{\text{eff}} \) for finite \( \lambda \) deviate from that for \( \lambda = 0 \). In other words, LDA is not a good approximation when the variation of \( \mu_{i}^{\text{eff}} \) is brisk. However, if the variation is slow and smooth over few lattice sites, the LDA is a good approximation.

The compressibility, i.e., the slope of \( \rho_i \) versus \( \mu_{i}^{\text{eff}} \) curve, has decreased from that of pure system. However, it is still finite everywhere in the system for small values of \( \lambda \), thus a superfluid. As \( \lambda \) increases, there are regions in the lattice where the compressibility \( \kappa_i \) is zero, which is reflected as a plateau.
in the $\rho_i$ versus $\mu_i^{\text{eff}}$ curve, thus the system goes to a Bose glass, BG-I phase.

The results for $U = 12$ and $6$ for $Q = \frac{1+\sqrt{5}}{2}$ are similar to the case of $Q = \frac{1+\sqrt{5}}{2}$ and are given in the Figs. 5.30 and 5.31, respectively. The difference is that the LDA is not a good approximation in this case as discussed above. The deviation from the pure system curve is large near the MI region. However, we can still see the SF to BG transition in the case of $U = 12$ and no such transition for $U = 6$ for smaller values of $\lambda$ and the conclusions are similar to the case with $Q = \frac{1+\sqrt{5}}{2}$. 
Figure 5.29: Variation of $\rho_i$ versus $\mu_i^{\text{eff}} = \mu - \lambda \cos(Q_i)$ for different values of $\lambda$ keeping $U = 2$ and $\rho = 1$. 
Figure 5.30: Variation of $\rho_i$ versus $\mu_i^{eff} = \mu - \lambda \cos(Qi)$ for different values of $\lambda$ keeping $U = 12$ and $\rho = 0.84$. 
Figure 5.31: Variation of $\rho_i$ versus $\mu_i^{eff} = \mu - \lambda \cos(Q_i)$ for different values of $\lambda$ keeping $U = 6$ and $\rho = 0.84$. 
\section*{5.6 \( \rho = 1, \ U > U_C \)}

When density \( \rho \) is exactly equal to an integer, say, \( \rho = 1 \), there are additional interesting features in the problem. This is because, the ground state of model (5.1) with \( \lambda = 0 \) has a Mott insulating phase for \( U > U_C \). The Mott insulating phase has finite gap given by \( G = \mu^+(U, \lambda = 0) - \mu^-(U, \lambda = 0) \).

When we vary the chemical potential \( \mu \), whenever, \( \mu \) satisfy the condition \( \mu^-(U, 0) < \mu < \mu^+(U, 0) \), the system enters into Mott insulating phase and the corresponding density is pinned to an integer value. For example see Fig. 2.7. The question then arises, what happened to this phase when incommensurate potential is introduced, which is addressed in this section.

We begin with a large value of \( U \), say \( U = 12 \) and increase the value of \( \lambda \) keeping \( Q = \frac{1+\sqrt{5}}{10} \). The variation of \( \rho_i \) and \( \kappa_i \) with respect to \( \mu_i^{\text{eff}} \) are calculated for each values of \( \lambda \). For two typical values of \( \lambda = 3.5 \) and 4, \( \rho_i \) and \( \kappa_i \) are given in Figs. 5.32 and 5.33 respectively. We have also plotted the behavior of pure system for comparison. From these calculation we observe that \( \kappa_i = 0 \) and \( \rho_i = 1 \) for all \( \mu_i^{\text{eff}} \) when \( \lambda \leq 3.5 \). In other words the system remains in the Mott insulator phase for \( \lambda \leq 3.5 \). However, as we increase \( \lambda \) further, there are regions in the lattice, where \( \kappa_i > 0 \) and in these regions the \( \rho_i \neq 1 \). The variation of \( \rho_i \) and \( \kappa_i \) across the lattice as a function of \( i \) is given in Figs. 5.34 for same set of parameters, i.e., \( \lambda = 3.5 \) and 4, which
confirms these observation. Thus when we increase $\lambda$ starting from a Mott insulator, we get transition from Mott insulator to Bose glass transition and the Bose glass is of type BG-II.

![Figure 5.32: Variation of $\rho_i$ versus $\mu_i^{eff} = \mu - \lambda \cos(Q_i)$ for different values of $\lambda$ keeping $U = 12$ and $\rho = 1$.](image)

Similar behavior is also seen for $U = 7$. For example $\rho_i$ versus $\mu_i$ as well as $\rho_i$ versus $i$ behaviors are plotted for different values of $\lambda$ in Figs. 5.35 and 5.36 respectively. The local compressibility for the corresponding system is plotted in Fig. 5.37. From these plots, we observe that system which was in the MI phase for $\lambda = 0$ continue to be in the MI phase for small values of $\lambda$. For example $\lambda = 0.2$ is in the MI phase. Increasing $\lambda$, say $\lambda = 1$, it goes over to BG-II. BG-II phase is very narrow for $U = 7$ unlike for $U = 12$. Further increase in $\lambda$, the MI region is completely smeared out and $\kappa_i$ is finite.
Figure 5.33: Variation of $\kappa_i$ versus $\mu_i^{\text{eff}} = \mu - \lambda \cos(Qi)$ for different values of $\lambda$ keeping $U = 12$ and $\rho = 1$.

