1. INTRODUCTION

We consider a differentiable manifold Vn of class $C^\infty$. Let there exist in $\mathcal{V}$ a vector valued $C^\infty$- linear function $F$, a $C^\infty$- vector field $T$ and $C^\infty$- 1-Form $A$ such that for any arbitrary vector field $X$, we have

\begin{align*}
F^2X &= c^2X + A(X) T \\
A(T) &= -c^2 \\
A(FX) &= 0 \\
F(T) &= 0
\end{align*}

where $c$ is a non-zero complex number. Let us agree to say that $F$ gives to $\mathcal{V}$ a differentiable structure defined by an algebraic structure (1.1). (Upadhyay and Srivastava, 1973) [108]. According as $c = \pm i$ or $c = \pm 1$. 
**Quartic Structure**

Let $V_n$ be a $C^\infty$- manifold equipped with a unified structure of almost contact and almost general conic structure satisfying (1.1) to (1.4). Let $V_{n-1}$ be a non-invariant hypersurface of $V_n$. Let $N$ be the affine normal to the surface $V_{n-1}$. Such that

\begin{align}
(1.5) \quad & FBX = BfX - u(X)N \\
(1.6) \quad & FN = BU \\
(1.7) \quad & T = BV + \mu N
\end{align}

and

\begin{align}
(1.8) \quad & A(BX) = v(X)
\end{align}

where $f$ is a tensor field of type $(1,1)$, $u, v$ the 1-form, $U, V$ two vector field and $\mu$ a $C^\infty$- function in $V_n$.

Operating (1.5), (1.6), (1.7) by $F$ and using (1.1) to (1.4) and taking the tangential and Normal parts separately, we get.

\begin{align}
(1.9) \quad & (a) \quad f^2X = c^2X + u(X) U + v(X) V \\
& (b) \quad fU = A(N)V, fV = -\mu U
\end{align}
(c) \( u(fX) = -\mu v(X), \)
\[ v(fX) = A(N) u(X) \]

(d) \( u(U) = -(c^2 + \mu A(N)), \)
\[ u(V) = 0 \]
and

(e) \( v(U) = 0, \quad v(V) = -(c^2 + \mu A(N)) \)

Now operating (1.9) (a) by \( f^2 \) and using (1.9(a),(b) we get-

(1.10) \[ f^4 - ((c^2 - \mu A(N)) f^2 - c^2 \mu A(N) I = 0 \]

which gives a quartic structure on \( V_{n-1} \).

The left hand side of the equation (1.10) can be factorised as

(1.11) \[ (f^2 - c^2 I) (f^2 + \mu A(N)) = 0 \]

where \( I \) is identity.

There may be three cases namely \((\mu=1, A(N) \neq 1)\)

\[ \mu A(N) = 1 \text{ and } A(N) = \mu. \]
Theorem (1.1)

Let $V_{n-1}$ be a non-invariant hypersurface of the $C^\infty$-manifold $V_n$. Then if $\mu=1$ and $A(N) \neq 1$, $V_{n-1}$ is not globally framed, i.e.

\begin{equation}
(1.12) \quad \begin{align*}
(a) & \quad f^2 X = c^2 X + u(X) \ U + v(X) \ V. \\
(b) & \quad fU = A(N) V, \\
& \quad fV = -U \\
(c) & \quad u(fX) = -v(X), \\
& \quad v(fX) = A(N) u(X) \\
(d) & \quad u(U) = -(c^2 + A(N)), \\
& \quad u(V) = 0
\end{align*}
\end{equation}

and

\begin{equation}
(1.12) \quad \begin{align*}
(e) & \quad v(U) = 0, \\
& \quad v(V) = -(c^2 + A(N)).
\end{align*}
\end{equation}

Proof:

Putting $\mu=1$, and $A(N) \neq 1$ in equation (1.9) we shall get required result.
**Theorem (1.2):**

Let $V_{n-1}$ be non-invariant hypersurface of the $C^\infty$-manifold $V_n$. Then if $\mu=1$, $A(N)=0$ we get following structure on $V_{n-1}$.

