Primeness, Semiprimeness and Separation Theorem in Posets

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Chapter-I

PRIMENESS, SEMIPRIMENESS AND
SEPARATION THEOREM IN POSETS

§1. INTRODUCTION.

Let $P$ be a set. A *partial order* on $P$ is a binary relation $\leq$ on $P$ such that, for all $x, y, z \in P$ the following conditions hold:

1. $x \leq x$.
2. $x \leq y$ and $y \leq x$ imply $x = y$.
3. $x \leq y$ and $y \leq z$ imply $x \leq z$.

These conditions are referred to, respectively, as *reflexivity*, *antisymmetry* and *transitivity*. A set $P$ equipped with a partial order relation $\leq$ is called a *partially ordered set*, in short *POSET*, and denoted by $(P; \leq)$, if it is necessary to specify the order relation, otherwise it will be denoted by $P$. We say that a poset $P$ is finite if $P$ has finite number of elements. For $a, b \in P$, we say that $a$ and $b$ are *comparable* if $a \leq b$ or $b \leq a$; otherwise, $a$ and $b$ are said to be *incomparable*, in notation, $a \not\leq b$. The inverse relation of $\leq$ is denoted by $\geq$. The set $(P; \geq)$ is also a poset, called the *dual* of $(P; \leq)$. If $\Phi$ is a statement about posets, and if in $\Phi$ we replace all occurances of $\leq$ by $\geq$, we get the *dual* of $\Phi$. 
We state a principle that halves the labour of proving some results.

**Duality Principle:** "If a statement $\Phi$ is true in all posets, then its dual is also true in all posets."

In the light of what we have said above, the concepts of lower bound, the greatest lower bound (or infimum) are simply dual statements of upper bound, the least upper bound (or supremum) respectively. Hence, we simply say that the notions of lower bound, the infimum are dually defined once the concepts of upper bound and the supremum are defined. Similar arguments are made for other concepts also.

We begin with necessary concepts and terminologies in posets. For undefined notations and terminologies, the reader is referred to Birkhoff [1967], Davey and Priestley [1990] and Grätzer [1998].

Let $P$ be a poset. A subset $X$ of $P$ is called a **down-set** if, for $x \in X$, $y \in P$ and $y \leq x$, we have $y \in X$. Dually, we have the concept of an **up-set**.

An element $x$ of a poset $P$ is an **upper bound** of $A \subseteq P$, if $a \leq x$ for all $a \in A$. A **lower bound** is defined dually. The set of all upper bounds of $A$ is denoted by $A^u$ (read as, *upper cone of $A$*), where $A^u = \{x \in P \mid x \geq a \text{ for every } a \in A\}$ and dually, we have the concept of a **lower cone** $A^l$ of $A$. By
Chapter-I  

1. Introduction

$A^{ul}$ we mean $\{A^u\}^l$ and $A^{lu}$ we mean $\{A^l\}^u$. The upper cone $\{a\}^u$ is simply denoted by $a^u$ and $\{a, b\}^u$ is denoted by $(a, b)^u$. Further, for $A, B \subseteq P$, $\{A \cup B\}^u$ is denoted by $\{A, B\}^u$ and for $x \in P$, the set $\{A \cup \{x\}\}^u$ is denoted by $\{A, x\}^u$. Similar notations are used for lower cones. We note that, for $A, B \subseteq P$ $A \subseteq A^{ul}$ and $A \subseteq A^{lu}$, if $A \subseteq B$, then $B^l \subseteq A^l$ and $B^u \subseteq A^u$. Moreover, $A^{lul} = A^l$, $A^{ulu} = A^u$ and $\{a^u\}^l = \{a\}^l = a^l$. Since $\leq$ is transitive, $A^u$ is always an up-set and $A^l$ is a down-set.

An upper bound $x$ of $A$ is the least upper bound of $A$ (or supremum of $A$, in short $\text{sup} \{A\}$) iff for any $b \in A^u$, we have $x \leq b$. We shall write $x = \text{sup} \{A\}$ or $x = \bigvee A$.

The concept of the greatest lower bound or infimum of $A$ is defined dually; the later is denoted by $\text{inf} \{A\}$ or $\bigwedge A$. For $x, y \in P$, if $\text{sup} \{x, y\}$ exists then we denote this element by $x \lor y$ and read as $x$ join $y$; denote $\text{inf} \{x, y\}$ when it exists by $x \land y$ and read as $x$ meet $y$.

A poset $P$ is called a lattice, if $x \lor y$ and $x \land y$ exist for all $x, y \in P$. If $x \lor y$ exists in $P$ for all $x, y \in P$ then $P$ is called a join semilattice and dually we have a meet semilattice.

A lattice $L$ is called a complete lattice if $\bigvee A$ and $\bigwedge A$ exist for every subset $A$ of $L$. A non empty subset $S$ of $L$ is called a sublattice of $L$ if $x, y \in S$ implies $x \lor y \in S$ and $x \land y \in S$. 
Ideals of a Lattice: Ideals are of fundamental importance in Algebra. Filters, the dual concept of ideals, have a variety of applications in Logic and Topology. This algebraic concept is also well studied in the case of Lattice Theory.

A non empty subset $I$ of a lattice $L$ is called an ideal of $L$ if

(i) $a, b \in I$ implies $a \lor b \in I$,
(ii) $a \in L, b \in I$ and $a \leq b$ imply $a \in I$.

A dual ideal is called a filter and defined dually. Clearly, every ideal $I$ of a lattice $L$ is a down-set. An ideal or a filter is called proper if it does not coincide with $L$. For each $a \in L$, the set $(a) = \{x \in L \mid x \leq a\}$ is an ideal of $L$; it is known as the principal ideal generated by $a$ and dually, $[a]$ is a principal filter. It is known that the set $Id(L)$, of all ideals of a lattice $L$ with 0, under set inclusion $\subseteq$, is a complete lattice. If $L$ has no 0, then $Id_0(L) = Id(L) \cup \{\phi\}$ is a complete lattice, where $\phi$ denotes empty set.

A proper ideal $I$ of a lattice $L$ is called prime if $a, b \in L$ and $a \land b \in I$ imply $a \in I$ or $b \in I$. A prime filter $F$ is defined dually. An ideal $I$ of a lattice $L$ is prime if and only if $L - I$ is a prime filter.

Krull [1929] introduced the concept of semiprime ideal. Accordingly, if $R$ is a commutative ring with unity, then an ideal $I$ of $R$ is semiprime whenever $a^n \in I$, $n$ is positive integer,
implies that \( a \in I \). Krull also proved, using well ordering theorem, that every semiprime ideal is the intersection of all the prime ideals which contain it. It is known fact that "if \( I \) is a semiprime ideal of a commutative ring \( R \) with unity then the principal ideal \( (I) \) is a semiprime ideal in the lattice of ideals of \( R \)". Moreover, M. H. Stone in his famous 1936 paper proved the Separation Theorem for prime ideals in the case of distributive lattices as follows.

**Theorem A** (Stone [1936]) Let \( L \) be a distributive lattice, let \( I \) be an ideal, let \( D \) be a dual ideal of \( L \), and let \( I \cap D = \emptyset \). Then there exists a prime ideal \( P \) of \( L \) such that \( P \supseteq I \) and \( P \cap D = \emptyset \).

Rav [1989] introduced the concept of a semiprime ideal in lattices. An ideal \( I \) of a lattice \( L \) is said to be semiprime if, \( x \wedge y \in I \) and \( x \wedge z \in I \) together imply \( x \wedge (y \vee z) \in I \) and dually we have the concept of semiprime filter. Also, he studied the lattice of semiprime ideals and proved that the set of all semiprime ideals of a lattice \( L \) is a complete lattice with respect to set inclusion. Further, he studied the relation between the semiprime ideals of a lattice \( L \) and those of its ideal lattice. Moreover, he proved the analogue of the Prime Separation Theorem for semiprime ideals in Lattice Theory as follows.
**Theorem B** (Rav [1989]) Let $L$ be a lattice containing an ideal $I$ and a filter $F$ such that $I \cap F = \emptyset$. If $F$ is semiprime, then there exists a semiprime ideal $J$ such that $I \subseteq J$ and $J \cap F = \emptyset$. A dual result holds if $I$ is semiprime.

Beran [1990] established some properties of semiprime ideals in lattices (see also Beran [1994] and [1998]) and also the connection between prime ideals and semiprime ideals in lattices. In fact, he characterized semiprime ideals to be prime by using the concept of meet irreducible elements. An element $x$ of a lattice $L$ is called a meet irreducible element if $x = a \wedge b$ implies that $x = a$ or $x = b$. Dually, we have a concept of a join irreducible element.

**Theorem C** (Beran [1994]) Let $I$ be a semiprime ideal of a lattice $L$. Then $I$ is prime if and only if $I$ is a meet irreducible element of $Id(L)$.

In section 2, we extend the concept of a semiprime ideal in lattices to general posets and study their properties. Also, we establish the interrelationships between the prime ideals and the semiprime ideals in posets. We observe that Theorem C is indeed true in the case of posets also.

In section 3, we study the relations between the semiprime ideals of a poset $P$ and the ideals of the set $Id(P)$ of all ideals of $P$. Moreover, we study the relations between prime ideals of
$P$ and prime ideals of the set $Id(P)$.

In section 4, we have extended Separation Theorem for semiprime ideals to finite posets with some constraint on the filter. Then the Prime Separation Theorem for finite posets $P$ for which $Id(P)$ is distributive is obtained.

Another results and properties of semiprime ideals as well as prime ideals in posets are established. Some counterexamples are also given wherever necessary.
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To
The Registrar
University of Pune
Ganeshkhind
Pune - 411 007


Sir,

Sub: Returning the thesis after examination

Ref: Your letter no. PLS/905 dated 16-10-2007

I am returning the thesis after examination as desired by you. The report is sent to The Deputy Registrar (Admissions), University of Pune, separately.

