CHAPTER 7

RELATIVE MEASUREMENT AND COVARIANCE ANALYSIS OF CROSS-SECTIONS FOR $^{58}\text{Ni}(n, p)^{58}\text{Co}$ REACTION

7.1 Introduction

In this chapter we present the covariance analysis in the context of relative cross-section measurement. Experimental details are presented in chapter 4, we irradiated three sets of samples [each set containing one unknown sample (natural nickel foil) and two monitor samples (natural uranium and thorium foils)] at each of the three effective neutron energies $E_n = 5.88 \pm 0.12, 10.11 \pm 0.06$ and $15.86 \pm 0.12 \text{ MeV}$, respectively.

Our aim is to determine neutron induced reaction cross-section of $^{58}\text{Ni}(n, p)^{58}\text{Co}$ reaction, normalized to the cross-section for the formation of $^{97}\text{Zr}$ in the $^{238}\text{Th}(n, f)$ and $^{232}\text{U}(n, f)$ reactions, respectively, at the three effective neutron energies. At each of
the effective neutron energy, we get two values for the $^{58}\text{Ni}(n, p)^{58}\text{Co}$ reaction cross-section.

This chapter is based on section III.C, III.D and III.E of [Shivashankar et al., 2015].

### 7.2 Mathematical Expression Relating Unknown and Monitor Cross-sections

The unknown and monitor cross-sections in the relative measurement are logically connected by the following mathematical expression:

$$
\sigma_u(E_n) = \left[ \frac{C_u(CL)_{u}\lambda_u}{C_m(CL)_{m}\lambda_m} \frac{N_m\epsilon_{m\gamma}I_{m\gamma}T(\lambda_m, t)\alpha_m}{N_u\epsilon_{u\gamma}I_{u\gamma}T(\lambda_u, t)\alpha_u} \right] \sigma_m(E_n). \tag{7.1}
$$

The derivation of Eq.(7.1) is presented in chapter 2, correction factor $\alpha$ is included to account for neutron spectrum features and correction factor $CL$ and $LT$ are detector clock time (counting time) and live time. The accurate measurement of neutron activation cross-sections [Semkova, 1999], requires additional correction factors in Eq.(7.1).

The factor

$$
r_{um} = \left[ \frac{C_u(CL)_{u}\lambda_u}{C_m(CL)_{m}\lambda_m} \frac{N_m\epsilon_{m\gamma}I_{m\gamma}T(\lambda_m, t)\alpha_m}{N_u\epsilon_{u\gamma}I_{u\gamma}T(\lambda_u, t)\alpha_u} \right], \tag{7.2}
$$

is the ratio of (un-normalized) reaction rates [Kobayashi, 1994]. Equation.(7.2) can be further written in simplified form

$$
r_{um} = \left[ \frac{Q_u}{Q_m} \right], \tag{7.3}
$$

Eq.(7.1) is recast in the form of Eq.(7.3), which helps in the covariance analysis.
7.3 Data and Attributes For The Covariance Analysis of Relative Measurement

7.3.1 Number of Nuclei in The Sample

Weights of the nickel, thorium and uranium foils are presented in Table 4.1 of chapter 4 and is recast in Table 7.1 along with isotope abundance. The data of isotope abundances are taken from [Chu et al., 2014, NNDC, 2014].

<table>
<thead>
<tr>
<th>Foil</th>
<th>Weight ($\times 10^{-2}$ g)</th>
<th>Foil</th>
<th>Weight ($\times 10^{-2}$ g)</th>
<th>Foil</th>
<th>Weight ($\times 10^{-2}$ g)</th>
<th>Isotope</th>
<th>Isotope abundance ($\times 10^{-3}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ni-1</td>
<td>42.62 ± 0.83</td>
<td>Th-1</td>
<td>28.56 ± 0.57</td>
<td>U-1</td>
<td>69.70 ± 1.39</td>
<td>58Ni</td>
<td>680.77 ± 0.09</td>
</tr>
<tr>
<td>Ni-2</td>
<td>18.13 ± 0.36</td>
<td>Th-2</td>
<td>32.90 ± 0.65</td>
<td>U-2</td>
<td>99.17 ± 1.98</td>
<td>232Th</td>
<td>999.99 ± 0.01</td>
</tr>
<tr>
<td>Ni-3</td>
<td>12.60 ± 0.25</td>
<td>Th-3</td>
<td>32.52 ± 0.65</td>
<td>U-3</td>
<td>57.05 ± 1.16</td>
<td>238U</td>
<td>992.75 ± 0.06</td>
</tr>
</tbody>
</table>

Number of atoms in the foils is estimated using the following expression:

$$N = \frac{\text{Weight of the sample} \times \text{isotope abundance} \times \text{Avagadro number}}{\text{Mass number}}. \quad (7.4)$$

The Avogadro number $N_{av} = 60221408.57 \pm 0.74 \times 10^{30} \text{ mol}^{-1}$ is taken from the National Institute of Standards and Technology (NIST) reference on Constants, Units and Uncertainty [NIST, 2016]. Mass number for the isotopes $^{58}N$i, $^{232}Th$ and $^{238}U$ are 58, 232 and 238, respectively. The weights of the samples and isotope abundance are presented in Table 7.1.

In order to reduce number of attributes in the covariance analysis of relative measurement, we provide covariance analysis of number of sample atoms (or nuclei) in this section itself. There are three attributes to consider: Weight of the sample, Avogadro number and isotope abundance with attribute number q=1,2 and 3, respectively.

The expectation value of number of sample atoms (or nuclei) $\langle N \rangle (g \text{ mol})$ are presented in Table 7.2.
Chapter 7. Relative measurement and covariance analysis.

Table 7.2: Expectation value of number of sample atoms (or nuclei).

<table>
<thead>
<tr>
<th>Foil</th>
<th>( N \times 10^{12} )</th>
<th>Foil</th>
<th>( N \times 10^{12} )</th>
<th>Foil</th>
<th>( N \times 10^{12} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ni-1</td>
<td>0.3012850123.79</td>
<td>Th-1</td>
<td>0.00741338892.04</td>
<td>U-1</td>
<td>0.01750740191.11</td>
</tr>
<tr>
<td>Ni-2</td>
<td>0.01281570431.55</td>
<td>Th-2</td>
<td>0.00833418985.04</td>
<td>U-2</td>
<td>0.0249117205.46</td>
</tr>
<tr>
<td>Ni-3</td>
<td>0.00890269503.61</td>
<td>Th-3</td>
<td>0.00844025749.77</td>
<td>U-3</td>
<td>0.0145559004.44</td>
</tr>
</tbody>
</table>

Table of partial uncertainties is presented in Table 7.3. In Table 7.3 the index \( i = 1, 2 \) and 3 corresponds to nickel foils with tag number Ni-1, Ni-2 and Ni-3, respectively. \( i = 4, 5 \) and 6 corresponds to thorium foils with tag number Th-1, Th-2 and Th-3, respectively. and \( i = 7, 8 \) and 9 corresponds to uranium foils with tag number U-1, U-2 and U-3, respectively.

Table 7.3: Table of partial uncertainties \( \times 10^{12} \) for the covariance analysis of number of sample atoms.

<table>
<thead>
<tr>
<th>( i )</th>
<th>( q=1 )</th>
<th>( q=2 )</th>
<th>( q=3 )</th>
<th>( \Delta N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( i=1 )</td>
<td>60257002.50</td>
<td>37.02</td>
<td>398308.54</td>
<td>6025318.92</td>
</tr>
<tr>
<td>( i=2 )</td>
<td>25630148.63</td>
<td>15.75</td>
<td>169419.44</td>
<td>25630708.56</td>
</tr>
<tr>
<td>( i=3 )</td>
<td>17805390.07</td>
<td>10.94</td>
<td>117696.51</td>
<td>17805779.06</td>
</tr>
<tr>
<td>( i=4 )</td>
<td>14826777.84</td>
<td>09.11</td>
<td>007413.46</td>
<td>14826779.69</td>
</tr>
<tr>
<td>( i=5 )</td>
<td>16768379.70</td>
<td>10.30</td>
<td>008384.27</td>
<td>16768381.80</td>
</tr>
<tr>
<td>( i=6 )</td>
<td>16880515.00</td>
<td>10.37</td>
<td>008440.34</td>
<td>16880517.11</td>
</tr>
<tr>
<td>( i=7 )</td>
<td>35014803.82</td>
<td>21.51</td>
<td>105811.54</td>
<td>35014963.70</td>
</tr>
<tr>
<td>( i=8 )</td>
<td>49822344.11</td>
<td>30.61</td>
<td>150558.58</td>
<td>49822571.60</td>
</tr>
<tr>
<td>( i=9 )</td>
<td>49822344.11</td>
<td>17.88</td>
<td>087971.33</td>
<td>29111313.01</td>
</tr>
</tbody>
</table>

It can be observed from the table of partial uncertainties, that the partial uncertainties in attribute 1 have significant contributions in the magnitude of total uncertainties.

Generate matrices of partial uncertainties \( E_1, E_2 \) and \( E_3 \) from the table of partial uncertainties and generate micro-correlation matrices \( S_1 = I, S_2 = J \) (Avogadro number is common within attribute 2) and \( S_3 = I \) (Correlation information on the isotopic abundance is not available, and hence we are assigning micro-correlation zero within attribute 3). we obtained macro-correlation matrix whose off-diagonal elements were in the range \( 221 \times 10^{-12} \) to \( 0.377 \times 10^{-12} \). Hence, in the covariance analysis of relative measurement, the micro-correlations within the attribute of sample atoms were assigned zero.
Alternate way to look at the above analysis is evident from Eq. (7.1), it can be observed that, what we need is the ratio of sample atoms in the unknown and monitor foils:

\[ N' = \frac{N_m}{N_u} = \frac{w_m a_m N_A}{A_m} \times \frac{A_u}{w_u a_u N_A}, \tag{7.5} \]

i.e., \( N' \) is independent of Avogadro number, and is irrelevant for the covariance analysis of the ratio measurement.