(1.13)  
(a) $f^2 X = c^2 X + u(X) U + v(X) V$

(b) $fU=0$, $fV=0$

(c) $u(fX) = -v(X)$, $v(fX) = 0$

(d) $u(U) = -c^2$

and

(e) $v(U) = 0$, $v(V) = -c^2$

**Proof:**

Putting $\mu=1$ and $A(N)=0$ in (1.9) we get (1.13). The structure given by equation (1.10) is called restricted quartic structure.

**Theorem (1.3):**

Let $V_{n-1}$ be a non-invariant hypersurface of $C^\infty$-manifold $V_n$ then if

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\( \mu A(N) = 1 \), we have

\begin{align*}
(1.14) & \quad (a) \quad f^2X = c^2X + u(X) U + v(X) V \\
& \quad (b) \quad fU = 1/\mu U, \\
& \quad \quad fV = -\mu U \\
& \quad (c) \quad u(fX) = -\mu v(X), \\
& \quad \quad v(fX) = 1/\mu u(X) \\
& \quad (d) \quad u(U) = -(c^2 + 1), \\
& \quad \quad u(V) = 0 \\
& \quad \quad \text{and} \\
& \quad (e) \quad v(U) = 0, \\
& \quad \quad v(V) = -(c^2 + 1)
\end{align*}

**Proof:**

Putting \( \mu A(N) = 1 \) in equation (1.9) we get equation (1.14).

**Theorem (1.4):**

Let \( V_{n-1} \) be a non-invariant hypersurface of the \( C^\infty \)-manifold \( V_n \). Then if \( A(N) = \mu \), we get the following structure on \( V_{n-1} \).
(1.15)  
(a) \( f^2X = c^2X + u(X) \ U + v(X) \ V, \)
(b) \( fU = \mu V, \)
\( fV = -\mu U \)
(c) \( u(fX) = -\mu v(X), \)
\( v(fX) = \mu u(X) \)
(d) \( u(U) = -(c^2 + \mu^2), \)
\( u(V) = 0 \)
and
(e) \( v(U) = 0, \)
\( v(V) = -(c^2 + \mu^2) \)
equivalently.

(1.16)  \( f^4 - (c^2I - \mu^2) f^2 - c^2\mu^2I = 0. \)

**Proof:**

Putting \( A(N) = \mu \) in the equation (1.9) we get equation (1.15) putting \( A(N) = \mu \) in (1.10), we get (1.16).
2. HYPER SURFACE OF $C^\infty$-MANIFOLD $V_n$ ADMITTING $(F,A,T)$ CONNEXION

Let us consider the manifold $V_n$ with $(F,A,T)$ connexion, then

$$\text{Ex } F = 0, \text{Ex } A = 0, \text{Ex } T = 0$$

where $E$ is a symmetric connexion on $V_n$.

Let $D$ be the induced connexion on $V_{n-1}$. The Gauss and Weingarten's equation are given by

\[(2.1)\quad E_{BX} \, BY = BD_X \, Y - c^2 \, h(X,Y) \, N\]

and

\[(2.2)\quad E_{BX}N = - BHX\]

respectively, where $h$ and $H$ are the second fundamental tensor of type $(0,2)$ and $(1,1)$.