Thanking you;

Yours sincerely,

S. Parameshwara Bhatla

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§2. PRIMENESS AND SEMIPRIMENESS IN POSETS.

In this section, we have introduced the concept of semiprime ideal in a general poset. Characterizations of semiprime ideals in a poset as well as characterizations of semiprime ideals to be prime are obtained. In fact, we succeeded to generalize Theorem C stated in section 1, to posets. Also, prime ideals in posets are characterized. It is proved that a prime ideal \( I \) of a poset \( P \) and its \( F_I \) make a separation of \( P \). Further, we have established some properties and characterizations of prime ideals and semiprime ideals in posets.

The concepts of an ideal in a poset are independently studied by Frink [1954] and Halaš [1994].

**Definition 2.1.** (i) Frink: A subset \( I \) of a poset \( P \) is called an *ideal* in \( P \) if for each finite subset (possibly empty) \( M \) of \( I \) we have \( M^{ul} \subseteq I \).

(ii) Halaš: A subset \( I \) of a poset \( P \) is called an *ideal* if \( a, b \in I \) implies \( (a, b)^{ul} \subseteq I \).

The following remarks are made by Halaš and Rachůnek [1995]:

(a) If \( P \) is a lattice then \( \phi \neq I \subseteq P \) is an ideal in the poset \( P \) if and only if \( I \) is an ideal in the lattice \( P \).

(b) If a poset has not the least element the empty subset \( \phi \) is an ideal in \( P \).
Chapter-I  2. Primeness and Semiprimeness in Posets

It can be observed that an ideal in the sense of Frink is necessarily an ideal in the sense of Halaš but the converse need not be true (for more details see Joshi and Waphare [2007]).

Here, we consider the concept of an ideal in the sense of Halaš.

A prime ideal in a poset introduced by Halaš and Rachůnek [1995] is as follows:

**Definition 2.2.** A proper ideal $I$ is called *prime* if $(a, b) \subseteq I$ implies that either $a \in I$ or $b \in I$.

Dually, we have the concepts of *filter* and a *prime filter*.

Let $a$ be an element of a poset $P$. The subset $\{x \in P \mid x \leq a\}$ of $P$ is an ideal of $P$ generated by $a$, denoted by $(a]$; we shall call $(a]$, a *principal ideal*. Dually, $[a)$ is a filter generated by $a$ and called a *principal filter*.

We generalize the concept of a semiprime ideal in lattices to posets as follows.

**Definition 2.3.** An ideal $I$ of a poset $P$ is called *semiprime* if $(a, b)^l \subseteq I$ and $(a, c)^l \subseteq I$ together imply $\{a, (b, c)^u\}^l \subseteq I$.

Dually, we have the concept of a *semiprime filter*.

The following result establishes a connection between prime ideals and semiprime ideals.

**Lemma 2.4.** *Every prime ideal of a poset $P$ is semiprime ideal.*
Proof: Suppose that $I$ is a prime ideal and for $a, b, c \in P$, $(a, b)^l \subseteq I$ and $(a, c)^l \subseteq I$. Since $I$ is prime, we have two cases:

(i) If $a \in I$, $\{a, (b, c)^u\}^l \subseteq a^l \subseteq I$.

(ii) If $a \notin I$, then both $b$ and $c$ are in $I$ and by definition of an ideal, $(b, c)^u \subseteq I$. Therefore $\{a, (b, c)^u\}^l = a^l \cap (b, c)^u \subseteq (b, c)^u \subseteq I$. Thus $I$ is semiprime.

Remark 2.5. The converse of Lemma 2.4 does not hold general. In the poset depicted in Figure 1, the ideal $I = \{0\}$ is semiprime but not prime, as $(a, b)^l \subseteq I$ and neither $a$ nor $b$ is in $I$. We have answered the question that when a semiprime ideal is prime in terms of different concepts in the following results including Theorem 2.16.

![Figure 1](image)

In what follows, $Id(P)$ denotes the set of all ideals of a poset $P$. It is known that $(Id(P), \subseteq)$ is a complete lattice with the least element $\phi$ and the greatest element $P$ in which meets
coincide with set intersections (see Halaš and Rachůnek [1995]).

**Theorem 2.6.** Every prime ideal of a poset $P$ is a meet irreducible element of $\text{Id}(P)$.

**Proof:** Let $I$ be a prime ideal such that $I = J \cap K$ for $J, K \in \text{Id}(P)$. We have to show that either $I = J$ or $I = K$.

Clearly $I \subseteq J$ and $I \subseteq K$. Suppose $I \neq J$ and $I \neq K$; then there exists $x, y \in P$ such that $x \in J - I$ and $y \in K - I$. But since $J$ and $K$ are ideals, we have $(x, y)^I \subseteq J \cap K \subseteq I$. Since $I$ is prime, either $x \in I$ or $y \in I$, a contradiction to the fact that $x, y \notin I$.

**Remark 2.7.** The converse of Theorem 2.6 is not true in general. Consider the poset $Q$ depicted in Figure 2 and its ideal lattice $\text{Id}(Q)$, depicted in Figure 3. Observe that $(a]$ is a
meet irreducible element of $Id(Q)$. But $(a]$ is not prime in $Q$ as $(b, c)^l \subseteq (a]$ and neither $b$ nor $c$ is in $(a]$.

In order to characterize semiprime ideals to be prime we consider the following sets that are considered and studied by Halaš [1995b]. For the ideals $I$ and $J$ of a poset $P$, denote $C_1(I, J) = \bigcup\{(a, b)^ul \mid a, b \in I \cup J\}$. Inductively, let $C_{n+1}(I, J) = \bigcup\{(a, b)^ul \mid a, b \in C_n(I, J)\}$ for each $n \in \mathbb{N}$, the set of natural numbers.

It is easy to observe that the sets $C_n(I, J)$ form a chain, in other words, $C_1 \subseteq \ldots \subseteq C_{n-1} \subseteq C_n \subseteq \ldots$. Also, observe that each $C_n(I, J)$ is a down-set, as if $x \in C_n(I, J)$ and $y \leq x$, then $x \in (a, b)^ul$ for some $a, b \in C_{n-1}(I, J)$, i.e., $y \in (a, b)^ul$ for some $a, b \in C_{n-1}(I, J)$. Thus $y \in C_n(I, J)$.

We use the following Lemma in the result followed by it which is a characterization of semiprime ideals to be prime ideals. In fact, it is a generalization of Theorem C.

**Lemma 2.8.** (Halaš [1995b]) Let $P$ be a poset and $I, J \in Id(P)$. Then $I \lor J = \bigcup\{C_n(I, J) \mid n \in \mathbb{N}\}$.

The following statement characterizes semiprime ideals that are prime.

**Theorem 2.9.** Let $I$ be a semiprime ideal in a poset $P$. Then $I$ is prime if and only if $I$ is a meet irreducible element of $Id(P)$. 
Proof: ($\Rightarrow$) Follows by Theorem 2.6.

($\Leftarrow$) Let $(a, b)^l \subseteq I$. We claim that $I = (I \lor (a)) \cap (I \lor (b))$. Clearly, it is enough to show that $(I \lor (a)) \cap (I \lor (b)) \subseteq I$. In fact, in view of Lemma 2.8, we have to show that $C_n (I, (a)) \cap C_m (I, (b)) \subseteq I$ holds for all $n, m \in N$, by using the induction.

(i) Assume $n + m = 2$ and let $z \in C_1 (I, (a)) \cap C_1 (I, (b))$. We have that $z \in (x_1, y_1)^{ul} \cap (x_2, y_2)^{ul}$ for $x_1, y_1 \in I \cup (a)$ and $x_2, y_2 \in I \cup (b)$. We distinguish two cases:

(1) If $x_1, y_1 \in I$ or $x_1, y_1 \in (a]$ or $x_2, y_2 \in I$ or $x_2, y_2 \in (b]$, by definition of ideals, then clearly $z \in I$.

(2) Assume $x_1 \in I$, $y_1 \in (a)$, $x_2 \in I$ and $y_2 \in (b]$, then $(x_1, x_2)^l \subseteq I$ and $(x_1, y_2)^l \subseteq I$. By semiprimeness of $I$, we have $(x_1, y_2)^l \subseteq I$. Since $z \in (x_2, y_2)^{ul}$, we get $(x_1, z)^l \subseteq I$. Similarly, since $(y_1, x_2)^l \subseteq I$, $(y_1, y_2)^l \subseteq I$ and $z \in (x_2, y_2)^{ul}$, we have $(y_1, z)^l \subseteq I$. Now, $(x_1, z)^l \subseteq I$ and $(y_1, z)^l \subseteq I$, together yield $(z, (x_1, y_1)^{ul})^l \subseteq I$. But $z \in (x_1, y_1)^{ul}$, thus $z \in I$. Therefore the statement is true for $n + m = 2$.

(ii) Suppose the statement is true for $n + m = r$; i.e., suppose $C_n (I, (a)) \cap C_m (I, (b)) \subseteq I$ holds for $n + m = r$. We show that the statement is also true for $n + m = r + 1$. Let $z \in C_n (I, (a)) \cap C_{m+1} (I, (b))$. Then for $x_1, y_1 \in C_{n-1} (I, (a))$ and $x_2, y_2 \in C_m (I, (b))$, we have $z \in (x_1, y_1)^{ul} \cap (x_2, y_2)^{ul}$. Observe that $(x_2, y_1)^l \subseteq C_{n-1} (I, (a)) \cap C_m (I, (b))$. 
Since the sets $C_n(I, (a])$ form a chain, we have $(x_2, y_1)^l \subseteq C_{n-1}(I, (a]) \cap C_m(I, (b]) \subseteq C_n(I, (a]) \cap C_m(I, (b])$. By the induction hypothesis, $(x_2, y_1)^l \subseteq I$. Similarly, $(x_2, x_1)^l \subseteq C_{n-1}(I, (a]) \cap C_m(I, (b]) \subseteq C_n(I, (a]) \cap C_m(I, (b]) \subseteq I$. By semiprimeness of $I$, we get $(x_2, (x_1, y_1)^u)^l \subseteq I$. But $z \in (x_1, y_1)^u$ and therefore $(x_2, z)^l \subseteq I$. Similarly, we get $(y_2, z)^l \subseteq I$. Again by semiprimeness of $I$ and the fact that $z \in (x_2, y_2)^u$, we have $z \in z^l \cap (x_2, y_2)^u = \{z, (x_2, y_2)^u\}^l \subseteq I$. Thus the statement is true for $n + m = r + 1$ and consequently by induction, it is true for each $n, m \in N$.