### 7.3.2 Irradiation, Cooling and Counting Time

The irradiation, cooling and counting time were presented in Table 4.2 in chapter 4. The uncertainty in time factors were not taken into consideration and are assumed to be negligibly small\(^1\).

### 7.3.3 Counting Data

The characteristic \( \gamma \)-lines of energy \( 810.77 \times 10^{-3} \, MeV \) and \( 743.36 \times 10^{-3} \, MeV \) from the daughter nuclei \(^{58}\)Co and the fission product yield \(^{97}\)Zr were measured (counted) in the HPGe detector. The count data are presented in Table 4.2 of chapter 4. As mentioned in the previous chapter, the micro-correlations within the attribute of \( \gamma \)-counts were assigned zero.

---

\(^1\)Attributes are considered sequentially in chapter 5, chapter 6 and chapter 7 for clarity. Attributes considered in chapter 5, chapter 6 and chapter 7 constitute overall attributes of the measurement of interest. The only attribute disregarded is time, effect of time uncertainty being negligible in the sense that even if it were included, there would be no substantial change in total uncertainty (assigned to quantity of interest, cross-section). It can be observed (from sandwich formula) that, if uncertainty assigned to an attribute is at least three times smaller than the larger uncertainty assigned to other attribute, then the attribute with smallest uncertainty can be disregarded, provided micro-correlation within that attribute is zero. Since we are dealing with relative measurement and maintained the same geometry in case of gamma counting of irradiated samples and monitors, we have not included additional correction factors in Eq. (7.2), and hence there are no additional attributes considered.
7.3.4 Efficiency of The HPGe Detector

Efficiency of the detector is discussed in chapter 6. The efficiencies with respect to characteristic γ-lines of energy $810.77 \times 10^{-3} MeV$ and $743.36 \times 10^{-3} MeV$ and correlation information are presented in Section 6.4.2 of chapter 6.

7.3.5 Decay Data

Decay data required in the relative measurement are: Half-life of daughter nuclei $^{58}Co$ and of the fission product $^{97}Zr$. The γ-abundance ($I_γ$) of the γ-lines of energy $810.77 \times 10^{-3} MeV$ and $743.36 \times 10^{-3} MeV$. Decay data are presented in Table 7.4.

<table>
<thead>
<tr>
<th>Isotope</th>
<th>$T_{1/2}$</th>
<th>$E_γ \times 10^{-3}$ MeV</th>
<th>$I_γ \times 10^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{58}Co$</td>
<td>70.86 ± 0.07 d</td>
<td>810.77</td>
<td>989.99 ± 0.01</td>
</tr>
<tr>
<td>$^{97}Zr$</td>
<td>16.91 ± 0.05 h</td>
<td>743.36</td>
<td>929.99 ± 0.01</td>
</tr>
</tbody>
</table>

The decay constant ($λ$) and γ-abundance ($I_γ$) are considered as two attributes in the relative measurement.

7.3.6 Correction Due to Neutron Spectrum Features

Correction due to neutron spectrum features ($α$) required for the calculations are presented in Table 5.3 of chapter 5. The uncertainty in the correction factors ($α$) due to neutron spectrum features were not taken into consideration, because the method adopted in this work for the correction due to neutron spectrum features is approximate.

7.3.7 Monitor Cross-section

Although neutron energy does not appear explicitly in Eq.(7.1) of relative measurement, neutron energy plays major role in the covariance analysis of relative measurements. The neutron induced reaction cross-sections are neutron energy dependent.
Discussion on the effective neutron energy adopted in this work is discussed in Section 5.3 of Chapter 5 and is presented in Table 5.2.

The monitor cross-sections adopted in this work are: cross-section for the formation of the fission product $^{97}\text{Zr}$ in $^{232}\text{Th}(n, f)$ reaction and cross-section for the formation of the fission product $^{97}\text{Zr}$ in $^{238}\text{U}(n, f)$ reaction.

The two monitor cross-sections adopted in this work are:

1. The product of $^{232}\text{Th}(n, f)$ cross-section at the effective neutron energy and fission yield of the fission product $^{97}\text{Zr}$ produced in the $^{232}\text{Th}(n, f)$ reaction.

2. The product of $^{238}\text{U}(n, f)$ cross-section at the effective neutron energy and fission yield of the fission product $^{97}\text{Zr}$ produced in the $^{238}\text{U}(n, f)$ reaction.

### 7.3.7.1 Fission cross-section

Among the two fission reaction cross-sections considered in this work, only the $^{238}\text{U}(n, f)$ reaction is listed in the IAEA neutron cross-section standards in the neutron energy range $2 - 200\ MeV$ [IAEA, 2014], the $^{238}\text{U}(n, f)$ reaction cross-section data is presented in Table C.1 of Appendix C. The $^{232}\text{Th}(n, f)$ reaction cross-section is taken from ENDF/B-VII.1 [ENDF, 2014] and the $^{232}\text{Th}(n, f)$ reaction cross-section data is presented in Table C.2 of Appendix C.

The universal plots of $^{232}\text{Th}(n, f)$ and $^{238}\text{U}(n, f)$ cross-sections are presented in Fig.7.1. It can observed in the universal plots that, there is no abrupt variation in cross-section in the vicinity of effective neutron energies considered in this work and thus $^{232}\text{Th}(n, f)$ and $^{238}\text{U}(n, f)$ cross-sections qualify as part of the monitor cross-section.

The fission cross sections taken from [IAEA, 2014] and [ENDF, 2014] were linearized using the PREPRO linear module [Cullen, 2014], and the fission cross sections used in the present work were obtained using linear-linear interpolation [Cullen, 2010] and are presented in Table 7.5.
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Figure 7.1: Universal plot of $^{232}\text{Th}(n,f)$ and $^{238}\text{U}(n,f)$ cross-sections [ENDF, 2014].

Table 7.5: $^{232}\text{Th}(n,f)$ and $^{238}\text{U}(n,f)$ reaction cross-sections ($\sigma_{\text{mTh}}$ and $\sigma_{\text{mU}}$) at effective neutron energies ($E_n$).

<table>
<thead>
<tr>
<th>$E_n$, MeV</th>
<th>$\sigma_{\text{mTh}} \times 10^{-2}$b</th>
<th>$\sigma_{\text{mU}} \times 10^{-2}$b</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.88 ± 0.12</td>
<td>15.02 ± 0.23</td>
<td>58.44 ± 0.36</td>
</tr>
<tr>
<td>10.11 ± 0.06</td>
<td>31.70 ± 0.56</td>
<td>100.14 ± 0.84</td>
</tr>
<tr>
<td>15.86 ± 0.12</td>
<td>44.99 ± 0.79</td>
<td>129.80 ± 1.09</td>
</tr>
</tbody>
</table>

Fission cross-section is one of the attributes in the process of normalization. The fission cross-sections presented in Table 7.5 are assumed un-correlated and hence, micro-correlation within the attribute fission cross-section are assigned zero.

7.3.7.2 Fission product yield

The graph of $^{232}\text{Th}$ and $^{238}\text{U}$ neutron induced fission yields for $E_n = 14$ are presented in Fig.7.2. It can be observed in the graph that, we can choose any of the fission products between the mass numbers $\sim 82 - 100$ or between mass numbers $\sim 132 - 150$ of $^{232}\text{Th}$, and mass numbers $\sim 90 - 104$ or between mass numbers $\sim 134 - 148$ of $^{238}\text{U}$, since fission yield is higher for those mass numbers.

In the present work, we selected the fission product $^{97}\text{Zr}$ in the neutron induced fission of $^{232}\text{Th}$, and the fission product $^{97}\text{Zr}$ in the neutron induced fission of $^{238}\text{U}$. The
cumulative fission yield of the fission product $^{97}Zr$ at $E_n = 14\, MeV$ are presented in Table 7.6, the cumulative fission yield data are obtained from [JAEA, 2014].

Table 7.6: Cumulative fission yield of the fission product $^{97}Zr$ at $E_n = 14\, MeV$ [JAEA, 2014, Shivashankar et al., 2015].

<table>
<thead>
<tr>
<th>Reaction</th>
<th>Fission product</th>
<th>Fission yield ($\times 10^{-2}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{232}Th(n,f)$</td>
<td>$^{97}Zr$</td>
<td>$3.40 \pm 0.14$</td>
</tr>
<tr>
<td>$^{238}U(n,f)$</td>
<td>$^{97}Zr$</td>
<td>$5.37 \pm 0.12$</td>
</tr>
</tbody>
</table>

Fission product is one of the attributes in the process of normalization. We assumed that the fission product yield is constant at three effective neutron energies considered in this work and is equal to fission yield of the fission product $^{97}Zr$ at $E_n = 14\, MeV$ as presented in Table 7.6. As a consequence of this assumption micro-correlations within the attribute of fission yield are assigned as follows [Shivashankar et al., 2015]:

1. The fission product yield of $^{97}Zr$ in the neutron induced fission of $^{232}Th$ at the three effective neutron energies are fully correlated.

2. The fission product yield of $^{97}Zr$ in the neutron induced fission of $^{238}U$ at the three effective neutron energies are fully correlated.

3. The fission product yield of $^{97}Zr$ in the neutron induced fission of $^{232}Th$ and $^{238}U$ at the three effective neutron energies are un-correlated.
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7.4 Covariance Analysis

First, we present covariance analysis of relative measurement at effective neutron energy $E_n = 5.88\, MeV$ in Section 7.4.1, and then generalize the analysis at the three effective energies considered in this work.

7.4.1 Covariance Analysis of Relative Measurement at Effective Neutron Energy $E_n = 5.88 \pm 0.12\, MeV$

1. Consider the expressions presented in Eq.(7.2) and Eq.(7.3), we presented the expressions in the simplified form to indicate that, however complex the expression may look, it is of the form $z = \frac{x}{y}$. Then it is straightforward to show that expectation value of $z$ is $\langle z \rangle = \frac{\langle x \rangle}{\langle y \rangle}$, and variance is $\text{var}(z) = \text{var}(x) + \text{var}(y) - 2\text{cov}(x, y)$. exactly same procedure is employed in the context of relative measurement.