Differentiating (1.5), (1.6), (1.7) and (1.8) covariantly and using (1.5), (1.6), (1.7), (1.8), (2.1) and (2.2), we get
(2.3)  
(a)  \((D_y f)(X) = -u(X) \ HY - c^2 h(X,Y) \ U.\)

(b)  \((D_y u)(X) = -c^2 h(Y, fX)\)

(c)  \((D_y v)(X) = c^2 h(X,Y) \ A(N)\)

(d)  \(c^2 h(Y,U) = -u(H,Y), \ c^2 h(Y,V) = D_y \mu.\)

**Theorem (2.1)**

Let \(V_{n-1}\) with \((f, g, u, v, c)\) structure be a non-invariant hypersurface of \(V_n\). Admitting \((F,A,T)\) – connexion. Then, induced connexion \(D\) on \(V_{n-1}\) is given by

(2.4)  
(a)  \((D_y f)(X) = -u(X) \ HY - c^2 h(X,Y) \ U.\)

(b)  \((D_x v)(X) = 0,\)

(c)  \((D_y u)(X) = -c^2 h(Y, fX).\)

(c)  \(D_y V = HY,\)

\(D_y U = -fHY\)

and

(d)  \(c^2 h(Y,U) = -u(H,Y), \ c^2 h(Y,V) = 0\)
Proof:

Putting $\mu=1$, and $A(N) = 0$ in (2.3), we get (2.4).

Theorem (2.2):

Let $V_{n-1}$ with $(f, g, u, v, c)$ – structure be a non-invariant totally umbilical hypersurface of $V_n$ admitting $(F,A,T)$ – connexion. Then $V_{n-1}$ with $(f, g, u, v, c)$ – Structure is totally geodesic.

Proof:

Let $V_{n-1}$ is totally umbilical $H=\eta I$ then (2.4)(d) implies $\eta u(Y) = 0$ but $u(Y) \neq 0$ so $\eta$ must vanish. Hence $H=0$.

Theorem (2.3):

Let $V_{n-1}$ with $(f,g,u,v,c)$-structure a non-invariant hypersurface of $V_n$ admitting $(F,A,T)$ connexion. Then if $f$ is a parallel field we have

$$ (2.5) \quad h(X,Y) = -\frac{1}{c^2} \eta \ u(X) \ u(Y) $$
Proof:

If \( f \) is a parallel field, putting \( D_yf=0 \) in (2.4)(a) and operating it by \( u \) and using (1.13)(d), we get

\[
(2.6) \quad c^4 h(X,Y) = u(X) u(HY)
\]

which gives us

\[
(2.7) \quad u(X) u(HY) = u(Y) u(HX)
\]

because \( h \) is symmetric. Substituting \( Y=U \) in (2.7) and using (2.4)(d), (2.6) setting \( \eta = h(U,U) \), we get (2.5).

**Theorem 2.4:**

Let \( V_{n-1} \) with \( (f,g,u,v,c) \)-structure be non-invariant totally geodesic hypersurface of \( V_n \) admitting \( (F,A,T) \)-connexion \( D \).

Proof:

Putting \( h=0 \) in (2.4)(a), we have \( (D_yf)(X) = -u(X) HY \).

**Theorem 2.5:**

Let \( V_{n-1} \) with \( (f,g,u,v,c) \)-structure be a non-invariant hypersurface of \( V_n \) admitting \( (F,A,T) \)-connexion. Then, if \( V_{n-1} \) with \( (f,g,u,v,c) \)-structure is totally umbilical, we have
where $\eta$ is a constant and

(b) $f'(X,Y) \overset{\text{def}}{=} g(fX,Y)$

**Proof:**

Putting $h = \eta g$ in (2.4)(b) and using (2.8)(b) we get (2.8)(a).

**Theorem 2.6:**

Let $V_{n-1}$ with $(f,g,u,v,c)$-structure be a non-invariant hypersurface of $V_n$ admitting $(F,A,T)$-connexion. Then, induced connexion-$D$ on $V_{n-1}$ satisfies

(2.9)(a) \hspace{1cm} (D_y f) (X) = -u(X) HY - c^2 h(X,Y) U,

(b) \hspace{1cm} (D_y u) (X) = -c^2 h(Y,fX);

(c) \hspace{1cm} D_y U = -f HY; \hspace{0.5cm} D_y V = \mu HY

(d) \hspace{1cm} c^2 h(Y,U) = -u(HY) ; \hspace{0.5cm} c^2 h(Y,V) = D_y \mu

and

(2.10) \hspace{1cm} v(HX) = -D_x \mu
**Proof:**

Putting $A(N) = \mu$ in (2.3), we get (2.9). Differentiating $A(N) = \mu$ covariantly with respect to $X$ and using (2.2), we get (2.10).