Now, since $I$ is a meet irreducible element of $Id(P)$ and $I = (I \lor (a]) \cap (I \lor (b])$, we have either $I = (I \lor (a])$ or $I = (I \lor (b])$. Therefore, $a \in I$ or $b \in I$ and thus $I$ is prime.

Now we present more properties of semiprime ideals in posets by using of the following sets. For an ideal $I$ and a non empty subset $A$ of a poset $P$, define a subset $I:A$ of $P$ as follows:

$$I:A = \left\{ z \in P \mid (a, z)^l \subseteq I \text{ } \forall a \in A \right\}.$$ 

If $A = \{x\}$, then we write $I:x$ instead of $I:\{x\}$. Note that, if $x \leq y$ for $x, y \in P$, then $I:y \subseteq I:x$. Indeed, if $z \in I:y$, then $(y, z)^l \subseteq I$, i.e., $(x, z)^l \subseteq (y, z)^l \subseteq I$, which yields $z \in I:x$.

Observe that $I \subseteq I:A$ and $I:A = \bigcap_{x \in A} I:x$, however $I:A$ need not be an ideal. In fact, $I:x$ need not be an ideal. In the
poset $Q$ depicted in Figure 2, $I = \{0, a\}$ is an ideal of $Q$ but $I:b = \{0, a, c\}$ is not an ideal. However, if $I$ is semiprime, then in Theorem 2.15 we prove that $I:x$ is an ideal for all $x \in P$.

Some properties of this set in the following result.

**Lemma 2.10.** Let $I$ be a semiprime ideal of a poset $P$. Then the following statements hold for $x, a, b \in P$:

(i) $(a, b)^l \subseteq I:x$ if and only if $(x, a, b)^l \subseteq I$.

(ii) $\{x, (a, b)^u\}^l \subseteq I$ if and only if $(a, b)^u \subseteq I:x$.

(iii) $I:x = P$ if and only if $x \in I$.

Note: (i) and (iii) don’t require the semiprimeness of $I$.

**Proof:**

(i) Suppose that $(x, a, b)^l \subseteq I$ and $z \in (a, b)^l$. It is clear that $(x, z)^l \subseteq (x, a, b)^l \subseteq I$, i.e., $(x, z)^l \subseteq I$. Thus $z \in I:x$.

Conversely, suppose that $(a, b)^l \subseteq I:x$ and $z \in (x, a, b)^l$. Since $z \in (x, a, b)^l \subseteq I:x$, we get $(x, z)^l \subseteq I$. But $z \leq x$, therefore $z \in I$.

(ii) Let $\{x, (a, b)^u\}^l \subseteq I$ and $z \in (a, b)^u$. We have $(x, z)^l \subseteq x^l \cap (a, b)^u = \{x, (a, b)^u\}^l \subseteq I$, hence $z \in I:x$.

Conversely, suppose that $(a, b)^u \subseteq I:x$. Clearly $a, b \in (a, b)^u \subseteq I:x$. Hence $(x, a)^l \subseteq I$ and $(x, b)^l \subseteq I$. By semiprimeness, we get $\{x, (a, b)^u\}^l \subseteq I$.

(iii) Observe that $x \in I$ iff $(x, z)^l \subseteq I \ \forall z \in P$ iff $I:x = P$. 

\[\bullet\]
Let $P$ be a poset. A non empty subset $Q$ of $P$ is called an \textit{up directed set}, if $Q \cap (x, y)^u \neq \emptyset$ for any $x, y \in Q$. Dually, we have the concept of a \textit{down directed set}. If an ideal $I$ (filter $F$) is up (down) directed set of $P$, then it is called an \textit{u-ideal} (l-filter).

It is easy to observe that every principal filter is an l-filter. But the converse is not true in general. Consider the poset depicted in Figure 4. Observe that the filter $\{x_1, x_2, \ldots\}$ is an l-filter but not principal.

However, in finite posets we have Lemma 2.12 in which we use the following definition.

\textbf{Definition 2.11.} Let $P$ be a poset and $Q \subseteq P$. Denote the set of all maximal elements of $Q$ by $Max(Q)$ and the set of all minimal elements of $Q$ by $Min(Q)$.

\textbf{Lemma 2.12.} Every l-filter of a finite poset $P$ is principal.
Proof: It is enough to show that every $l$-filter $F$ has the least element. If not, suppose $\text{Min}(F) = \{d_1, d_2\}$. Since $d_1, d_2 \in F$, then by definition of $l$-filter, there exists $z \in P$ such that $z \in (d_1, d_2)\uparrow$ and $z \in F$. By definition of $d_1$ and $d_2$, we get that $d_1 = z = d_2$. Hence $\text{Min}(F)$ has the least element say $d$. Therefore $F = [d]$ is a principal filter.

We prove some characterizations of prime ideals and semiprime ideals in the case of finite posets. For this we introduce the concept of an $I$-atom in posets. Beran [1995b] defined the concept of an $I$-atom in lattices and has shown that this concept plays very crucial role in the study of prime ideals. Before stating the definition of an $I$-atom in posets, for the sake of completeness we note that an element $p$ of a poset $P$ is called an atom if

(i) $0 \prec p$ if $0$ is the least element of $P$, or

(ii) $p$ is a minimal element of $P$ if $P$ has no least element,

where $0 \prec p$ means there is no element $x \in P$ such that $0 < x < p$ holds. Dually, we have a concept of a coatom of $P$.

Definition 2.13. Let $I$ be an ideal of a poset $P$. An element $i \in P$ is called an $I$-atom if the following conditions hold.

(i) $i \notin I$, and

(ii) for $x \in P$, if $x < i$, then $x \in I$.

Dually, we have an $F$-coatom for a given filter $F$ of $P$. 
Remarks 2.14. (1) Consider the ideal $I = (a]$ of the poset $Q$ depicted in Figure 2. Observe that $b$ is an $I$-atom of $Q$ but not an atom. Also, $a$ is an atom of $Q$ but not an $I$-atom.

(2) Let $P$ be a poset. From the definitions of an atom and an $I$-atom we observe the following.
(i) If $P$ has the least element $0$, then $i \in P$ is a $(0]$-atom if and only if $i$ is an atom of $P$.
(ii) If $P$ has no least element, then $i \in P$ is a $\phi$-atom if and only if $i$ is an atom of $P$.

(3) Note that for every finite poset $P$ and $I$ a proper ideal of $P$, an $I$-atom exists. However, an $I$-atom need not exist infinite poset. For example in the poset $Q$ depicted in Figure 4, observe that for the ideal $I = \{0\}$, there is no $I$-atom.

(4) Note that two distinct $I$-atoms of a poset $P$ are incomparable. Indeed, if $i_1, i_2$ are two distinct $I$-atoms of a poset $P$ and $i_1 < i_2$, then by (ii) of the Definition 2.13, we get that $i_1 \in I$, which is a contradiction to part (i) of the same definition.

Theorem 2.15. Let $I$ be an ideal of a poset $P$. Then $I$ is semiprime if and only if $I:x$ is an ideal for all $x \in P$, in fact, a semiprime ideal. Moreover, if $P$ is finite, then $I$ is semiprime if and only if $I:i$ is a principal prime ideal for all $I$-atoms of $P$.

Proof: First we show that $I:x$ is an ideal for all $x \in P$. For
this assume that $I$ is semiprime and $a, b \in I:x$. We have to show that $(a, b)^{ul} \subseteq I:x$. Since $a, b \in I:x$, we have $(x, a)^l \subseteq I$ and $(x, b)^l \subseteq I$. By semiprimeness, we get $\{x, (a, b)^u\}^l \subseteq I$. By Lemma 2.10(ii), we have $(a, b)^{ul} \subseteq I:x$.

Now, we show that $I:x$ is semiprime. Suppose that $(a, b)^l \subseteq I:x$ and $(a, c)^l \subseteq I:x$. We obtain $(x, a, b)^l \subseteq I$ and obtain $(x, a, c)^l \subseteq I$, thus by Lemma 2.10(i) we obtain $(x, a)^l \subseteq I:b$ and $(x, a)^l \subseteq I:c$. We claim that $\{x, a, (b, c)^u\}^l \subseteq I$. Indeed, let $z \in \{x, a, (b, c)^u\}^l$. Since $z \in (x, a)^l \subseteq I:b$ and $z \in (b, c)^{ul}$, we have $z \in I:b$ and $z \in I:c$, i.e., $b, c \in I:z$. Since $I:z$ is an ideal, $(b, c)^{ul} \subseteq I:z$ and $z \in (b, c)^{ul}$, which yields $z \in I:z$. Thus $z \in I$.

Now, to prove that $\{a, (b, c)^u\}^l \subseteq I:x$, let $z \in \{a, (b, c)^u\}^l$. Clearly we have $(x, z)^l \subseteq (x, a, (b, c)^u)^l \subseteq I$. Hence $z \in I:x$, so $\{a, (b, c)^u\}^l \subseteq I:x$. Therefore $I:x$ is semiprime.

Conversely, suppose $I:x$ is an ideal for all $x \in P$. We show that $I$ is semiprime. Let $(x, y)^l \subseteq I$ and $(x, z)^l \subseteq I$. Since $y, z \in I:x$ and $I:x$ is an ideal, we have $(y, z)^{ul} \subseteq I:x$. By using of Lemma 2.10(ii), we get $\{x, (y, z)^u\}^l \subseteq I$ as required.