2. Since we irradiated one unknown and two monitor samples at each of the effective neutron energies considered in this work. we get two ratios of reaction rates at each of the effective neutron energies, below we consider ratios of reaction rates at $E_n = 5.88\, MeV$:

(a) $r_{12} = \frac{Q_1}{Q_2}$ is the ratio of reaction rate of irradiated Ni-1 foil, relative to reaction rate of Th-1 foil.

(b) $r_{13} = \frac{Q_1}{Q_3}$ is the ratio of reaction rate of irradiated Ni-1 foil, relative to reaction rate of U-1 foil.

3. Obtain expectation values $\langle Q_i \rangle$, where $i=1,2,3$:

In Section 7.3, we discussed about data required for the ratio measurement. Based on the discussion presented in Section 7.3 obtain expectation values of reaction rates as presented in Table 7.7.

The column of Tag in Table 7.7 is used as an identifier, to identify corresponding reaction rate in column 2. For example, $\langle Q_1 \rangle = 700.32 \times 10^{-17}$ is the reaction
Table 7.7: Expectation value of reaction rate $Q$.

<table>
<thead>
<tr>
<th>Tag</th>
<th>$i$</th>
<th>$\langle Q_i \rangle \times 10^{-17}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ni-1</td>
<td>i=1</td>
<td>700.32</td>
</tr>
<tr>
<td>Th-1</td>
<td>i=2</td>
<td>008.25</td>
</tr>
<tr>
<td>U-1</td>
<td>i=3</td>
<td>042.31</td>
</tr>
</tbody>
</table>

rate of irradiated Ni-1 foil ($i=1$), $\langle Q_2 \rangle = 008.25 \times 10^{-17}$ is the reaction rate of irradiated Th-1 foil ($i=2$) and $\langle Q_3 \rangle = 042.31 \times 10^{-17}$ is the reaction rate of irradiated U-1 foil ($i=3$). Further, Ni-1, Th-1 and U-1 are the unknown and monitor samples (or foils) irradiated at effective neutron energy $E_n = 5.88 \pm 0.12 \text{ MeV}$. 

4. Assign attribute number to the different physical quantities considered in the ratio measurement.

Based on the discussion presented in Section 7.3, we selected attributes in the following order:

(a) The attribute $q=1$ for the counting measurement. Counts (registered in the HPGe detector) of the $\gamma$-rays emitted from the reaction product. $^{58}Co$ is the reaction product in the $^{58}Ni(n, p)$ reaction, and $^{97}Zr$ is the fission product in $^{232}Th(n, f)$ reaction and $^{97}Zr$ is the fission product in $^{232}U(n, f)$ reaction.

(b) The attribute $q=2$ for the number of sample atoms (or nuclei).

(c) The attribute $q=3$ for the decay constant of the reaction product.

(d) The attribute $q=4$ for the $\gamma$-abundance. The $\gamma$-abundance of $E_{\gamma_u} = 810.77 \times 10^{-3} \text{MeV}$ $\gamma$-rays emitted from the reaction product $^{58}Co$ and $\gamma$-abundance of $E_{\gamma_s} = 743.36 \times 10^{-3} \text{MeV}$ $\gamma$-rays emitted from the fission product $^{97}Zr$.

(e) The attribute $q=5$ for efficiency of the HPGe detector with respect to the (characteristic) $\gamma$-rays of energy $E_{\gamma_u} = 810.77 \times 10^{-3} \text{MeV}$ and $E_{\gamma_s} = 743.36 \times 10^{-3} \text{MeV}$.

5. Prepare table of partial uncertainties:

The last column in Table 7.8 represents total uncertainty $\Delta Q_i$. In Table 7.8 partial and total uncertainties are represented with scientific notation $10^{-21}$, where as
expectation values of reaction rates ($< Q_i >$) in Table 7.7 were represented with scientific notation $10^{-17}$, to preserve non zero decimal places in column 5.

6. Generate the covariance matrix $V_Q$ using $V_Q = \sum_{q=1}^{q=5} E_q S_q E_q$:

In $V_Q = \sum_{q=1}^{q=5} E_q S_q E_q$, $E_q$ are diagonal matrices of partial uncertainties generated using the table of partial uncertainties as presented in Table 7.8. The matrices of micro-correlations are generated based on the discussion presented in Section 7.3, and are as follows:

$$S_{q=1} = S_{q=2} = I,$$

$$S_{q=3} = S_{q=4} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

$$S_{q=5} = \begin{bmatrix} 1 & 0.99 & 0.99 \\ 0.99 & 1 & 1 \\ 0.99 & 1 & 1 \end{bmatrix},$$

and the covariance matrix $V_Q$ generated is:

$$V_Q = 10^{-37} \times \begin{bmatrix} 2584572.25 & 0006781.47 & 0034772.88 \\ 0006781.47 & 0000464.99 & 0000471.36 \\ 0034772.88 & 0000471.36 & 0005594.27 \end{bmatrix}. \quad (7.9)$$

7. Obtain expectation value and covariance for the ratio of reaction rates:
There are two ratios to consider: \( r_{12} = \frac{Q_1}{Q_2} \) is the ratio of reaction rates corresponding to \( ^{58}\text{Ni}(n,p)^{58}\text{Co} \) reaction and \( ^{232}\text{Th}(n,f) \) reaction, and \( r_{13} = \frac{Q_1}{Q_3} \) is the ratio of reaction rates corresponding to \( ^{58}\text{Ni}(n,p)^{58}\text{Co} \) reaction and \( ^{238}\text{U}(n,f) \) reaction.

By using the expectation values of reaction rates \( \langle Q_i \rangle \) presented in Table 7.7, obtain expectation values \( \langle r_{ij} \rangle \) as presented in Table 7.9:

<table>
<thead>
<tr>
<th>( r_{ij} )</th>
<th>( \langle r_{ij} \rangle )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_{12} )</td>
<td>84.86</td>
</tr>
<tr>
<td>( r_{13} )</td>
<td>16.54</td>
</tr>
</tbody>
</table>

In order to obtain covariance for the ratio of reaction rates, it is better and easy to work with relative covariance matrices. Transform the covariance matrix \( V_Q \) into relative covariance matrix using

\[
R_Q = D * V_Q * D, \tag{7.10}
\]

where \( V_Q \) is the absolute covariance matrix as presented in Eq.(7.9), \( D \) is the diagonal matrix generated using

\[
D = \text{diag}\left[ \frac{1}{\langle Q_1 \rangle}, \frac{1}{\langle Q_2 \rangle}, \frac{1}{\langle Q_3 \rangle} \right]. \tag{7.11}
\]

The relative covariance matrix \( R_Q \) is given by:

\[
R_Q = \begin{bmatrix}
R_{Q_{11}} & R_{Q_{12}} & R_{Q_{13}} \\
R_{Q_{21}} & R_{Q_{22}} & R_{Q_{23}} \\
R_{Q_{31}} & R_{Q_{32}} & R_{Q_{33}}
\end{bmatrix}
= \begin{bmatrix}
\text{cov}(Q_1,Q_1) & \text{cov}(Q_1,Q_2) & \text{cov}(Q_1,Q_3) \\
\text{cov}(Q_2,Q_1) & \text{cov}(Q_2,Q_2) & \text{cov}(Q_2,Q_3) \\
\text{cov}(Q_3,Q_1) & \text{cov}(Q_3,Q_2) & \text{cov}(Q_3,Q_3)
\end{bmatrix}, \tag{7.12}
\]

\[
R_Q = 10^{-4} \begin{bmatrix}
52.70 & 11.73 & 11.73 \\
11.73 & 68.28 & 13.50 \\
11.73 & 13.50 & 31.24
\end{bmatrix}. \tag{7.13}
\]
Chapter 7. *Relative measurement and covariance analysis.*

By using sandwich rule of error propagation, elements of relative covariance matrix for the ratio of reaction rates $R_r$ is given by the following set of equations:

\[
\begin{align*}
R_{r_{11}} &= R_{Q_{11}} + R_{Q_{22}} - 2R_{Q_{12}} \\
R_{r_{22}} &= R_{Q_{11}} + R_{Q_{33}} - 2R_{Q_{13}} \\
R_{r_{12}} &= R_{Q_{11}} - R_{Q_{12}} - R_{Q_{13}} + R_{Q_{23}}.
\end{align*}
\]  

(7.14)

The relative covariance matrix for the ratio of reaction rates $R_r$ is:

\[
R_r = 10^{-4} \begin{bmatrix} 97.51 & 42.72 \\ 42.72 & 60.47 \end{bmatrix}.
\]  

(7.15)

8. Obtain expectation value and covariance for the $^{58}Ni(n, p)^{58}Co$ reaction cross-section (Normalization):

With reference to Eq.(7.1) to Eq(7.3), the ratio of reaction rates $r_{ij}$ is normalized with respect to monitor cross-section $\sigma_j$ to yield unknown cross-section [$^{58}Ni(n, p)^{58}Co$ reaction cross-section] $\sigma_i$ in accordance with Eq.(7.16):

\[
\sigma_i = r_{ij} \sigma_j.
\]  

(7.16)

In our case, the monitor cross-section $\sigma_j$ is the cross-section for the formation of the fission product yield $^{97}Zr$ in $^{232}Th(n, f)$ reaction and the cross-section for the formation of the fission product yield $^{97}Zr$ in $^{238}U(n, f)$ reaction. Hence, Eq.(7.16) takes the form of Eq.(7.17):

\[
\begin{align*}
\sigma_i &= r_{ij} Y_{fj} \sigma_{fj}, \text{and} \\
\sigma_k &= r_{kl} Y_{fl} \sigma_{fl},
\end{align*}
\]  

(7.17)

where, the symbols $\sigma_i$ and $\sigma_k$ represents $^{58}Ni(n, p)^{58}Co$ reaction cross-section at effective neutron energy $E_n = 5.88 \pm 0.12$ MeV. The symbols $Y_{fj} \sigma_{fj}$ and $Y_{fl} \sigma_{fl}$ represents the cross-section for the formation of the fission product yield $^{97}Zr$ in $^{232}Th(n, f)$ and cross-section for the formation of the fission product yield $^{97}Zr$ in $^{238}U(n, f)$ reaction, respectively. The symbol $r_{ij}$ and $r_{kl}$ represents the
ratio of reaction rates corresponding to $^{58}\text{Ni}(n,p)^{58}\text{Co}$ reaction and $^{232}\text{Th}(n,f)$ reaction and ratio of reaction rates corresponding to $^{58}\text{Ni}(n,p)^{58}\text{Co}$ reaction and $^{238}\text{U}(n,f)$ reaction, respectively.