**Theorem 2.7:**

Let $V_{n-1}$ with $(f,g,u,v,c)$-structure be a non-invariant totally umbilical hypersurface of $V_n$ admitting $(F,A,T)$-connexion. Then, $V_{n-1}$ with $(f,g,u,v,\mu,c)$-structure is totally geodesic, iff $\mu$ is constant.

**Proof:**

Let $V_{n-1}$ be totally umbilical then putting $H=\eta I$ in (2.10) we get

$$v(HX) = \eta \cdot v(X) = -D_x \mu$$

If $\mu$ is a covariant constant, then we have

$$\eta \cdot v(X) = 0 \Rightarrow \eta = 0$$
Since

\[ v(X) \neq 0. \]

Hence the hypersurface is totally geodesic.

Conversely, if it is totally geodesic \( \eta = 0 \Rightarrow D_x \mu = 0. \)

Thus \( \mu \) is covariant constant.

**Theorem 2.8:**

Let \( V_{n-1} \) with \( (f, g, u, v, \mu, c) \)-structure be a non-invariant hypersurface of \( V_n \) admitting \( (F, A, T) \)-connexion. Then if the tensor field \( f \) is parallel, we get

\[ c^2 (c^2 + \mu^2)^2 h(X, Y) = \eta u(X) u(Y) \]

where \( \eta = h(U, U). \)

**Proof:**

If the tensor field \( f \) is parallel field, putting \( D_y f = 0 \) in (2.9)(a) and operating it by \( u \) and using (1.15)(d), we get,

\[ c^2 (c^2 + \mu^2) h(X, Y) = u(X) u(HY). \]
Which gives us

\[(2.13) \quad u(Y) u(HX) = u(X) u(HY)\]

since \(h\) is symmetric. Putting \(Y=U\) in (2.13) and using (1.15)(d) (2.12) and setting \(\mu=h\ (U,U)\), we get (2.11).

Conversely putting (2.11) in (2.9)(a), we get \(D_y f = 0\).

**Theorem 2.9**

On a non-invariant hypersurface \(V_{n-1}\) with \((f,g,u,v,\mu,c)\)-structure of \(V_n\) admitting \((F,A,T)\)-connexion, we have

\[(2.14) \quad (D_x f') (Y,Z) + (D_y f') (Z,X) + (D_z f') (X,Y) = -(1+c^2) [u(Y) h(X,Z) + u(X) h(Y,Z) + u(Z) h(X,Y)]\]

**Proof:**

We know that

\[(D_x f') (Y,Z) = g((D_x f') (Y),Z).\]

Using (2.9)(a) we get

\[(2.15) \quad (D_x f') (Y,Z) = -u(Y) h(Z,X) - c^2 h(X,Y) u(Z).\]
(2.16) \[(D_y f ') (Z,X) = -u(Z) h(X,Y) - \alpha^2 h(Z,Y) u(X)\]

and

(2.17) \[(D_z f ') (X,Y) = -u(X) h(Y,Z) - \alpha^2 h(X,Z) u(Y).\]

Adding (2.15), (2.16) and (2.17), we get (2.14).

3. **NORMAL QUARTIC STRUCTURE**

Let S and N be the torsion tensor and Nijenhuis tensor of \( V_{n-1} \) having polynomial structure

\[ f^4 - (\alpha^2 I - \mu A(N)) f^2 - \alpha^2 \mu A(N) I = 0 \]

which is invariant hypersurface of \( V_n \) Admitting \( (F,A,T) \) connexion. Then

(3.1) \[(a) \quad S(X,Y) \overset{\text{def}}{=} N(X,Y) - du(X,Y) U - dv(X,Y) V \]

where

(b) \[du(X,Y) = (D_x u)(Y) - (D_y u)(X)\]

and

(c) \[dv(X,Y) = (D_x v)(Y) - (D_y v)(X)\]
when the torsion tensor $S$ vanishes, the structure is said to be Normal.