Further, let $P$ be a finite, $I$ be a semiprime ideal of $P$ and $i$ be an $I$-atom of $P$. First we show that $I:i$ is prime. For this assume $(x, y)^l \subseteq I:i$ and $x \notin I:i$. As $x \notin I:i$, we have $(x, i)^l \subseteq I$, there exists an element $b \in (x, i)^l$ and $b \notin I$, and consequently, there exists $I$-atom $j$ of $P$ such that $j \leq b \leq i$. Thus we must have
$j = i$, and therefore $(x, i)^* = i^*$, i.e., $i \leq x$. Since $(x, y)^* \subseteq I:i$, by Lemma 2.10 (i), we obtain $(x, y, i)^* \subseteq I$. Therefore $(y, i)^* \subseteq I$, $y \in I:i$ and thus $I:i$ is prime.

We show that $I:i$ is principal. In view of the statement dual to Lemma 2.12, it suffices to show that $I:i$ is $u$–ideal. Suppose on the contrary that $I:i$ is not an $u$–ideal. Then there exists $b, c \in I:i$ such that there is no $x \in (b, c)^u$ for which $(i, x)^* \subseteq I$. Denote $(b, c)^u = \{x_1, x_2, \ldots, x_n\}$. Then $(i, x_j)^* \not\subseteq I$, for all $x_j$, $j = 1, 2, \ldots, n$. Therefore $(i, x_1)^* = (i, x_2)^* = \ldots = (i, x_n)^* = i^*$ and hence $i \leq x_j$ for all $x_j \in (b, c)^u$. Since $I$ is semiprime, we have $\{i, (b, c)^u\}^* \subseteq I$. Therefore $i^* = \{i, (b, c)^u\} \subseteq I$, which is a contradiction to the fact that $i \not\in I$. Thus $I:i$ is an $u$–ideal.

Conversely, suppose that $(a, b)^* \subseteq I$, $(a, c)^* \subseteq I$, and suppose $x \in\{a, (b, c)^u\}^*$ but $x \not\in I$. Then there exists an $I$-atom $i$ of $P$ such that $i \leq x$. Hence $(i, b)^* \subseteq (a, b)^* \subseteq I$ and $(i, c)^* \subseteq (a, c)^* \subseteq I$, which yields $b, c \in I:i$. Since $I : i$ is an ideal and $i \in (b, c)^{ul}$, we have $i \in (b, c)^{ul} \subseteq I:i$, and thus $i \in I$, a contradiction to the fact that $i$ is an $I$-atom.

As noted in Remark 2.5, we give the converse of Lemma 2.4 in terms of maximal ideals and so we have one more characterization of a semiprime ideal to be prime. Let $I$ be a proper ideal of a poset $P$. Then $I$ is said to be a maximal ideal of $P$.
if the only ideal properly containing $I$ is $P$. A maximal filter more usually known as an ultrafilter, is defined dually. Also, we have the concepts of minimal ideal and minimal filter.

The following statement is another characterization of semiprime ideals to be prime ideals.

**Theorem 2.16.** Every maximal semiprime ideal of a poset $P$ is a prime ideal.

**Proof:** Let $I$ be a maximal semiprime ideal of $P$ and $(x, y)^I \subseteq I$. We have $x \in I: y$. Since $I \subseteq I: y$, we have following two cases:

(i) If $I = I: y$, then $x \in I$.

(ii) Suppose $I \subset I: y$. But by Theorem 2.15, $I: y$ is also a semiprime ideal and since $I$ is maximal semiprime, we must have $I: y = P$. So, by Lemma 2.10(iii), $y \in I$.

As a consequence, we have

**Corollary 2.17.** Let $I$ be a maximal ideal of a poset $P$. Then $I$ is semiprime if and only if $I$ is prime.

By using Theorem 2.16 and Theorem 2.6, we have

**Corollary 2.18.** Every maximal semiprime ideal of a poset $P$ is a meet irreducible element of $Id(P)$.

The following is a characterization of an ideal $I$ to be prime in terms of an $I$-atom in a finite poset $P$.

**Theorem 2.19.** Let $I$ be an ideal of a finite poset. Then
I is prime if and only if \( P \) has exactly one \( I \)-atom.

**Proof:** Suppose \( I \) is a prime ideal. If \( I \) has two different atoms \( i_1 \) and \( i_2 \), then \( (i_1, i_2)^I \subseteq I \), but \( i_1, i_2 \notin I \), a contradiction.

Conversely, suppose \( P \) has exactly one \( I \)-atom and \( I \) is not a prime ideal. Then there exist \( a, b \in P \) such that \( (a, b)^I \subseteq I \) and neither \( a \) nor \( b \) is in \( I \). However, there exist two \( I \)-atoms of \( P \), say \( i_1 \) and \( i_2 \) for which \( i_1 \leq a, i_2 \leq b, i_1 \neq i_2 \), a contradiction to the fact that \( P \) has exactly one \( I \)-atom.

Next is also a characterization of an ideal \( I \) to be prime in the terms of \( I:x \), where \( x \notin I \) in a poset \( P \).

**Theorem 2.20.** Let \( I \) be an ideal of a poset \( P \). Then \( I \) is prime if and only if \( I:x=I \) for all \( x \in P - I \).

**Proof:** Suppose that \( I \) is prime, \( z \in I:x \) and \( x \in P - I \). Since \( I \) is prime, \( (z, x)^I \subseteq I \) and \( x \notin I \), we have \( z \in I \).

Conversely, assume \( I:x=I \) for all \( x \in P - I \) and let \( (x, y)^I \subseteq I \). If \( x \notin I \), then \( y \in I:x=I \). Thus \( I \) is prime.

Corollary 2.22 is an analogue of the Krull's result [1929] in the case of finite posets. In fact, it is a representation of a semiprime ideal \( I \) as an intersection of all prime ideals determined by \( I \)-atoms which, by definition, contain \( I \).

**Lemma 2.21.** Let \( I \) be a proper ideal of a finite poset \( P \).
Then $I = \bigcap_i I:i$ for all $I$-atoms $i$ of $P$.

**Proof:** We show that $\bigcap_i I:i \subseteq I$, as the converse inclusion always holds. Suppose on the contrary that $z \in \bigcap_i I:i$ and $z \notin I$. Then there exists an $I$-atom $j \in P$ such that $j \leq z$ and $j \notin I$. Since $z \in \bigcap_i I:i$, we have $z \in I:j$, which gives $(j, z)^I \subseteq I$, i.e., $j \in I$, as $j \leq z$, a contradiction to the fact that $j$ is an $I$-atom.

An immediate consequence of Theorem 2.15 and Lemma 2.21 is

**Corollary 2.22.** Every proper semiprime ideal $I$ of a finite poset $P$ is representable as an intersection of prime ideals.

**Lemma 2.23.** The intersection of any non empty family of prime ideals of a poset $P$ is a semiprime ideal.

**Proof:** Suppose $I = \bigcap_n J_n$, $n \in \Gamma$, where $\Gamma$ is some indexed set and where each $J_n$ is a prime ideal and $(x, y)^I \subseteq I$, $(x, z)^I \subseteq I$. We have to show that $\{x, (y, z)^u\}^I \subseteq I$. Since $(x, y)^I \subseteq J_n$ and $(x, z)^I \subseteq J_n$ for each $n$, by primeness of $J_n$’s we have $x \in J_n$ or $y, z \in J_n$ for each $n$. In either case, we have $\{x, (y, z)^u\}^I \subseteq J_n$ for each $n$. Hence $\{x, (y, z)^u\}^I \subseteq \bigcap_n J_n = I$. Thus $I$ is semiprime.

As an immediate consequence of Theorem 2.15, Corollary
2.22 and Lemma 2.23 in the case of finite posets we obtain

**Theorem 2.24.** Let I be a proper ideal of a finite poset. Then I is semiprime if and only if I is representable as an intersection of prime ideals.

We characterize the distributive posets in terms of semiprime ideals in the following results. We consider the following definition of a distributive poset which is essentially due to Larmerová and Rachůnek [1988].

**Definition 2.25.** A poset $P$ is called *distributive*, if for all $a, b, c \in P$, \( \{a, (b, c)^u\}^l = \{(a, b)^l, (a, c)^l\}^u \).

![Figure 5](image)

**Figure 5**

Note that in a distributive poset every ideal need not be a semiprime ideal. Consider the distributive poset $Q$ depicted in Figure 5 and the ideal $I = \{0, a, b, c\}$ which is not semiprime as, $\langle d', b' \rangle^l \subseteq I$ and $\langle d', c' \rangle^l \subseteq I$ but $d'^l = \langle d', (b', c')^u \rangle^l \nsubseteq I$. 

But for principal ideals of $P$ we have the following result.

**Theorem 2.26.** Let $P$ be a poset. Then $P$ is distributive if and only if $(x]$ is a semiprime ideal for all $x \in P$.

**Proof:** ($\Rightarrow$) Suppose that $(a, b)^l \subseteq (x]$, $(a, c)^l \subseteq (x]$ and $z \in \{a, (b, c)^u\}^l$. We have $z^l = z^l \cap (b, c)^u = \{z, (b, c)^u\}^l$ and by distributivity, we get that $z^l = \{(z, b)^l, (z, c)^l\}^u$. Since $z \leq a$, we obtain $(z, b)^l \subseteq (a, b)^l \subseteq (x]$ and $(z, c)^l \subseteq (a, c)^l \subseteq (x]$. Therefore $(z, b)^l \cup (z, c)^l \subseteq (x]$ , i.e., $\{(z, b)^l, (z, c)^l\}^u \subseteq (x]$. Thus $z^l \subseteq (x]$ and we get $z \in (x]$.