Figure 5.34: Variation of $\rho_i$ and $\kappa_i$ versus $i$ for different values of $\lambda$ keeping $U = 12$ and $\rho = 1$. (a) $\rho_i$ for $\lambda = 3.5$, (b) $\kappa_i$ for $\lambda = 3.5$, (c) $\rho_i$ for $\lambda = 4$, (d) $\kappa_i$ for $\lambda = 4$. 
everywhere in the system making it a superfluid. And finally for very high values of $\lambda$, say $\lambda = 8$, the superfluity is lost in some regions in the lattice which have $\rho_i = 0$ and the lattice is a BG-I phase.

So when $U >> U_C$, we get the following phase transitions increase the strength of incommensurate potential keeping $\rho = 1$. Starting from a MI phase, the system goes over to BG-II as we increase $\lambda$. The parameter range where we get BG-II phase depends on the strength of $U$ with larger the value of $U$ leads to a bigger BG-II region. If $U$ is increased further, the MI region is completely lost and the system is now a superfluid with non zero local compressibility everywhere in the system. Finally at a large value of $\lambda$ the system goes over to BG-I phase.

Question then arises, whether there is a direct transition from superfluid to Mott insulator when we have a incommensurate potential. To answer this question we obtain $\rho_i$ versus $\mu_i$ plots keeping $\lambda$ fixed and varying $U$. We may recollect that the Bose glass phase intervening SF and MI is of the type BG-II where the entire system breaks into superfluid and Mott insulator pockets. First we consider the case $Q = \frac{1+\sqrt{5}}{10}$. In Fig. 5.38 plots of $\rho_i$ versus $\mu_i$ for keeping fixed $\lambda = 1.5$ and varying $U$ are given. In this figure we observe that all the curves corresponds to different values of $U$ cross at the same point at $\mu_i = 0$. As we increase $U$, the slope of each curve, which is nothing but the compressibility decreases continuously and became zero for $U > 6$. If there is
Figure 5.35: Variation of $p_i$ versus $\mu_i = \lambda \cos(Q_i)$ for different values of $\lambda$, but for $U = 7$. 
Figure 5.36: Variation of $\rho_i$ versus $i$ for same set of parameter as Fig. 5.35.
Figure 5.37: Variation of $\kappa_1$ versus $\mu_1 = \lambda \cos(\Omega t)$ for same set of parameters as Fig. 5.35.
Figure 5.38: Variation of $\rho_i$ versus $\mu_i$ keeping $\lambda = 1.5$, but varying $U$ as given in the figure.

Figure 5.39: Variation of $\rho_i$ versus $\mu_i$ keeping $\lambda = 2$, but varying $U$ as given in the figure.
a BG-II phase intervening SF and MI phases, we should have got (i) a region of \( m\mu_i \)'s such that \( \rho_i = 1 \) and (ii) all other values of \( \mu_i \)'s, \( \rho_i \neq 1 \). We do not see this behavior is happening in Fig. 5.38. For \( U \leq 6 \), \( \rho_i \neq 1 \) except at \( \mu_i = 0 \) and for \( U > 6 \), \( \rho_i = 1 \) for all \( \mu_i \). This suggest that the transition from SF to MI is direct. For small values of \( \lambda \), BG-II phase does not intervene between SF and MI phases. This result is consistent with the recent work which obtained the phase diagram based on global properties of different phases [85]. We observe similar result for \( Q = \frac{1 + \sqrt{5}}{2} \), with the differences only being that the Mott insulator phase get stabilized only for a higher values of \( U \) compare to the case when \( Q = \frac{1 + \sqrt{5}}{10} \). For example \( \rho_i \) versus \( \mu_i \) plots are given in the Fig. 5.39 for \( \lambda = 2 \) varying \( U \). We observe that the Mott insulator phase is not stabilized even for \( U = 12 \), which, however, was in the Mott insulator for \( Q = \frac{1 + \sqrt{5}}{10} \). This means that for a Mott insulator phase to get stabilized, the local chemical potential should be very smoothly across the lattice. If \( \mu_i \) vary very fast as in the case of \( Q = \frac{1 + \sqrt{5}}{2} \), higher value of \( U \) is required to stabilize the MI phase. These results have important contribution in the understanding of the disorder Bose Hubbard model which is discussed in the next chapter.
5.7 Summary

We now summarize our results for incommensurate potential. When $\lambda$ is finite, the effective local chemical potential $\mu_i^{\text{eff}}$ is not homogeneous. The effect of this is that the average local density $\rho_i$ is also not uniform across the lattice. The value of $\rho_i$ depends on the value of $\mu_i^{\text{eff}}$. The relation between $\rho_i$ and $\mu_i^{\text{eff}}$ obey local density approximation, LDA, i.e., Eq. 5.4, when the variation of $\mu_i$ is smooth over many sites in the lattice as well as the system is not close to the Mott insulator phase.