The fission cross-sections required for the calculation are presented in Table 7.5 of Section 7.3.7.1 and the fission product yields required for the calculation are presented in Table 7.6 of Section 7.3.7.2.

It is straightforward to obtain expectation values of the unknown cross-sections $\sigma_i$ and $\sigma_k$ using:

$$
\langle \sigma_i \rangle = \langle r_{ij} \rangle \langle Y_{fj} \rangle \langle \sigma_{fj} \rangle,
\langle \sigma_k \rangle = \langle r_{kl} \rangle \langle Y_{fl} \rangle \langle \sigma_{fl} \rangle.
$$

(7.18)

The expectation values $\langle r_{ij} \rangle$ and $\langle r_{kl} \rangle$ of the ratio of reaction rates presented in Table 7.9. The expectation values $\langle \sigma_{fj} \rangle$ and $\langle \sigma_{fl} \rangle$ of fission cross-sections are presented in Table 7.5 of Section 7.3.7.1 and the expectation values $\langle Y_{fj} \rangle$ and $\langle Y_{fl} \rangle$ of the fission product yields required for the calculation are presented in Table 7.6 of Section 7.3.7.2.

<table>
<thead>
<tr>
<th>$E_n$ MeV</th>
<th>Monitor reaction</th>
<th>cross-section ($\sigma$) (mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.88 ± 0.12</td>
<td>$^{232}\text{Th}(n,f)$</td>
<td>433.36</td>
</tr>
<tr>
<td>5.88 ± 0.12</td>
<td>$^{238}\text{U}(n,f)$</td>
<td>519.36</td>
</tr>
</tbody>
</table>

With reference to Eq.(7.16) and Eq.(7.17), it is straightforward to show that relative covariance matrix ($R_\sigma$) for the $^{58}\text{Ni}(n,p)^{58}\text{Co}$ reaction cross-section normalized to the monitor cross-sections is sum of the relative covariance matrices for the ratio of reaction rates ($R_r$), cross-section for the formation of the fission product $^{97}\text{Zr}$ in $^{232}\text{Th}(n,f)$ and in $^{238}\text{U}(n,f)$ reaction ($R_{Yf} + R_{\sigma f}$):

$$
R_\sigma = R_r + R_{Yf} + R_{\sigma f},
$$

(7.19)
where
\[
R_{\sigma f} = \begin{bmatrix}
\frac{\text{cov}(\sigma_{fj}, \sigma_{fl})}{\langle \sigma_f \rangle \langle \sigma_l \rangle} & \frac{\text{cov}(\sigma_{fj}, \sigma_{fl})}{\langle \sigma_f \rangle \langle \sigma_l \rangle} \\
\frac{\text{cov}(\sigma_{fj}, \sigma_{fl})}{\langle \sigma_f \rangle \langle \sigma_l \rangle} & \frac{\text{cov}(\sigma_{fj}, \sigma_{fl})}{\langle \sigma_f \rangle \langle \sigma_l \rangle}
\end{bmatrix}
= 10^{-4} \begin{bmatrix} 2.34 & 0 \\ 0 & 0.38 \end{bmatrix}
\] (7.20)

\[
R_{Y_f} = \begin{bmatrix}
\frac{\text{cov}(Y_{fj}, Y_{fl})}{\langle Y_f \rangle \langle Y_l \rangle} & \frac{\text{cov}(Y_{fj}, Y_{fl})}{\langle Y_f \rangle \langle Y_l \rangle} \\
\frac{\text{cov}(Y_{fj}, Y_{fl})}{\langle Y_f \rangle \langle Y_l \rangle} & \frac{\text{cov}(Y_{fj}, Y_{fl})}{\langle Y_f \rangle \langle Y_l \rangle}
\end{bmatrix}
= 10^{-4} \begin{bmatrix} 16.96 & 0 \\ 0 & 0.499 \end{bmatrix}
\] (7.21)

With reference to discussion presented in Section 7.3.7.1 and Section 7.3.7.2, the off-diagonal elements of \( R_{\sigma f} \) in Eq.(7.20) and off-diagonal elements of \( R_{Y_f} \) in Eq.(7.21), \( \rho_{\sigma_{fj}, \sigma_{fl}} \Delta \sigma_j \Delta \sigma_l \) and \( \rho_{Y_{fj}, Y_{fl}} \Delta Y_j \Delta Y_l \) were assigned zero by assigning \( \rho_{\sigma_{fj}, \sigma_{fl}} = 0 \) and \( \rho_{Y_{fj}, Y_{fl}} = 0 \), respectively.

By substituting \( R_r, R_{\sigma f} \) and \( R_{Y_f} \) presented in Eq.(7.15), Eq.(7.20) and Eq.(7.21), respectively in Eq.(7.19) results in \( R_{\sigma} \) presented in Eq.(7.22):

\[
R_{\sigma} = \begin{bmatrix}
\frac{\text{cov}(\sigma_{fj}, \sigma_{fl})}{\langle \sigma_f \rangle \langle \sigma_l \rangle} & \frac{\text{cov}(\sigma_{fj}, \sigma_{fl})}{\langle \sigma_f \rangle \langle \sigma_l \rangle} \\
\frac{\text{cov}(\sigma_{fj}, \sigma_{fl})}{\langle \sigma_f \rangle \langle \sigma_l \rangle} & \frac{\text{cov}(\sigma_{fj}, \sigma_{fl})}{\langle \sigma_f \rangle \langle \sigma_l \rangle}
\end{bmatrix}
= 10^{-4} \times \begin{bmatrix} 116.81 & 42.72 \\ 42.72 & 65.84 \end{bmatrix}
\] (7.22)

### Table 7.11: The $^{58}$Ni($n, p$)$^{58}$Co reaction cross-section normalized to the monitor cross-section at the effective neutron energy $E_n = 5.88 \pm 0.12$ MeV, with relative covariance matrix $R_{\sigma}$.

<table>
<thead>
<tr>
<th>$E_n$ (MeV)</th>
<th>Monitor reaction</th>
<th>cross-section ($\sigma$) (mb)</th>
<th>$R_{\sigma} \times 10^{-4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.88 ± 0.12</td>
<td>$^{212}$Th($n, f$)</td>
<td>433 ± 046</td>
<td>116.81 42.72</td>
</tr>
<tr>
<td>5.88 ± 0.12</td>
<td>$^{238}$U($n, f$)</td>
<td>519 ± 042</td>
<td>42.72 65.84</td>
</tr>
</tbody>
</table>

The expectation values presented in Table 7.10 and relative covariance matrix presented in Eq.(7.22) constitutes the result of the $^{58}$Ni($n, p$)$^{58}$Co reaction cross-section at $E_n = 5.88 \pm 0.12$ MeV, and the result can be summarized as presented in Table 7.11. The uncertainty in the $^{58}$Ni($n, p$)$^{58}$Co reaction cross-section presented in Table 7.11 were obtained from the diagonal elements (relative variances)
9. Weighted average of equivalent data:

Note that both \( \sigma_i \) and \( \sigma_k \) represent the \( ^{58}Ni(n, p)^{58}Co \) reaction cross-section at \( E_n = 5.88 \pm 0.12 \) MeV, the true value of which is otherwise an unknown constant. The only difference is that, \( \sigma_i \) is normalized with respect to the cross-section for the formation of the fission product \( ^{97}Zr \) in \( ^{232}Th(n, f) \) reaction and \( \sigma_k \) is normalized with respect to the cross-section for the formation of the fission product \( ^{97}Zr \) in \( ^{238}U(n, f) \) reaction.

Further, the separation between the expectation values

\[
|\langle \sigma_i \rangle - \langle \sigma_k \rangle| = |433 - 519| = 86 \text{ mb},
\]

is smaller than the sum of corresponding uncertainties

\[
\Delta \sigma_i + \Delta \sigma_k = 46 + 42 = 88 \text{ mb},
\]

indicating consistent data (error bars overlap).

If \(|\langle \sigma_i \rangle - \langle \sigma_k \rangle| >\Delta \sigma_i + \Delta \sigma_k\), then the data are in-consistent\(^2\) [Froehner, 2000].

Since \( \sigma_i \) and \( \sigma_k \) are equivalent (Both represents the \( ^{58}Ni(n, p)^{58}Co \) reaction cross-section at \( E_n = 5.88 \pm 0.12 \) MeV) and consistent (error bars overlap\(^3\)), we can extract best value \( \langle \sigma_{ni} \rangle \) representing the \( ^{58}Ni(n, p)^{58}Co \) reaction cross-section at

\(^2\)In-consistencies are caused by un-recognized or mal-corrected experimental effects such as back-grounds, dead time of the counting electronics, instrumental resolution, sample impurities, calibration errors, etc. A popular quick fix consists in increasing all input errors by a common factor until chi-square has the expected value (and all error bars overlap). This, however, does not change the relative weights, hence there is the same penalty for overoptimistic as for conservative uncertainty assignments [Froehner, 2000].