($\Leftarrow$) It is enough to prove $\{a, (b, c)^u\}^l \subseteq \{(a, b)^l, (a, c)^l\}^u$, as the converse inclusion is always true. Let $x \in \{a, (b, c)^u\}^l$ and $y \in \{(a, b)^l, (a, c)^l\}^u$. We claim that $x \leq y$. Since $\{(a, b)^l, (a, c)^l\}^u \subseteq y^l$, we have $(a, b)^l \subseteq y^l$ and $(a, c)^l \subseteq y^l$. By semiprimeness of $y$ we get that $x \in \{a, (b, c)^u\}^l \subseteq y^l$. Thus $x \leq y$ as required.

An immediate consequence of Theorem 2.24 and Theorem 2.26 is

**Corollary 2.27.** Let $P$ be a finite poset. Then $P$ is distributive if and only if every proper principal ideal is representable as an intersection of prime ideals.

In what follows are the sets that are used to characterize prime ideals in a poset and to study the separation of a poset.
For an ideal $I$ of a poset $P$, consider the set $F_I$ as follows:

$$F_I = \{ z \in P \mid I:z \subseteq I \}.$$ 

In the following result, we establish some properties of $F_I$ and its connections with $I$.

**Lemma 2.28.** Let $I$ be a proper semiprime ideal of a poset $P$. Then $I:x \cap F_I = \emptyset$ for all $x \in P - I$.

**Proof:** Let $x \in P - I$ and $z \in I:x \cap F_I$. We have $(z, x)^I \subseteq I$ and $I:z \subseteq I$. Hence $x \in I:z = I$, a contradiction.

The following Theorem is the first result toward a Separation of a poset $P$.

**Theorem 2.29.** Let $I$ be an ideal of a poset $P$. Then $F_I$ is a filter. Moreover, if $P$ is finite and $I$ is semiprime, then $F_I$ is semiprime.

**Proof:** Suppose $x, y \in F_I$ and $z \in (x, y)^I$. To show that $z \in F_I$, it is enough to show that $I:z \subseteq I$. Let $a \in I:z$. Then $(z, a)^I \subseteq I$. Since $(x, y)^I \subseteq z^I$, we obtain $(x, y, a)^I \subseteq (z, a)^I \subseteq I$. This yields by Lemma 2.10(i) $(y, a)^I \subseteq I:x$. Since $x \in F_I$, we get $(y, a)^I \subseteq I = I:x$. Hence $a \in I:y = I$, as $y \in F_I$.

Now, suppose that $I$ is a semiprime ideal of a finite poset $P$. We have to show that $F_I$ is semiprime. Suppose that $(x, y)^u \subseteq F_I$ and $(x, z)^u \subseteq F_I$. Let $a \in \{ x, (y, z)^I \}^u$ and $a \notin F_I$. Then $I \supset I:a$, therefore there exists an element $b \in P$ such that
b \in I:a$ and $b \notin I$. Consequently, there exists an $I$-atom $i$ of $P$ such that $i \leq b$. Clearly $i \in I:a$, and so $a \in I:i$. Now, since $a \in \{x, (y, z)^l\}^u$, we have $x \in I:i$ and $(y, z)^l \subseteq I:i$. By Theorem 2.15, $I:i$ is a prime principal ideal, hence either $y \in I:i$ or $z \in I:i$.

Suppose $y \in I:i$. Since $I:i$ is a principal ideal, it is $u$-ideal and so $x, y \in I:i$ implies $I:i \cap (x, y)^u \neq \phi$. But $(x, y)^u \subseteq F_I$. Therefore $I:i \cap F_I \neq \phi$, a contradiction with Lemma 2.28.

**Remark 2.30.** Consider the infinite poset $Q$ depicted in Figure 6, and $I = \{0, a, b, c\}$ which is a semiprime ideal. Observe that $F_I = \{y_j | j \in N\}$ and $F_I$ is a filter but it is not semiprime as $(a, b)^u \subseteq F_I$ and $(a, c)^u \subseteq F_I$ but $a^l = \{a, (b, c)^l\}^u \not\subseteq F_I$. 

\[\text{Figure 6}\]
Thus, Theorem 2.29 is not true if we drop the condition of finiteness.

However, if \( P \) is a join semilattice then we have Theorem 2.32 for which we need the following Lemma.

**Lemma 2.31.** Let \( I \) be a semiprime ideal of a join semilattice \( P \) and let \( x, y \in P \). Then \( I:(x \vee y) = I:x \cap I:y \).

**Proof:** We have \( a \in I:(x \vee y) \) if and only if \( (a, x \vee y)^l \subseteq I \) if and only if \( (a, x)^l \subseteq I \) and \( (a, y)^l \subseteq I \) if and only if \( a \in I:x \cap I:y \).

**Theorem 2.32.** Let \( I \) be a proper semiprime ideal of a join semilattice \( P \). Then \( F_I \) is a semiprime filter.

**Proof:** Let \( x \vee y, x \vee z \in F_I \) and \( a \in \{x, (y, z)^l\}^u \). If \( a \notin F_I \), there exists an element \( b \in P \) such that \( b \in I:a \) and \( b \notin I \), i.e., \( (a, b)^l \subseteq I \) and \( b \notin I \). Since \( x \leq a \) and \( (y, z)^l \subseteq a^l \), we have

\[(2.32.1) \quad b \in I:x \text{ and } (y, b)^l \subseteq I:z.\]

As \( b \in I:x \), we have \( (y, b)^l \subseteq I:x \). By Lemma 2.31 and (2.32.1), we have \( (y, b)^l \subseteq I:x \cap I:z = I:(x \vee z) = I \). Thus \( b \in I:y \) and by (2.32.1), \( b \in I:x \), which yields \( b \in I:(x \vee y) = I \), a contradiction.

**Lemma 2.33.** Let \( I \) be a proper ideal of a poset \( P \). Then \( I \cap F_I = \phi \).

**Proof:** Suppose \( x \in I \cap F_I \). We have \( I:x = I \). Since \( x \in I \), by
Lemma 2.10(iii), \( I:x=P \) and consequently \( I=I:x=P \), which is a contradiction to the fact that \( I \) is proper.

The following Theorem characterizes prime ideals in a poset.

**Theorem 2.34.** Let \( I \) be an ideal of a poset \( P \). Then \( I \) is prime if and only if \( I \cup F_I = P \).

**Proof:** Suppose \( I \) is prime, \( x \in P \) and \( x \notin F_I \). Since \( x \notin F_I \), we have that \( I \subseteq I:x \), i.e., there exists an element \( y \in I:x \) such that \( y \notin I \). In other words, \( (y,x)^I \subseteq I \) and \( y \notin I \). By primeness of \( I \), we get \( x \in I \) as required.

Conversely, suppose \( I \cup F_I = P \), \( (y,x)^I \subseteq I \) and \( x \notin I \). Clearly, \( x \in F_I \) and hence \( I:x=I \). Since \( y \in I:x \), we have \( y \in I \). Thus, \( I \) is Prime.

Observe that Lemma 2.33 and Theorem 2.34 show that a prime ideal \( I \) and the corresponding filter \( F_I \) separate elements of a poset \( P \).
§3. Ideal Lattice of Semiprime Ideals.

Recall that the set of all ideals of a poset $P$, namely $Id(P)$, is a complete lattice which is a generalization of the result for lattices. Also, for a given lattice $L$, the set of all semiprime ideals $SId(L)$ is a lattice with greatest element as $L$ itself, where the meet operation coincides with set-theoretical intersection (see Rav [1989]). In fact, Rav proved that $SId(L)$ is complete.

In this section, we have proved that the set of all semiprime ideals $SId(P)$ of a poset $P$ is also a complete lattice. The relations between semiprime ideals of $P$ and ideals of $Id(P)$ have been studied. Moreover, we establish the relation between the prime ideals of $P$ and the prime ideals of the lattice $Id(P)$.

A prime ideal $I$ in a lattice is characterized by the property that if $a \land b \in I$, then either $a \in I$ or $b \in I$. Rav proved that semiprime ideals enjoy a certain "half prime ideal" property using homomorphism and congruence relation. In fact, he proved the following.

**Lemma 3.1.** (Rav [1989]) Let $I$ be a semiprime ideal of a lattice $L$ and suppose that for some $a, b \in L$, $a \land b \in I$. Then there exist semiprime ideals $A$ and $B$ (possibly improper) such that $a \in A$, $b \in B$ and $I = A \cap B$.

We prove the analogue of this result (Theorem 3.13) for finite posets by using the concepts of an extension and the contrac-
tion (see Rav [1989]) which are generalizations of the respective concepts in lattices described as below:

If $I$ is an ideal of a lattice $L$ and $\lambda$ is an ideal of the lattice $Id(L)$, define its extension and the contraction, denoted by $I^e$ and $\lambda^c$ respectively, as follows:

$$I^e = \{J \in Id(L) | J \subseteq I\}, \quad \lambda^c = \bigcup\{J | J \in \lambda\}.$$

We begin by proving that the set $SId(P)$ is closed under the set-theoretical intersection, in fact, $SId(P)$ is a complete lattice.

**Lemma 3.2.** Let $X$ be a family of members of $SId(P)$. Then $\bigcap_{I \in X} I$ is also in $SId(P)$.

**Proof:** Let $(x, y)^l \subseteq \bigcap_{I \in X} I$ and $(x, z)^l \subseteq \bigcap_{I \in X} I$. We have $(x, y)^l \subseteq I$ and $(x, z)^l \subseteq I$ for all $I \in X$. By semiprimeness of $I$, we get that $\{x, (y, z)^u\}^l \subseteq I$ for all $I \in X$. Therefore $\bigcap_{I \in X} I \in SId(P)$.

From Lemma 3.2, Theorem 3.3 follows immediately.

**Theorem 3.3.** Let $P$ be a poset. Then $(SId(P), \subseteq)$ forms a complete lattice in which meet coincides with set intersection and the join is defined as follows.

For $I, J \in SId(P)$, $I \lor J = \bigcap_{Y \in SId(P)} Y$, where $I \cup J \subseteq Y$.