As we increase the values of $\lambda$, the range in which $\mu_i^{\text{eff}} = \mu - \lambda \cos(Qi)$ vary in the lattice increases. $\mu_i^{\text{eff}}$ vary between $\mu - \lambda$ to $\mu + \lambda$. Accordingly the $\rho_i$ and $\kappa_i$ vary across the lattice. When $\kappa_i$ is finite for all $i$, we get a superfluid phase. However, as we increase $\lambda$, there are regions in the lattice where the compressibility vanishes and the entire lattice breaks into pockets of superfluid and incompressible regions and we get the Bose glass phase. The nature of these incompressible regions depends upon the value of $\rho$ and $U$.

For $U < U_C$, for any value of $\rho$, these regions are incompressible because, $\mu_i$, the local chemical potential is large enough to hold any bosons and the Bose glass phase is of the type BG-I. However, the scenario is different for $U > U_C$.

When the density is an integer, say $\rho = 1$, the system has a finite gap given by Eq. 5.5 and this gap decreases as a function of $\lambda$. For density $\rho \neq 1$, the
system is in the superfluid phase when \( \lambda = 0 \). As we increase \( \lambda \), \( \mu^e_{i ff} \) vary over \( \mu(\lambda) \pm \lambda \) and whenever the condition \( \mu^-(U, \lambda) < \mu^e_{i ff} < \mu^+(U, \lambda) \) is satisfied, \( \rho_i = 1 \) and \( \kappa_i = 0 \). That is we get a incompressible Mott insulator phase. The entire system now breaks into pockets of compressible superfluids and incompressible Mott insulator, thus a Bose glass, BG-II, phase. The critical value of \( \lambda \) for the superfluid to Bose glass transition depends both on \( \rho \) and \( U \). If \( U \) is large and \( \rho \) is close to 1, \( \lambda_C \) is small. However, if \( \rho \) is far from unity, then a larger value of \( \lambda \) is required for the condition \( \mu^-(U, \lambda) < \mu^e_{i ff} < \mu^+(U, \lambda) \) to be satisfied. Thus \( \lambda_C \) is higher.

However, when \( U \) is not very large from \( U_C \), we get BG-II phase only for densities very close to unity. However, when the densities are far from unity, large value of \( \lambda \) is required for the condition \( \mu^-(U, \lambda) < \mu^e_{i ff} < \mu^+(U, \lambda) \) to be satisfied. Since gap is small when \( U \) is close to \( U_C \), large value of \( \lambda \) destroys the gap and the condition \( \mu^-(U, \lambda) < \mu^e_{i ff} < \mu^+(U, \lambda) \) is never satisfied as the case for \( \rho = 0.84 \) keeping \( U = 6 \). So we do not get BG-II phase here. However, for a much higher values of \( \lambda \) we get BG-I for the same reasons as discussed for the case \( U < U_C \).

When the density is exactly equal to an integer, say \( \rho = 1 \) and \( U > U_C \), the scenario is the following. We start with a Mott insulator phase for \( \lambda = 0 \). When \( \lambda \) is finite and if \( \mu^e_{i ff} \) for all the values of \( i \) are such that, \( \mu^-(U, \lambda) < \mu^e_{i ff} < \mu^+(U, \lambda) \), then \( \rho_i = 1, \kappa_i = 0 \) for all the values of \( i \) and
the system continues to be in the Mott insulator phase. As soon as few sites have \( \mu_{i}^{\text{eff}} < \mu^{-}(U, \lambda) \) or \( \mu_{i}^{\text{eff}} > \mu^{+}(U, \lambda) \), those sites go over to superfluid phase and the system is in the BG-II phase. However, when \( U \) is not very far from \( U_C \), the gap get smeared out when \( \lambda \) is increased and the Mott insulator region is completely lost. In this scenario, we get \( \kappa_i > 0 \) for all \( i \), thus a superfluid phase. This is the reentrant phenomena discussed in the earlier work [67]. Eventually for a higher value of \( \lambda \) the system goes over to BG-I phase. Thus we have a case where we get three phase transitions (i) MI to BG-II, (ii) BG-II to SF and (iii) SF to BG-I. This is the scenario for \( U >> U_C \). However, \( U - U_C \) is small, we get a direct SF to MI transition with no BG-II phase intervening between them.

So for density \( \rho = 1 \), we have (i) when \( U < U_C \), only SF to BG-I transition, (ii) when \( U > U_C \), we get different possibilities depends on the value of \( U \). For \( U \) close to \( U_C \) we get MI to SF and then to BG-I transition and when \( U >> U_C \), MI to BG-II, BG-II to SF and SF to BG-I transitions.