\(^3\)The uncertainty assigned to physical quantity is commonly referred to as “error bars” in literature. The nouns ‘uncertainty’ and ‘error’ are used synonyms in literature, even though they are related to different concepts [D’Agostini, 2003].
$E_n = 5.88 \pm 0.12 \text{ MeV}$ by weighted averaging of equivalent data using least-squares approximation [Peelle, 1982, Smith, 1987] as presented in Section 1.2.4 of chapter 1:

Define

$$X_\alpha \equiv [X_{\alpha i} = \langle \sigma_i \rangle \ X_{\alpha j} = \langle \sigma_k \rangle]^T \approx [\langle \sigma_{n_1} \rangle \ \langle \sigma_{n_1} \rangle]^T$$

$$\approx [1 \ 1]^T \langle \sigma_{n_1} \rangle$$

$$X_\alpha \approx A_\alpha \langle \sigma_{n_1} \rangle,$$

(7.23)

where $X_{\alpha i} = \langle \sigma_i \rangle$ and $X_{\alpha j} = \langle \sigma_k \rangle$ are the elements of the vector $X_\alpha$, superscript $^T$ represents vector transpose and $A_\alpha = [1 \ 1]^T$ is the design matrix.

The least-square approach to extract the best value $\langle \sigma_{n_1} \rangle$ is to minimize the $\chi_\alpha^2$ given by:

$$\chi_\alpha^2 = [X_\alpha - A_\alpha \langle \sigma_{n_1} \rangle]W[X_\alpha - A_\alpha \langle \sigma_{n_1} \rangle]^T,$$

(7.24)

with respect to $\langle \sigma_{n_1} \rangle$,

$$\frac{\partial \chi_\alpha^2}{\partial \langle \sigma_{n_1} \rangle} = 0,$$

(7.25)

where $W = V_\alpha^{-1}$ is the weight matrix, superscript $^{-1}$ represents matrix inversion. The absolute covariance matrix $V_\alpha$ corresponding to normalized $^{58}\text{Ni}(n,p)^{58}\text{Co}$ reaction cross-section at $E_n = 5.88 \pm 0.12 \text{ MeV}$ is obtained from relative covariance matrix $R_\sigma$ presented in Eq.(7.22), using the transformation

$$V_\alpha = M R_\sigma M^T,$$

(7.26)

where

$$M = \begin{bmatrix} <\sigma_i> & 0 \\ 0 & <\sigma_k> \end{bmatrix}.$$

(7.27)

The minimization condition presented in Eq.(7.25) yields the following set of solutions [Smith, 1987] for the expectation value $\langle \sigma_{n_1} \rangle$, and covariance matrix $V_{n_1}$.
\begin{equation}
\langle \sigma_n \rangle = B^T X, \tag{7.28}
\end{equation}

and

\begin{equation}
V_n = B^T V B, \tag{7.29}
\end{equation}

where

\begin{align}
B &= [C A^{-1} V^{-1}]^T, \\
C &= [A^{-1} V^{-1} A]. \tag{7.30}
\end{align}

After simplification we obtained following result:

\begin{align}
\sigma_n &= \langle \sigma_n \rangle \pm \Delta \sigma_n \\
\sigma_n &= 485 \pm 038 \text{ mb or} \\
\sigma_n &= 485(38) \text{ mb or} \\
\sigma_n &= 485(7.8\%) \text{ mb}, \tag{7.31}
\end{align}

where \( \Delta \sigma_n = \sqrt{V_n} \), with

\begin{equation}
\chi^2 = 3.61. \tag{7.32}
\end{equation}

One can easily verify the result presented in Eq.(7.31) using the expressions Eq.(1.41) and Eq.(1.42) presented in Section 1.2.4 of chapter 1.

Note that Eq.(7.32) indicates slightly higher \( \chi^2 \) value than expected (\( \chi^2 = \text{number of data points-number of parameters}=1 \)). Higher \( \chi^2 \) value indicates that uncertainties assigned to \( \sigma_i \) and \( \sigma_k \) are underestimated.

10. Chi-square scaling :

An ad-hock method to obtain \( \chi^2 = 1 \) is to adjust the covariance matrix \( V \) by a factor of 3.61, which is nothing but scaling the uncertainties assigned to \( \sigma_i \) and \( \sigma_k \) by a factor \( \sqrt{3.61} \) [Froehner, 2000]:

\begin{align}
\Delta \sigma_{is} &= \sqrt{3.61} \times \Delta \sigma_i = 87 \text{ mb}, \\
\Delta \sigma_{ks} &= \sqrt{3.61} \times \Delta \sigma_k = 80 \text{ mb}. \tag{7.33}
\end{align}
Generate adjusted covariance matrix \( V_{\alpha s} \) based on the scaled uncertainties presented in Eq. (7.33):

\[
V_{\alpha s} = \begin{bmatrix}
\Delta \sigma_{is}^2 & \rho \times \Delta \sigma_{is} \times \Delta \sigma_{ks} \\
\rho \times \Delta \sigma_{is} \times \Delta \sigma_{ks} & \Delta \sigma_{ks}^2
\end{bmatrix} = 10^{-5} \times \begin{bmatrix}
78.97 & 34.62 \\
34.62 & 63.94
\end{bmatrix},
\]

(7.34)

where correlation coefficient \( \rho = 0.48 \) is calculated from \( V_{\alpha} \) presented in Eq. (7.26).

The result of chi-square minimization is:

\[
\sigma_{n_{1s}} = \langle \sigma_{n_{1s}} \rangle \pm \Delta \sigma_{n_{1s}}
\]

\[
\sigma_{n_{1s}} = 485 \pm 0.72 \text{ mb or} \quad \sigma_{n_{1s}} = 485(72) \text{ mb or}
\]

\[
\sigma_{n_{1s}} = 485(14.84\%) \text{ mb},
\]

with

\[
\chi_{\alpha s}^2 = 1.00.
\]

(7.35) (7.36)

### 7.4.2 Covariance Analysis of Relative Measurement at Effective Neutron Energies \( E_n = 5.88 \pm 0.12, 10.11 \pm 0.06 \text{ and } 15.86 \pm 0.12 \text{ MeV} \)

In this section we extend the discussion of Section 7.4.1 to the covariance analysis of relative measurement at effective neutron energies \( E_n = 5.88 \pm 0.12, 10.11 \pm 0.06 \text{ and } 15.86 \pm 0.12 \text{ MeV} \).

Following are the ratios of reaction rates we consider for the covariance analysis:

1. \( r_{12} = \frac{Q_1}{Q_2} \) and \( r_{13} = \frac{Q_1}{Q_3} \) are the ratio of reaction rate of irradiated Ni-1 foil, relative to reaction rate of Th-1 foil and U-1 foil, respectively at effective energy \( E_n = 5.88 \pm 0.12 \).

2. \( r_{45} = \frac{Q_4}{Q_5} \) and \( r_{46} = \frac{Q_4}{Q_6} \) are the ratio of reaction rate of irradiated Ni-2 foil, relative to reaction rate of Th-2 foil and U-2 foil, respectively at effective energy \( E_n = 10.11 \pm 0.06 \).
3. \( r_{78} = \frac{Q_7}{Q_8} \) and \( r_{79} = \frac{Q_7}{Q_9} \) are the ratio of reaction rate of irradiated Ni-3 foil, relative to reaction rate of Th-3 foil and U-3 foil, respectively at effective energy \( E_n = 15.86 \pm 0.12 \).

Below we present covariance analysis with respect to the discussion presented in Section 7.4.1:

The expectation values of reaction rates are presented in Table 7.12. The nickel, thorium and uranium foils with tag Ni-1, Th-1 and U-1, respectively, are irradiated at effective neutron energy \( E_n = 5.88 \pm 0.12 \). Correspondingly, \( \langle Q_1 \rangle \), \( \langle Q_2 \rangle \) and \( \langle Q_3 \rangle \) are expectation values of reaction rates of the nickel, thorium and uranium foils with tag Ni-1, Th-1 and U-1 at effective neutron energy \( E_n = 5.88 \pm 0.12 \). Similarly, \( \langle Q_4 \rangle \), \( \langle Q_5 \rangle \) and \( \langle Q_6 \rangle \) are expectation values of reaction rates of the nickel, thorium and uranium foils with tag Ni-2, Th-2 and U-2 at effective neutron energy \( E_n = 10.11 \pm 0.06 \) and \( \langle Q_7 \rangle \), \( \langle Q_8 \rangle \) and \( \langle Q_9 \rangle \) are expectation values of reaction rates of the nickel, thorium and uranium foils with tag Ni-3, Th-3 and U-3 at effective neutron energy \( E_n = 15.86 \pm 0.12 \).

<table>
<thead>
<tr>
<th>Tag</th>
<th>( i )</th>
<th>( \langle Q_i \rangle \times 10^{-17} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ni-1</td>
<td>i=1</td>
<td>700.32</td>
</tr>
<tr>
<td>Th-1</td>
<td>i=2</td>
<td>008.25</td>
</tr>
<tr>
<td>U-1</td>
<td>i=3</td>
<td>042.31</td>
</tr>
<tr>
<td>Ni-2</td>
<td>i=4</td>
<td>792.71</td>
</tr>
<tr>
<td>Th-2</td>
<td>i=5</td>
<td>014.38</td>
</tr>
<tr>
<td>U-2</td>
<td>i=6</td>
<td>062.83</td>
</tr>
<tr>
<td>Ni-3</td>
<td>i=7</td>
<td>162.82</td>
</tr>
<tr>
<td>Th-3</td>
<td>i=8</td>
<td>013.65</td>
</tr>
<tr>
<td>U-3</td>
<td>i=9</td>
<td>057.12</td>
</tr>
</tbody>
</table>

Prepare table of partial uncertainties as presented in Table 7.13. The attributes \( q = 1 \), \( q = 2 \), \( q = 3 \), \( q = 4 \) and \( q = 5 \) corresponds to \( \gamma \)-ray counts, number of sample atoms (or nuclei), decay constant of the reaction product, \( \gamma \)-abundance and efficiency of the HPGe detector, respectively. Additional details are presented in Section 7.4.1.