(May be $I \lor J$ is an improper.)

Let $P$ be a given poset. Define an extension of an ideal $I$
of $P$, denoted by $I^e$, as

$$I^e = \{ J \in \text{Id}(P) | J \subseteq I \}$$

and for an ideal $\lambda$ of the lattice $(\text{Id}(P), \subseteq)$, define the contraction of $\lambda$, denoted by $\lambda^e$, as

$$\lambda^e = \bigcup \{ J | J \in \lambda \}.$$

We prove in Theorem 3.4 that $I^e$ is an ideal of $\text{Id}(P)$ for every ideal $I$ of a poset $P$ and in Theorem 3.5 that $\lambda^e$ is an ideal of $P$ for every ideal $\lambda$ of $\text{Id}(P)$.

As a proposition toward our contention that the semiprime ideals in posets also enjoy the “half prime ideal” property, we establish the connection between the elements of $\text{Id}(P)$ and $\text{SId}(P)$.

**Theorem 3.4.** Let $I$ be a semiprime ideal of a poset $P$. Then $I^e$ is a semiprime ideal of the lattice $\text{Id}(P)$.

**Proof:** First, we observe that $I^e$ is an ideal of $\text{Id}(P)$. Indeed,

(i) if $J_1, J_2 \in I^e$, then $J_1 \subseteq I, J_2 \subseteq I$ and so $J_1 \lor J_2 \subseteq I$, consequently $J_1 \lor J_2 \in I^e$,

(ii) if $J_1 \subseteq J_2$ and $J_2 \in I^e$, then we have $J_1 \subseteq J_2 \subseteq I$ and thus $J_1 \in I^e$.

Now, we show that $I^e$ is semiprime. Suppose $J_1 \cap J_2 \in I^e$ and $J_1 \cap J_3 \in I^e$, so we have $J_1 \cap J_2 \subseteq I$ and $J_1 \cap J_3 \subseteq I$. We have to prove that $J_1 \cap (J_2 \lor J_3) \in I^e$. In view of Lemma 2.8,
it is sufficient to show by induction that $J_1 \cap C_n (J_2, J_3) \subseteq I$ for each $n \in \mathbb{N}$.

(1) Suppose $n = 1$ and $x \in J_1 \cap C_1 (J_2, J_3)$. Then $x \in J_1$ and $x \in (a, b)^u$ for $a, b \in J_2 \cup J_3$. If $a, b \in J_2$ or $J_3$, then obviously $x \in I$. So suppose, without loss of generality, that $a \in J_2$ and $b \in J_3$. We have $(x, a)^l \subseteq J_1 \cap J_2 \subseteq I$ and $(x, b)^l \subseteq J_1 \cap J_3 \subseteq I$. By semiprimeness of $I$, we get $\{x, (a, b)^u\}^l \subseteq I$. Since $x \in (a, b)^u$, we have $x^l = x^l \cap (a, b)^u = \{x, (a, b)^u\}^l \subseteq I$, therefore $x \in I$. Thus the statement is true for $n = 1$.

(2) Suppose now that $J_1 \cap C_n (J_2, J_3) \subseteq I$ holds for some $n \in \mathbb{N}$. We will prove that it also holds for $n + 1$.

Suppose on the contrary that there exists some element $x \in P$ such that $x \in J_1 \cap C_{n+1} (J_2, J_3)$ but $x \notin I$. We have, $x \in J_1$ and $x \in (a, b)^u$ for some $a, b \in C_n (J_2, J_3)$. Also, we claim that $(x, a)^l \subseteq I$ and $(x, b)^l \subseteq I$ can not hold simultaneously. Otherwise, if $(x, a)^l \subseteq I$ and also $(x, b)^l \subseteq I$, then by semiprimeness of $I$, we have $\{x, (a, b)^u\}^l \subseteq I$, i.e., $x^l = x^l \cap (a, b)^u \subseteq I$, a contradiction to the fact that $x \notin I$. Now, suppose that $(x, a)^l \not\subseteq I$, so there exists an element $y \in P$ such that $y \in (x, a)^l$ and $y \notin I$. Observe that $y \leq x$ implies $y \in J_1$. Since $a \in C_n (J_2, J_3)$, we have $a \in (a_1, b_1)^u$ for some $a_1, b_1 \in C_{n-1} (J_2, J_3)$. But $y \leq a$, hence $y \in (a_1, b_1)^u$ for some $a_1, b_1 \in C_{n-1} (J_2, J_3)$. Thus, $y \in C_n (J_2, J_3)$ and therefore
By induction hypothesis, $y \in I$, which is a contradiction to the fact that $y \notin I$. Therefore $x \in I$ as required.

Therefore $J_1 \cap C_n(J_2, J_3) \subseteq I$ holds for each $n \in N$ and consequently $I^e$ is semiprime.

**Theorem 3.5.** Let $P$ be a finite poset and let $\lambda$ be a semiprime ideal of $\text{Id}(P)$. Then $\lambda^c$ is a semiprime ideal of $P$.

**Proof:** First we prove that $\lambda^c$ is an ideal. Consider the elements $x, y \in \lambda^c$. If $x$ and $y$ belong to some $J \in \lambda$, then the result is obvious. Suppose there exist $J_1, J_2 \in \lambda$ such that $x \in J_1$ and $y \in J_2$, $J_1 \neq J_2$, then we have $(x, y)^u \subseteq J_1 \lor J_2 \in \lambda$, as $\lambda$ is an ideal. Thus $\lambda^c$ is an ideal of $P$.

Now, we show that $\lambda^c$ is semiprime. Suppose $\langle x, y \rangle^l \subseteq \lambda^c$ and $\langle x, z \rangle^l \subseteq \lambda^c$ and we claim that $\{x, (y, z)^u\}^l \subseteq \lambda^c$. Since $\langle x, y \rangle^l \subseteq \lambda^c$, we have $\langle x, y \rangle^l \subseteq J_1 \cup \ldots \cup J_n$, where $J_1, \ldots, J_n \in \lambda$, as $P$ is finite. Hence $\langle x \rangle \cap \langle y \rangle \subseteq J_1 \lor \ldots \lor J_n \in \lambda$. Therefore $\langle x \rangle \cap \langle y \rangle \in \lambda$. Similarly, $\langle x \rangle \cap \langle z \rangle \in \lambda$. By semiprimeness of $\lambda$, we get $\langle x \rangle \cap \{(y) \lor (z)\} \in \lambda$. Thus $\{x, (y, z)^u\}^l \subseteq \lambda^c$.

**Remark 3.6.** The finiteness condition in the statement of the Theorem 3.5 is necessary. For, consider the infinite poset
Consider $\lambda = \{\{y_i\}\}$, where $i = 1, 2, \ldots$, which is a semiprime ideal of $Id(Q)$. Now, $\lambda^c = \{y_i\}, i = 1, 2, \ldots$ and which is not a semiprime ideal of $Q$, as $(a, b)^l \subseteq \lambda^c$ and $(a, c)^l \subseteq \lambda^c$ but $a^l = \{a, (b, c)^u\}^l \not\subseteq \lambda^c$.

However, if the poset is a meet semilattice, then we have

**Theorem 3.7.** Let $P$ be a meet semilattice and $\lambda$ be a semiprime ideal of $Id(P)$. Then $\lambda^c$ is a semiprime ideal of $P$.

**Proof:** Suppose $\lambda$ is a semiprime ideal of $Id(P)$ and suppose that $x \land y \in \lambda^c$ and $x \land z \in \lambda^c$. Then we have $x \land y \in J_1$ and $x \land z \in J_2$ for some $J_1, J_2 \in \lambda$. Since $(x) \cap (y) = (x \land y) \subseteq J_1$
and $J_1 \in \lambda$, we have $[x] \cap [y] \in \lambda$ and similarly $[x] \cap [z] \in \lambda$. By semiprimeness of $\lambda$, we have $[x] \cap \{(y] \lor (z]\} \in \lambda$. Thus $\{x, (y, z)^u\} \subseteq \lambda^c$ and therefore $\lambda^c$ is semiprime.

In the next two Theorems we investigate the relationships between prime ideals of $P$ and those of $Id(P)$.

**Theorem 3.8.** Let $I$ be a prime ideal of a poset $P$. Then $I^e$ is a prime ideal of the lattice $Id(P)$.

**Proof:** Suppose $I$ is a prime ideal of a poset $P$ and $J \cap L \in I^e$ for $J, L \in Id(P)$. Then we must have either $J$ or $L$ must be in $I^e$. Indeed, on the contrary if both $J, L \notin I^e$, then $J \nsubseteq I$ and $L \nsubseteq I$ and so there exist $x, y \in P$ such that $x \in J$, $y \in L$ and $x, y \notin I$. But $(x, y)^l \subseteq J \cap L \subseteq I$, which is a contradiction to the fact that $I$ is prime.

**Theorem 3.9.** Let $P$ be a finite poset and let $\lambda$ be a prime ideal of $Id(P)$. Then $\lambda^c$ is a prime ideal of $P$.

**Proof:** Suppose that $\lambda$ is a prime ideal of $Id(P)$ where $P$ is a finite poset. By Theorem 3.5, $\lambda^c$ is an ideal of $P$. To prove that $\lambda^c$ is prime, suppose $(x, y)^l \subseteq \lambda^c$ for $x, y \in P$. As in the proof of Theorem 3.5, we have $(x] \cap [y] \in \lambda$ and since $\lambda$ is prime we must have $(x] \in \lambda$ or $(y] \in \lambda$. Consequently $(x] \subseteq \lambda^c$ or $(y] \subseteq \lambda^c$ and therefore $x \in \lambda^c$ or $y \in \lambda^c$. 
Remark 3.10. The statement of Theorem 3.9 is not necessarily true if the poset $P$ is not finite. Consider the infinite poset $Q$ depicted in Figure 7 and its ideal lattice $Id(Q)$ in Figure 8. Observe that $\lambda = \{(y_i)\}, i = 1, 2, ..., $ is a prime ideal of $Id(Q)$. Also, see that $\lambda^c = \{y_i\}$, where $i = 1, 2, ...$, is an ideal of $Q$ but not prime, as $(b, c)^l \subseteq \lambda^c$ and neither $b$ nor $c$ is in $\lambda^c$.