**Theorem (3.1):**

A non-invariant hypersurface $V_{n-1}$ with quartic structure of a differentiable manifold $V_n$ admitting $(F,A,T)$ connexion is Normal iff

$$-(c^2 + \mu A(N)) (Hf - fH) = u \otimes \left( \frac{A(N)}{\mu} D_v V + D_u U \right)$$

**Proof:**

Putting the value of $N$, $du$, $dv$ in (3.1)(a) using (2.3)(a) & (2.3)(b) we get

$$S(X,Y) = u(X) \{HfY - fHY\} + u(Y) \{fHX - HfX\}$$

Putting the value of $Hf - fH$ from (3.2) in (3.3) we get $S=0$. Hence the structure is Normal.

Conversely, if the structure is Normal. Then from (3.3), we get
(3.4) \[ u(X) \{ H_f Y - f H Y \} = u(Y) \{ H_f X - f H X \}. \]

Putting \( X = U \) in (3.4) and using (1.6)(b),(d) and (2.3)(c) we get (3.2).

**Corollary (3.1):**

Let \( V_{n-1} \) with \((f, g, u, v, c)\) – Structure be a non-invariant hypersurface of \( V_n \) admitting \((F,A,T)\) – connexion. Then with \( V_{n-1} \) with \((f, g, u, v, c)\) – structure is Normal iff

\[ (3.5) \quad -c^2 (H_f - f H) = u \odot D_u U \]

**Proof:**

Putting \( A(N) = 0 \), and \( \mu = 1 \) in (3.2) we get (3.5).

**Corollary (3.2):**

Let \( V_{n-1} \) with \((f, g, u, v, \mu, c)\) structure of \( V_n \) admitting \((F,A,T)\) connexion. Then \( V_{n-1} \) with \((f, g, u, v, \mu, c)\) – structure is normal iff
\[(3.6) \quad -(c^2 + \lambda^2) \left\{H_f - fH\right\} = u \otimes (D_v V + D_u U).\]

\textbf{Proof:}

Putting \( A(N) = \mu \) in (3.2) we get (3.6). Let \( V_{n-1} \) be the normal, then from (3.1) and (2.3)(b) we get

\[(3.7) \quad N(x,y) - du(X,Y) U = 0.\]

\textbf{Theorem 3.2:}

The non-invariant hypersurface \( V_{n-1} \) with \( (f, g, u, v, c) \) – Structure of manifold \( V_n \) Admitting \( (F,A,T) \) connexion will be normal if

\[(3.8) \quad du(fX, fY) + c^2 du(X,Y) = 0\]

\textbf{Proof:}

From (3.7) we have

\[(3.9) \quad (D_{fx} f)(Y) - (D_{fy} f)(X) + f(D_y f)(X) - f(D_x f)(Y) - du(X,Y) U = 0\]

operating (3.9) by \( u \) and using (1.10)(c),(d) we get
(3.10) \[ u ((D_{fx}f) (Y)) - u (D_{fy}f) (X) - v ((D_yf) (X)) \]
\[ + v((D_xf) (Y)) + c^2 du(X,Y) = 0 \]

Now differentiating \( u(fy) = -v(Y) \) covariantly with respect to \( fX \). We get

(3.11) \[ u((D_{fx}f) (Y)) = - (D_{fx}u) (fy) - (D_{fx}v) (Y). \]

and differentiating \( v(fy) = 0 \), covariantly with respect to \( X \), we get

(3.12) \[ v((D_xf) (Y)) = -(D_xv) (fy), \]

using (3.11), (3.12) and \( dv = 0 \) in (3.10) we get (3.8).

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