The last column in Table 7.13 represents total uncertainty \( \Delta Q_i \). In Table 7.13 partial and total uncertainties are represented with scientific notation \( 10^{-21} \), where as expectation
values of reaction rates \( (Q_i) \) in Table 7.12 were represented with scientific notation \( 10^{-17} \), to preserve non zero decimal places in column 5.

Generate the covariance matrix \( V_Q \) using \( V_Q = \sum_{q=1}^{5} E_q S_q E_q \), where \( E_q \) are diagonal matrices of partial uncertainties generated using the table of partial uncertainties as presented in Table 7.13. The matrices of micro-correlations are generated based on the discussion presented in Section 7.3, and are presented in Eq.(7.37)-Eq.(7.39):

\[
S_{q=1} = S_{q=2} = I_{9 \times 9},
\]

\[
S_{q=3} = S_{q=4} = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 
\end{bmatrix},
\]
The covariance matrix of reaction rates $V_Q$ generated using $V_Q = \sum_{q=1}^{5} E_q S_q E_q$ is presented in Eq. (7.40) [Shivashankar et al., 2015]:

$V_Q = 10^{-37}$

Transform the (absolute) covariance matrix of reaction rates $V_Q$ into relative covariance matrix of reaction rates $R_Q$ as presented in Eq. 7.41 (also refer Eq. (7.10) of Section 7.4.1) [Shivashankar et al., 2015]:

$R_Q = 10^{-4}$
By using the expectation values of reaction rates $\langle Q_i \rangle$ presented in Table 7.12, obtain expectation values of ratio of reaction rates $\langle r_{ij} \rangle = \frac{\langle Q_i \rangle}{\langle Q_j \rangle}$ as presented in Table 7.14.

**Table 7.14: Expectation value of ratio of reaction rates $r_{ij}$.**

<table>
<thead>
<tr>
<th>$r_{ij}$</th>
<th>$\langle r_{ij} \rangle$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_{12}$</td>
<td>84.86</td>
</tr>
<tr>
<td>$r_{13}$</td>
<td>16.54</td>
</tr>
<tr>
<td>$r_{45}$</td>
<td>55.12</td>
</tr>
<tr>
<td>$r_{46}$</td>
<td>12.62</td>
</tr>
<tr>
<td>$r_{78}$</td>
<td>11.92</td>
</tr>
<tr>
<td>$r_{79}$</td>
<td>02.85</td>
</tr>
</tbody>
</table>

Obtain relative covariance matrix of ratio of reaction rates $R_r$ from the elements of $R_Q$ [Mannhart, 2011] presented in Eq.(7.41) using the set of equations Eq.(7.42) to Eq.(7.44).

Diagonal elements of $R_r$ are obtained using set of equations presented in Eq(7.42):

\[
R_{r11} = R_Q(1,1) + R_Q(2,2) - 2 \times R_Q(1,2),
\]
\[
R_{r22} = R_Q(1,1) + R_Q(3,3) - 2 \times R_Q(1,3),
\]
\[
R_{r33} = R_Q(4,4) + R_Q(5,5) - 2 \times R_Q(4,5),
\]
\[
R_{r44} = R_Q(4,4) + R_Q(6,6) - 2 \times R_Q(4,6),
\]
\[
R_{r55} = R_Q(7,7) + R_Q(8,8) - 2 \times R_Q(7,8),
\]
\[
R_{r66} = R_Q(7,7) + R_Q(9,9) - 2 \times R_Q(7,9).
\]
Off-diagonal elements \((j > i)\) of \(R_r\) are obtained using set of equations presented in Eq(7.43):

\[
\begin{align*}
R_{r12} &= R_Q(1, 1) + R_Q(2, 3) - R_Q(1, 2) - R_Q(1, 3), \\
R_{r13} &= R_Q(1, 4) + R_Q(2, 5) - R_Q(1, 5) - R_Q(2, 4), \\
R_{r14} &= R_Q(1, 4) + R_Q(2, 6) - R_Q(1, 6) - R_Q(2, 4), \\
R_{r15} &= R_Q(1, 7) + R_Q(2, 8) - R_Q(1, 8) - R_Q(2, 7), \\
R_{r16} &= R_Q(1, 7) + R_Q(2, 9) - R_Q(1, 9) - R_Q(2, 7), \\
R_{r23} &= R_Q(1, 4) + R_Q(3, 5) - R_Q(1, 5) - R_Q(3, 4), \\
R_{r24} &= R_Q(1, 4) + R_Q(3, 6) - R_Q(1, 6) - R_Q(3, 4), \\
R_{r25} &= R_Q(1, 7) + R_Q(3, 8) - R_Q(1, 8) - R_Q(3, 7), \\
R_{r26} &= R_Q(1, 7) + R_Q(3, 9) - R_Q(1, 9) - R_Q(3, 7), \\
R_{r34} &= R_Q(4, 4) + R_Q(5, 6) - R_Q(4, 5) - R_Q(4, 6), \\
R_{r35} &= R_Q(4, 7) + R_Q(5, 8) - R_Q(4, 8) - R_Q(5, 7), \\
R_{r36} &= R_Q(4, 7) + R_Q(5, 9) - R_Q(4, 9) - R_Q(5, 7), \\
R_{r45} &= R_Q(4, 7) + R_Q(6, 8) - R_Q(4, 8) - R_Q(6, 7), \\
R_{r46} &= R_Q(4, 7) + R_Q(6, 9) - R_Q(4, 9) - R_Q(6, 7), \\
R_{r56} &= R_Q(7, 7) + R_Q(8, 9) - R_Q(7, 9) - R_Q(8, 7), \\
\end{align*}
\]

and off-diagonal elements \((i > j)\) are obtained using set of equations presented in Eq(7.44):

\[
\begin{align*}
R_{r21} &= R_{r12}; R_{r31} = R_{r13}; R_{r41} = R_{r14}; R_{r51} = R_{r15}; R_{r61} = R_{r16}, \\
R_{r32} &= R_{r23}; R_{r42} = R_{r24}; R_{r52} = R_{r25}; R_{r62} = R_{r26}, \\
R_{r43} &= R_{r34}; R_{r53} = R_{r35}; R_{r63} = R_{r36}, \\
R_{r54} &= R_{r45}; R_{r64} = R_{r46}, \\
R_{r65} &= R_{r56}. \\
\end{align*}
\]

The relative covariance matrix for the ratio of reaction rates \(R_r\) is presented in Eq(7.45) [Shivashankar et al., 2015]

\[
R_r = 10^{-4} \begin{bmatrix} 97.51 & 42.72 & 0.47 & 0.47 & 0.47 & 0.47 \\ 42.72 & 60.47 & 0.47 & 0.47 & 0.47 & 0.47 \\ 0.47 & 0.47 & 66.63 & 28.03 & 0.47 & 0.47 \\ 0.47 & 0.47 & 28.03 & 33.44 & 0.47 & 0.47 \\ 0.47 & 0.47 & 0.47 & 0.47 & 75.34 & 48.91 \\ 0.47 & 0.47 & 0.47 & 0.47 & 48.91 & 54.57 \end{bmatrix}. 
\]

Next step is to obtain expectation value and covariance for the \(^{58}Ni(n, p){^{58}}Co\) reaction
cross-section (Normalization). The fission cross-sections required for the calculation are presented in Table 7.5 of Section 7.3.7.1 and the fission product yields required for the calculation are presented in Table 7.6 of Section 7.3.7.2.

The expectation value of $^{58}\text{Ni}(n,p)^{58}\text{Co}$ reaction cross-section ($\langle \sigma \rangle$) normalized to the monitor cross-section at the effective neutron energy $E_n = 5.88 \pm 0.12, 10.11 \pm 0.06$ and $15.86 \pm 0.12$ MeV are presented in Table 7.15.

**Table 7.15**: The expectation value of $^{58}\text{Ni}(n,p)^{58}\text{Co}$ reaction cross-section ($\langle \sigma \rangle$) normalized to the monitor cross-section at the effective neutron energy $E_n = 5.88 \pm 0.12, 10.11 \pm 0.06$ and $15.86 \pm 0.12$ MeV.

<table>
<thead>
<tr>
<th>$E_n$ MeV</th>
<th>Monitor reaction</th>
<th>cross-section ($\langle \sigma \rangle$) (mb)</th>
</tr>
</thead>
<tbody>
<tr>
<td>05.88 ± 0.12</td>
<td>$^{232}\text{Th}(n,f)$</td>
<td>433.36</td>
</tr>
<tr>
<td>05.88 ± 0.12</td>
<td>$^{238}\text{U}(n,f)$</td>
<td>519.36</td>
</tr>
<tr>
<td>10.11 ± 0.06</td>
<td>$^{232}\text{Th}(n,f)$</td>
<td>594.12</td>
</tr>
<tr>
<td>10.11 ± 0.06</td>
<td>$^{238}\text{U}(n,f)$</td>
<td>678.44</td>
</tr>
<tr>
<td>15.86 ± 0.12</td>
<td>$^{232}\text{Th}(n,f)$</td>
<td>182.42</td>
</tr>
<tr>
<td>15.86 ± 0.12</td>
<td>$^{238}\text{U}(n,f)$</td>
<td>198.67</td>
</tr>
</tbody>
</table>

The relative covariance matrix ($R_\sigma$) for the $^{58}\text{Ni}(n,p)^{58}\text{Co}$ reaction cross-section normalized to the monitor cross-sections is sum of the relative covariance matrices for the ratio of reaction rates ($R_r$), cross-section for the formation of the fission product $^{97}\text{Zr}$ in $^{232}\text{Th}(n,f)$ and in $^{238}\text{U}(n,f)$ reaction ($R_{Y_f} + R_{\sigma_f}$):

$$ R_\sigma = R_r + R_{Y_f} + R_{\sigma_f}, $$

where, relative covariance matrix for the fission cross-sections ($R_{\sigma_f}$) and fission yields ($R_{Y_f}$) are presented in Eq.(7.46) and Eq.(7.47), respectively:

$$ R_{\sigma_f} = 10^{-4} \begin{bmatrix} 2.34 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.38 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.12 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.70 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.08 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.71 \end{bmatrix}, \quad (7.46) $$
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The matrix addition of $R_y R_f$ and $R_Y$ presented in Eq. (7.45), Eq. (7.46) and Eq. (7.47), respectively, results in $R_\sigma$ presented in Eq. (7.48):

$$R_\sigma = 10^{-4} \begin{bmatrix}
116.81 & 42.72 & 17.42 & 0.46 & 17.42 & 0.46 \\
42.72 & 65.84 & 00.46 & 05.46 & 00.46 & 05.46 \\
17.42 & 00.46 & 86.71 & 28.03 & 17.42 & 0.46 \\
0.46 & 05.46 & 28.03 & 39.14 & 00.46 & 05.46 \\
17.42 & 00.46 & 17.42 & 00.46 & 95.38 & 48.91 \\
0.46 & 05.46 & 00.46 & 05.46 & 48.91 & 60.27
\end{bmatrix}. \tag{7.48}$$

The expectation values presented in Table 7.15 and relative covariance matrix presented in Eq. (7.48) constitutes the result of the $^{58}\text{Ni}(n, p)^{58}\text{Co}$ reaction cross-section at $E_n = 5.88 \pm 0.12, 10.11 \pm 0.06$ and $15.86 \pm 0.12$ MeV, respectively, and the result can be summarized as presented in Table 7.16.