However, if the poset is a meet semilattice, then we have

**Theorem 3.11.** Let $P$ be a meet semilattice and $\lambda$ be a prime ideal of $Id(P)$. Then $\lambda^c$ is a prime ideal of $P$.

**Proof:** Suppose that $\lambda$ is a prime ideal of $Id(P)$ where $P$ is a meet semilattice and $x \land y \in \lambda^c$ for $x, y \in P$. Then $x \land y \in J$ for some $J \in \lambda$. Therefore $(x) \cap (y) \subseteq J$ and since $J \in \lambda$, we have $(x) \cap (y) \in \lambda$. Now, by primeness of $\lambda$, we must have $(x) \in \lambda$ or $(y) \in \lambda$ and so $x \in \lambda^c$ or $y \in \lambda^c$. Therefore $\lambda^c$ is prime.

**Lemma 3.12.** Let $I$ be an ideal of a poset $P$. Then $I^{ec} = I$.

**Proof:** Suppose $I$ is an ideal of a poset $P$ and $x \in I^{ec}$ where $x \in P$. Then $x \in J$, for some $J \in I^e$. In other words, $(x) \subseteq J$ and $J \in I^e$. Hence $(x) \in I^e$, which yields $(x) \subseteq I$, i.e., $x \in I$.

On the other hand, if $x \in I$ then $(x) \subseteq I$. Thus we have $(x) \in I^e$ and hence $(x) \subseteq I^{ec}$, which yields $x \in I^{ec}$.
The following Theorem is a generalization of Lemma 3.1 for finite posets.

**Theorem 3.13.** Let $I$ be a semiprime ideal of a finite poset $P$ and suppose that for some $a, b \in P$, $(a, b)^I \subseteq I$. Then there exist semiprime ideals $X$ and $Y$ (possibly improper) such that $a \in X$, $b \in Y$ and $I = X \cap Y$.

**Proof:** Suppose $I$ is a semiprime ideal of a finite poset $P$ and suppose that for some $a, b \in P$, $(a, b)^I \subseteq I$. If $a, b \in I$, then set $I = X = Y$ and the result is true. Also, if one of $a$ and $b$ is in $I$, say $a \in I$ but $b \notin I$, then set $X = I$, $Y = P$ and in this case also the result is true. So, suppose that $a, b \notin I$.

By Theorem 3.4, $I^e$ is a semiprime ideal of a finite poset $P$ and by definition of $I^e$, $(a, b)^I \subseteq I$ implies $(a] \cap [b] \in I^e$. Now, by Lemma 3.1, there exist semiprime ideals $A$ and $B$ of $Id(P)$ such that $(a] \in A$ and $(b] \in B$ and $I^e = A \cap B$. Since $I = I^{ec}$ by Lemma 3.12, we have $I = (A \cap B)^c$.

Since $A$ and $B$ are semiprime ideals of $Id(P)$, by Theorem 3.5, $A^c$ and $B^c$ are semiprime ideals of $P$. Observe that $a \in A^c$ and $b \in B^c$. If we show $(A \cap B)^c = A^c \cap B^c$, then proof will complete by taking $X = A^c$ and $Y = B^c$.

For, let $x \in (A \cap B)^c$ for some $x \in P$. Then there exists some $J \in A \cap B$ such that $x \in J$. Consequently, $x \in J$ and
$J \in A$, $J \in B$ and thus $x \in A^c \cap B^c$. Now, suppose that $x \in A^c \cap B^c$ for some $x \in P$. We have $x \in J_1 \cap J_2$, where $J_1 \in A$ and $J_2 \in B$. Thus $x \in J_1 \cap J_2 = J$ and $J \in A \cap B$. Therefore $x \in (A \cap B)^c$ as required.

Remark 3.14. The statement of Theorem 3.13 is not necessarily true if we drop the finiteness condition on $P$. For, consider the infinite poset $Q$ depicted in Figure 9 and the semiprime ideal $I = \{x_j\} \cup \{y_j\}$, where $j = 1, 2, \ldots$. Observe that $(a, b)^I \subseteq I$, 

\begin{center}
\begin{tikzpicture}
    \node (a) at (0,0) {$a$};
    \node (b) at (0,2) {$b$};
    \node (c) at (0,4) {$c$};
    \node (x1) at (-1,6) {$x_1$};
    \node (x2) at (-1,7) {$x_2$};
    \node (x3) at (-1,8) {$x_3$};
    \node (y1) at (1,6) {$y_1$};
    \node (y2) at (1,7) {$y_2$};
    \node (y3) at (1,8) {$y_3$};
    \draw (a) -- (c);
    \draw (b) -- (c);
    \draw (x1) -- (x2) -- (x3);
    \draw (y1) -- (y2) -- (y3);
\end{tikzpicture}
\end{center}

Figure 9
$X = (a]$ and $Y = (b]$ are semiprime ideals of $Q$. Thus $a \in X$, $b \in Y$, but $X \cap Y = \{y_j\} \neq I$.

However, if $P$ is a meet semilattice, then by Theorem 3.7 we have

**Corollary 3.15.** Let $I$ be a semiprime ideal of a meet semilattice $P$ and suppose that for some $a, b \in P$, $(a, b)^l \subseteq I$. Then there exist semiprime ideals $X$ and $Y$ (possibly improper) such that $a \in X$, $b \in Y$ and $I = X \cap Y$. 

§4. SEPARATION THEOREM IN POSETS.

We recall from the introduction that Stone [1936] proved the Separation Theorem (Theorem A) for prime ideals in the case of distributive lattices and Rav [1989] proved the Separation Theorem (Theorem B) for semiprime ideals in general lattices.

In this section, we obtain a filter \( F \) of a poset \( P \) with the help of given filter \( \beta \) of \( Id(P) \) and study the semiprimeness connection between them. Further, we generalize Theorem A and Theorem B for finite posets and we also show by means of counterexamples that in the case of infinite posets Theorem A and Theorem B need not be true.

Let \( P \) be a poset and \( \beta \) be a filter of \( Id(P) \). Define a subset \( F \) of \( P \) as follows.

\[
F = \{ h \in P \mid (h) \in \beta \}
\]

We have the following Lemma.

**Lemma 4.1.** Let \( P \) be a poset and \( F \) be a subset of \( P \) defined as in (\(*\)). Then \( F \) is a filter of \( P \).

**Proof:** Let \( x, y \in F \) and \( z \in (x, y)_{lu} \). We have \( (x), (y) \in \beta \) and \( (x, y)^l \subseteq z^l \). Since \( \beta \) is a filter, \( (x) \cap (y) \in \beta \) and \( (x) \cap (y) \subseteq (z) \), we get \( (z) \in \beta \) and therefore \( z \in F \).

In the case of finite posets we have the following.

**Lemma 4.2.** Let \( P \) be a finite poset and \( F \) be a filter of \( P \).
defined as in (*). Then the following statements hold.

(i) If \((x, y)^u \subseteq F\), then \(x \vee y \in \beta\), for \(x, y \in P\).

(ii) If \(\beta\) is semiprime, then \(F\) is semiprime.

**Proof:** (i) Suppose \(P\) is a finite poset and \((x, y)^u \subseteq F\) for \(x, y \in P\). Since \(P\) is a finite poset, we have \((a_i) \in \beta\) for all \(a_i \in (x, y)^u\), where \(i = 1, 2, \ldots, n\). Also \(\beta\) is a filter of a finite lattice \(Id(P)\) implies that \(\bigcap_{i=1}^{n} (a_i) \subseteq \beta\). To prove \(x \vee y \in \beta\), it is sufficient to show that \(\bigcap_{i=1}^{n} (a_i) \subseteq (x \vee y)\). Let \(z \in \bigcap_{i=1}^{n} (a_i)\). It is clear that \(z \leq a_i\) for all \(a_i \in (x, y)^u\), i.e., \(z\) is a lower bound for the set \((x, y)^u\). Hence \(z \in (x, y)^u\), which yields \(z \in (x \vee y)\), as required.

(ii) Let \((x, y)^u \subseteq F\) and \((x, z)^u \subseteq F\). We have to prove that \(\{x, (y, z)^l\}^u \subseteq F\). Suppose \(a \in \{x, (y, z)^l\}^u\). We have \(x \leq a\) and \(a \in (y, z)^{lu}\), i.e., \((x) \subseteq (a)\) and \((y) \cap (z) \subseteq (a)\). Therefore \((x) \vee ((y) \cap (z)) \subseteq (a)\).

Now \((x, y)^u \subseteq F\) and \((x, z)^u \subseteq F\), by (i) we have, \((x) \vee (y) \in \beta\) and \((x) \vee (z) \in \beta\). By semiprimeness of \(\beta\), we get \((x) \vee ((y) \cap (z)) \in \beta\) and since \(\beta\) is a filter, we have \((a) \in \beta\). Consequently \(a \in F\) and semiprimeness of \(F\) follows.

\[\Box\]

**Remark 4.3.** We give an example to show that the assertion of Lemma 4.2 is not necessarily true if we drop the finiteness condition. Consider the poset \(Q^d\) that is the dual of the infinite
poset $Q$ depicted in Figure 7 and its ideal lattice $Id(Q^d)$ which is the dual of the ideal lattice $Id(Q)$ depicted in Figure 8. Consider the filter $\beta = \{(y_i)\}$ of $Id(Q^d)$, where $i = 1, 2, ...$. Observe that $F = \{y_i\}$, $i = 1, 2, ...$, is a filter of $Q^d$ and $(a, b)^u \subseteq F$ but $(a) \lor (b) = \{(y_1, y_2, ...)\} \notin \beta$. Moreover, $\beta$ is a semiprime filter of $Id(Q^d)$. But $F$ is not semiprime, as $(a, b)^u \subseteq F$ and $(a, c)^u \subseteq F$, but $a^u = \{a, (b, c)^l\}^u \subseteq F$.