**Table 7.16:** The $^{58}\text{Ni}(n, p)^{58}\text{Co}$ reaction cross-section normalized to the monitor cross-section at $E_n = 5.88 \pm 0.12, 10.11 \pm 0.06$ and $15.86 \pm 0.12$ MeV, with relative covariance matrix $R_\sigma$.

<table>
<thead>
<tr>
<th>$E_n$, MeV</th>
<th>Monitor reaction</th>
<th>cross-section $\sigma$ (mb)</th>
<th>$R_\sigma \times 10^{-3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.88 ± 0.12</td>
<td>$^{232}\text{Th}(n, f)$</td>
<td>432 ± 046</td>
<td>116.81 42.72 17.42 0.46 17.42 0.46</td>
</tr>
<tr>
<td>05.88 ± 0.12</td>
<td>$^{238}\text{U}(n, f)$</td>
<td>519 ± 042</td>
<td>42.72 65.84 00.46 05.46 00.46 05.46</td>
</tr>
<tr>
<td>10.11 ± 0.06</td>
<td>$^{232}\text{Th}(n, f)$</td>
<td>594 ± 055</td>
<td>17.42 00.46 86.71 28.03 17.42 0.46</td>
</tr>
<tr>
<td>10.11 ± 0.06</td>
<td>$^{238}\text{U}(n, f)$</td>
<td>678 ± 042</td>
<td>00.46 05.46 28.03 39.14 00.46 05.46</td>
</tr>
<tr>
<td>15.86 ± 0.12</td>
<td>$^{232}\text{Th}(n, f)$</td>
<td>182 ± 018</td>
<td>17.42 00.46 17.42 00.46 95.38 48.91</td>
</tr>
<tr>
<td>15.86 ± 0.12</td>
<td>$^{238}\text{U}(n, f)$</td>
<td>198 ± 019</td>
<td>00.46 05.46 00.46 05.46 48.91 60.27</td>
</tr>
</tbody>
</table>

The absolute covariance matrix $V_\sigma$ obtained from the relative covariance matrix $R_\sigma$ is presented in Eq. (7.49), also see Eq. (7.26) of Section 7.4.1 for additional details:
\[ V_\sigma = 10^{-5} \begin{bmatrix}
0.0219.36 & 0.0096.17 & 0.0044.86 & 0.0001.37 & 0.0013.77 & 0.0000.40 \\
0.0096.17 & 0.0017.61 & 0.0001.44 & 0.0019.24 & 0.0000.44 & 0.0005.63 \\
0.0044.86 & 0.0001.44 & 0.0036.06 & 0.0012.99 & 0.0018.88 & 0.0000.55 \\
0.0001.37 & 0.0019.24 & 0.0011.29 & 0.0018.01 & 0.0000.58 & 0.0007.36 \\
0.0013.77 & 0.0000.44 & 0.0018.88 & 0.0000.58 & 0.0031.74 & 0.0017.72 \\
0.0000.40 & 0.0005.63 & 0.0000.55 & 0.0007.36 & 0.0017.72 & 0.0023.78 \\
\end{bmatrix}. \] (7.49)

Next step is to extract best set of values representing $^{58}\text{Ni}(n,p)^{58}\text{Co}$ reaction cross-section at $E_n = 5.88 \pm 0.12, 10.11 \pm 0.06$ and $15.86 \pm 0.12$ MeV, using the set of equivalent data presented in Table 7.16.

Before proceeding to the discussion of extraction of best values by least square approximation, one should check consistency of equivalent data presented in Table 7.16. It can be observed in Table 7.17, that the equivalent data presented in Table 7.16 are consistent.

**Table 7.17:** Consistency of equivalent data presented in Table 7.16.

| $E_n$ MeV | $|\langle \sigma_i \rangle - \langle \sigma_j \rangle| \text{ mb}$ | $\Delta \sigma_i + \Delta \sigma_j \text{ mb}$ |
|---------|--------------------|---------------------|
| $5.88 \pm 0.12$ | 86 | 88 |
| $10.11 \pm 0.06$ | 84 | 98 |
| $15.86 \pm 0.12$ | 16 | 33 |

Let $X_\alpha$, $X_\beta$ and $X_\gamma$ be the vectors, representing set of equivalent data presented in Table 7.16 at $E_n = 5.88 \pm 0.12, 10.11 \pm 0.06$ and $15.86 \pm 0.12$ MeV, respectively.

\[
X_\alpha = [433 \ 519]^T \text{ mb},
\]
\[
X_\beta = [594 \ 678]^T \text{ mb}, \quad (7.50)
\]
\[
X_\gamma = [182 \ 198]^T \text{ mb}.
\]
Let $\boldsymbol{\sigma}_n = [\sigma_{n_1} \sigma_{n_2} \sigma_{n_3}]^T$ be the vector representing set of best values of $^{58}\text{Ni}(n,p)^{58}\text{Co}$ reaction cross-section at $E_n = 5.88 \pm 0.12, 10.11 \pm 0.06$ and $15.86 \pm 0.12 \text{ MeV}$, respectively.

The vector of equivalent data $X = [X_\alpha \ X_\beta \ X_\gamma]^T$ and the vector of best values $\boldsymbol{\sigma}_n = [\sigma_{n_1} \sigma_{n_2} \sigma_{n_3}]^T$ are related to each other by

$$X \approx A\boldsymbol{\sigma}_n,$$

where the design matrix $A$ is given by

$$A = \begin{bmatrix} A_\alpha & 0 & 0 \\ 0 & A_\beta & 0 \\ 0 & 0 & A_\gamma \end{bmatrix},$$

where, $A_\alpha = A_\beta = A_\gamma = [1 \ 1]^T$, symbol 0 represents a $2 \times 1$ column vector of zeroes.

Accordingly absolute covariance matrix of equivalent data $V_\sigma$ presented in Eq. (7.49) can be partitioned as

$$V_\sigma = \begin{bmatrix} V_\alpha & V_{\alpha\beta} & V_{\alpha\gamma} \\ V_{\beta\alpha} & V_\beta & V_{\beta\gamma} \\ V_{\gamma\alpha} & V_{\gamma\beta} & V_\gamma \end{bmatrix},$$

where,

$$V_\alpha = 10^{-5} \begin{bmatrix} 0.021936 & 0.009617 \\ 0.009617 & 0.017761 \end{bmatrix},$$

$$V_\beta = 10^{-5} \begin{bmatrix} 0.030606 & 0.011299 \\ 0.011299 & 0.018018 \end{bmatrix},$$

$$V_\gamma = 10^{-5} \begin{bmatrix} 0.003174 & 0.001772 \\ 0.001772 & 0.002378 \end{bmatrix},$$

$$V_{\alpha\beta} = 10^{-5} \begin{bmatrix} 0.004486 & 0.000137 \\ 0.000144 & 0.001924 \end{bmatrix}; \ V_{\beta\alpha} = V_{\alpha\beta}^T,$$
Chapter 7. Relative measurement and covariance analysis.

\[ V_{\alpha\gamma} = 10^{-5} \begin{bmatrix} 0.001377 & 0.000040 \\ 0.000044 & 0.000563 \end{bmatrix} ; V_{\gamma\alpha} = V_{\alpha\gamma}^T, \]  
\[ V_{\beta\gamma} = 10^{-5} \begin{bmatrix} 0.001888 & 0.000055 \\ 0.000058 & 0.000736 \end{bmatrix} ; V_{\gamma\beta} = V_{\beta\gamma}^T. \]  
(7.58)
(7.59)

The least-square approach to extract set of best values \( \sigma_n \) is to minimize the deviation presented in Eq.(7.51) in \( \chi^2 \) presented in Eq.(7.62):

\[ \chi^2 = [X - A\sigma_n]^T * V^{-1} * [X - A\sigma_n], \]  
(7.60)

with respect to the partition of matrices presented in Eq.(7.50)-Eq.(7.59), \( \chi^2 \) in Eq.(7.62) is partitioned accordingly, as presented in Eq.(7.61):

\[ \chi^2 = [\chi^2_{\alpha} \chi^2_{\beta} \chi^2_{\gamma}]^T, \]  
(7.61)

where

\[ \chi^2_{\alpha} = [X_{\alpha} - A_{\alpha}\langle \sigma_{n1} \rangle]^T * V_{\alpha}^{-1} * [X_{\alpha} - A_{\alpha}\langle \sigma_{n1} \rangle], \]
\[ \chi^2_{\beta} = [X_{\beta} - A_{\beta}\langle \sigma_{n2} \rangle]^T * V_{\beta}^{-1} * [X_{\beta} - A_{\beta}\langle \sigma_{n2} \rangle], \text{ and} \]
\[ \chi^2_{\gamma} = [X_{\gamma} - A_{\gamma}\langle \sigma_{n3} \rangle]^T * V_{\gamma}^{-1} * [X_{\gamma} - A_{\gamma}\langle \sigma_{n3} \rangle]. \]  
(7.62)

The chi-square minimization condition can be presented by following set of equations:

\[ \frac{\partial \chi^2_{\alpha}}{\partial \langle \sigma_{n1} \rangle} = 0, \quad \frac{\partial \chi^2_{\beta}}{\partial \langle \sigma_{n2} \rangle} = 0, \quad \text{and} \quad \frac{\partial \chi^2_{\gamma}}{\partial \langle \sigma_{n3} \rangle} = 0. \]  
(7.63)

The solution of the chi-square minimization condition can be simplified by defining following set of equations:

\[ C_{\alpha} = [A_{\alpha}^T V_{\alpha}^{-1} A_{\alpha}]^{-1}, \]
\[ C_{\beta} = [A_{\beta}^T V_{\beta}^{-1} A_{\beta}]^{-1}, \text{ and} \]
\[ C_{\gamma} = [A_{\gamma}^T V_{\gamma}^{-1} A_{\gamma}]^{-1}, \]  
(7.64)
and,

\[ B_\alpha = [C_\alpha A_\alpha^T V_\alpha^{-1}]^T, \]
\[ B_\beta = [C_\beta A_\beta^T V_\beta^{-1}]^T, \]
\[ B_\gamma = [C_\gamma A_\gamma^T V_\gamma^{-1}]^T. \] (7.65)

The solution of chi-square minimization, representing best values of \(^{58}\text{Ni}(n,p)^{58}\text{Co}\) reaction cross-section at \(E_n = 5.88 \pm 0.12, 10.11 \pm 0.06\) and \(15.86 \pm 0.12\) MeV, respectively, is presented in the following set of equations:

\[ \langle \sigma_{n_1} \rangle = B_{n_1}^T X_\alpha, \]
\[ \langle \sigma_{n_2} \rangle = B_{n_2}^T X_\beta, \]
\[ \langle \sigma_{n_3} \rangle = B_{n_3}^T X_\gamma. \] (7.66)

The elements of covariance matrix \(V_n\) corresponding to the best values of \(^{58}\text{Ni}(n,p)^{58}\text{Co}\) reaction cross-section at \(E_n = 5.88 \pm 0.12, 10.11 \pm 0.06\) and \(15.86 \pm 0.12\) MeV, respectively, are presented in the following set of equations:

The diagonal elements of \(V_n\) are

\[ V_{n11} = B_{n1}^T V_\alpha B_\alpha, \]
\[ V_{n22} = B_{n2}^T V_\beta B_\beta, \]
\[ V_{n33} = B_{n3}^T V_\gamma B_\gamma. \] (7.67)

The off diagonal elements \((j > i)\) of \(V_n\) are

\[ V_{n12} = B_{n1}^T V_{\alpha\beta} B_\beta, \]
\[ V_{n13} = B_{n1}^T V_{\alpha\gamma} B_\gamma, \]
\[ V_{n23} = B_{n2}^T V_{\beta\gamma} B_\gamma. \] (7.68)
The off diagonal elements \((i > j)\) of \(V_n\) are

\[
\begin{aligned}
V_{n1} &= V_{n2}, \\
V_{n31} &= V_{n32}, \\
V_{n32} &= V_{n23}.
\end{aligned}
\]  

(7.69)

By substituting the data presented in Eq. (7.50) and Eq. (7.54)–Eq. (7.59) in the solutions presented in Eq. (7.66) and Eq. (7.67)–Eq. (7.69), respectively, we obtained the result presented in Table 7.18:

**Table 7.18**: The best values of \(^{58}Ni(n, p)^{58}Co\) reaction cross-section at effective neutron energies \(E_n = 5.88 \pm 0.12, 10.11 \pm 0.06\) and \(15.86 \pm 0.12\) \(MeV\), with covariance matrix \(V_n\).

<table>
<thead>
<tr>
<th>(E_n) MeV</th>
<th>cross-section (\sigma_n) (mb)</th>
<th>(V_n \times 10^{-4})</th>
<th>(\chi^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>05.88 ± 0.12</td>
<td>485 ± 38</td>
<td>14.52 01.38 00.42</td>
<td>3.61</td>
</tr>
<tr>
<td>10.11 ± 0.06</td>
<td>656 ± 40</td>
<td>01.38 16.28 00.55</td>
<td>2.73</td>
</tr>
<tr>
<td>15.86 ± 0.12</td>
<td>194 ± 14</td>
<td>00.42 00.55 02.20</td>
<td>1.31</td>
</tr>
</tbody>
</table>

In Table 7.18, \(\Delta \sigma_n = \sqrt{V_{ni}}\), the \(\chi^2\) values were obtained by substituting the best values of \(^{58}Ni(n, p)^{58}Co\) reaction cross-section at effective neutron energies \(E_n = 5.88 \pm 0.12, 10.11 \pm 0.06\) and \(15.86 \pm 0.12\) \(MeV\) in Eq. (7.62). The \(\chi^2\) values are slightly higher than the expected \((= 1)\).

The \(\chi^2\) values in Table 7.18 indicates to scale the uncertainties assigned to equivalent data \((^{58}Ni(n, p)^{58}Co\) reaction cross-section) at effective neutron energies \(E_n = 5.88 \pm 0.12, 10.11 \pm 0.06\) and \(15.86 \pm 0.12\) \(MeV\) presented in Table 7.16 by scaling factors \(\sqrt{3.61}, \sqrt{2.73}\) and \(\sqrt{1.31}\), respectively. The original and scaled uncertainties (\(\Delta \sigma\) and \(\Delta \sigma_s\)) are presented in Table 7.19.

With the scaled uncertainties (\(\Delta \sigma_s\)) assigned to equivalent data \((^{58}Ni(n, p)^{58}Co\) reaction cross-section) at effective neutron energies \(E_n = 5.88 \pm 0.12, 10.11 \pm 0.06\) and \(15.86 \pm 0.12\) \(MeV\), we generated adjusted covariance matrix \(V_{\sigma_s}\) and after repeating the least-square analysis, we obtained the adjusted best values of \(^{58}Ni(n, p)^{58}Co\) reaction cross-section at effective neutron energies \(E_n = 5.88 \pm 0.12, 10.11 \pm 0.06\) and \(15.86 \pm 0.12\) \(MeV\) as presented in Table 7.20:
TABLE 7.19: The original and scaled uncertainties ($\Delta \sigma$ and $\Delta \sigma_s$) assigned to equivalent data ($^{58}$Ni($n$, $p$)$^{58}$Co reaction cross-section) at effective neutron energies $E_n = 5.88 \pm 0.12, 10.11 \pm 0.06$ and $15.86 \pm 0.12$ MeV.

<table>
<thead>
<tr>
<th>$E_n$ MeV</th>
<th>$\Delta \sigma$ (mb)</th>
<th>$\Delta \sigma_s$ (mb)</th>
<th>scaling factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>05.88 ± 0.12</td>
<td>46</td>
<td>87</td>
<td>$\sqrt{3.61}$</td>
</tr>
<tr>
<td>05.88 ± 0.12</td>
<td>42</td>
<td>80</td>
<td>$\sqrt{3.61}$</td>
</tr>
<tr>
<td>10.11 ± 0.06</td>
<td>55</td>
<td>91</td>
<td>$\sqrt{2.73}$</td>
</tr>
<tr>
<td>10.11 ± 0.06</td>
<td>42</td>
<td>69</td>
<td>$\sqrt{2.73}$</td>
</tr>
<tr>
<td>15.86 ± 0.12</td>
<td>18</td>
<td>20</td>
<td>$\sqrt{1.31}$</td>
</tr>
<tr>
<td>15.86 ± 0.12</td>
<td>15</td>
<td>17</td>
<td>$\sqrt{1.31}$</td>
</tr>
</tbody>
</table>

TABLE 7.20: The adjusted best values of $^{58}$Ni($n$, $p$)$^{58}$Co reaction cross-section at effective neutron energies $E_n = 5.88 \pm 0.12, 10.11 \pm 0.06$ and $15.86 \pm 0.12$ MeV, with covariance matrix $V_{ns}$.

<table>
<thead>
<tr>
<th>$E_n$ MeV</th>
<th>cross-section $\sigma_n$ (mb)</th>
<th>$V_{ns} \times 10^{-4}$</th>
<th>$\chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>05.88 ± 0.12</td>
<td>485 ± 72</td>
<td>52.41 04.34 00.92</td>
<td>$\approx 1$</td>
</tr>
<tr>
<td>10.11 ± 0.06</td>
<td>656 ± 66</td>
<td>04.34 44.45 01.04</td>
<td>$\approx 1$</td>
</tr>
<tr>
<td>15.86 ± 0.12</td>
<td>193 ± 16</td>
<td>00.92 01.04 02.88</td>
<td>$\approx 1$</td>
</tr>
</tbody>
</table>

In the adhoc method discussed above, we simply scaled the total uncertainties to obtain desired $\chi^2$ value. The advanced methods to deal with discrepant data can be found in [Froehner, 2000, Cacuci and Ionescu-Bujor, 2010].

### 7.5 Conclusion

The data presented in Section 7.3 and the table of partial uncertainties presented in Table 7.13 are the main ingredients required for the EXFOR compilation, so that one can reproduce the result (the $^{58}$Ni($n$, $p$)$^{58}$Co reaction cross-section at $E_n = 5.88 \pm 0.12, 10.11 \pm 0.06$ and $15.86 \pm 0.12$ MeV) presented in Table 7.15 and relative covariance matrix presented in Eq.(7.48). The covariance analysis presented in chapter 6 and chapter 7 can be employed in the different experimental context and in the covariance analysis of the experimental data available in EXFOR database.