However, in the case of join semilattices we have

**Corollary 4.4.** Let $P$ be a join semilattice and $F$ be a filter of $P$ defined as in ($\ast$). Then the following statements hold:

(i) If $x \lor y \in F$, then $(x) \lor (y) \in \beta$.

(ii) If $\beta$ is semiprime, then $F$ is semiprime.

Now, let $K$ be an $l$-filter of a poset $P$. Define a subset $\alpha$ of $Id(P)$ as follows

($\ast\ast$) $\alpha = \{J \in Id(P) | J \cap K \neq \phi\}$.

We establish the following result.

**Lemma 4.5.** Let $P$ be a poset and $K$ be an $l$-filter of $P$ and let $\alpha$ be a subset of $Id(P)$ as defined in ($\ast\ast$). Then $\alpha$ is a filter of $Id(P)$.

**Proof:** (i) Let $J_1, J_2 \in \alpha$. We have to show that $J_1 \cap J_2 \in \alpha$. Since $J_1 \cap K \neq \phi$ and $J_2 \cap K \neq \phi$, there exist $x, y \in P$ such that $x \in J_1 \cap K$ and $y \in J_2 \cap K$. In other words, $(x, y)^l \subseteq J_1 \cap J_2$ and $x, y \in K$. Since $K$ is an $l$-filter, there exists an element
\(z \in (x, y)^I\) such that \(z \in K\). Now \(z \in J_1 \cap J_2\) and \(z \in K\), i.e., \((J_1 \cap J_2) \cap K \neq \emptyset\), which yields \(J_1 \cap J_2 \in \alpha\).

(ii) Let \(J_1 \subseteq J_2\) and \(J_1 \in \alpha\). It is easy to observe that \(J_2 \in \alpha\).

From (i) and (ii), \(\alpha\) is a filter of \(\text{Id}(P)\).

Now, we extend Theorem B for finite posets with some constraint on the filter.

**Theorem 4.6.** Let \(I\) be a semiprime ideal and \(K\) be an \(l\)-filter of a finite poset \(P\) for which \(I \cap K = \emptyset\). Then there exists a semiprime filter \(F\) of \(P\) such that \(K \subseteq F\) and \(I \cap F = \emptyset\).

**Proof:** Suppose \(I\) is a semiprime ideal and \(K\) is an \(l\)-filter of a finite poset \(P\) such that \(I \cap K = \emptyset\). By Theorem 3.4, \(I^e\) is a semiprime ideal of \(\text{Id}(P)\). The set \(\alpha\) defined as in (**) is a filter of \(\text{Id}(P)\) by Lemma 4.5. We claim that \(I^e \cap \alpha = \emptyset\). Were this false, then there exists \(J \in I^e \cap \alpha\). Thus \(J \subseteq I\) and \(J \cap K \neq \emptyset\). In other words, \(I \cap K \neq \emptyset\), which contradicts the hypothesis.

Now, since \(\text{Id}(P)\) is a lattice, by Theorem B, there exists a semiprime filter \(\beta\) of \(\text{Id}(P)\) such that \(\alpha \subseteq \beta\) and \(I^e \cap \beta = \emptyset\). Consider the set \(F = \{h \in p|(h) \in \beta\}\), which is a semiprime filter of \(P\) by Lemma 4.2(ii). Observe that \(K \subseteq F\); for, if
Chapter-I 4. Separation Theorem in Posets

$x \in K$ then \((x] \cap K \neq \phi\), i.e., \((x] \in \alpha \subseteq \beta\) and thus \(x \in F\). Further, we must have \(I \cap F = \phi\); otherwise if \(I \cap F \neq \phi\), then there exists \(x \in P\) such that \(x \in I\) and \(x \in F\), i.e., \((x] \subseteq I\) and \((x] \in \beta\). In other words, \((x] \in I^e \cap \beta\), which is a contradiction to the fact that \(I^e \cap \beta = \phi\).

The statement of Theorem 4.6 is not necessarily true if we remove the finiteness condition or if \(K\) is not an \(l\)-filter.

Remarks 4.7. (i) Consider the poset \(Q^d\) that is dual of the infinite poset \(Q\) depicted in Figure 7. Let \(I = \{0, a, b, c\}\) and \(K = \{y_i\}\), where \(i = 1, 2, \ldots\). Observe that \(I\) is a semiprime ideal and \(K\) is an \(l\)-filter such that \(I \cap K = \phi\). But there does not exist a semiprime filter \(F\) for which \(K \subseteq F\) and \(I \cap F = \phi\).

(ii) In the finite poset \(Q\) depicted in Figure 5, consider the semiprime ideal \(I = \{0, a\}\) and the filter \(K = \{b', c', d', 1\}\), which is not an \(l\)-filter. Observe that \(I \cap K = \phi\), but there does not exist a semiprime filter \(F\) such that \(K \subseteq F\) and \(I \cap F = \phi\).

However, in the case of join semilattices we have

Theorem 4.8. Let \(I\) be a semiprime ideal and \(K\) be an \(l\)-filter of a join semilattice for which \(I \cap K = \phi\). Then there exists a semiprime filter \(F\) of \(P\) such that \(K \subseteq F\) and \(I \cap F = \phi\).

Now, we extend Theorem A for a finite poset whose ideal
lattice is distributive. First we recall the following definition.

A poset $P$ is called *distributive*, if for all $a, b, c \in P$, 
\[
\{a, (b, c)^u\}^l = \{(a, b)^l, (a, c)^l\}^u.
\]

The following Lemma follows immediately from the definition.

**Lemma 4.9.** If $P$ is a poset such that $Id(P)$ is a distributive lattice, then $P$ is distributive.

**Remark 4.10.** The converse of Lemma 4.9 is not true in general. The poset $Q$ depicted in Figure 5 is distributive but $Id(Q)$ is not so. For, let $I = (a', b)$, $J = \{0, a, b, c\}$ and $L = (c')$ and observe that $I \cap (J \lor L) = I \cap P = I \neq J = J \lor \{0, a, b\} = (I \cap J) \lor (I \cap L)$.

**Theorem 4.11.** Let $P$ be a finite poset such that $Id(P)$ is distributive. Let $I$ be an ideal, $K$ be an $I$-filter of $P$ such that $I \cap K = \phi$. Then there exists a prime ideal $G$ of $P$ such that $I \subseteq G$ and $G \cap K = \phi$.

**Proof:** Suppose $I$ is an ideal, $K$ is an $I$-filter of a finite poset $P$ of which $Id(P)$ is distributive such that $I \cap K = \phi$. Observe that $I^e$ is an ideal of $Id(P)$ and also $\alpha = \{J \in Id(P) | J \cap K \neq \phi\}$ is a filter of $Id(P)$ by Lemma 4.5. Note that $I^e \cap \alpha = \phi$ as in the proof of Theorem 4.6. Since $Id(P)$ is a distributive lattice, by Theorem A, there exists a prime ideal $\lambda$ of $Id(P)$ such that $I^e \subseteq \lambda$ and $\lambda \cap \alpha = \phi$. By Theorem 3.9, $\lambda^e$ is a prime ideal
of \( P \), where \( \lambda^c = \bigcup \{ J \mid J \in \lambda \} \). Further, \( I \subseteq \lambda^c \), as if \( x \in I \), then \( [x] \in I^c \subseteq \lambda \). Thus \( [x] \in \lambda \) and by definition of \( \lambda^c \), we have \( x \in \lambda^c \). Also, we must have \( \lambda^c \cap K = \phi \). Otherwise, if \( \lambda^c \cap K \neq \phi \), then there exists \( x \in P \) such that \( x \in \lambda^c \cap K \). Hence \( [x] \subseteq J \), where \( J \in \lambda^c \) and \( [x] \in \alpha \). In other words, \( [x] \in \lambda \cap \alpha \) which is not true.

\[ \begin{array}{c}
1 \\
(\alpha) \quad (\beta) \\
(\alpha \cap \beta) \\
(\gamma_1) \\
(\gamma_2) \\
(\gamma_3)
\end{array} \]

**Figure 10**

**Remarks 4.12.** (i) The statement of Theorem 4.11 is not necessarily true if we drop the condition that \( P \) is finite. Con-
sider the infinite poset $Q$ depicted in Figure 9 and its $Id(Q)$ depicted in Figure 10. Observe that $Id(Q)$ is distributive. Let $I = \{y_i\} \cup \{x_i\}$, where $i = 1, 2, \ldots$, and $K = \{a, 1\}$. Observe that $I$ is an ideal but not prime as $(a, b)^I \subseteq I$ and neither $a$ nor $b$ is in $I$. Also, $K$ is an $l$-filter of $Q$ and $I \cap K = \phi$. But there does not exist a prime ideal $G$ of $Q$ for which $I \subseteq G$ and $G \cap K = \phi$.

(ii) Also, the condition of $l$-filter on $K$ cannot be dropped in the statement Theorem 4.11. Consider the finite poset $Q$ depicted in Figure 11 which is distributive. Observe that $Id(Q)$ in Figure 12 is also distributive. Consider the ideal $I = \{0, a, b\}$, which is not prime and a filter $K = \{c, d, 1\}$ which is not an $l$-filter. Observe that $I \cap K = \phi$, but there is no prime ideal, say $G$ of $Q$ such that $I \subseteq G$ and $G \cap K = \phi$.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig11_12}
\caption{Q and Id(Q)}
\end{figure}
(iii) Also, the condition of distributivity of \( \text{Id}(P) \) is necessary in Theorem 4.11. Consider the finite poset \( Q \) depicted in Figure 5 which is distributive but \( \text{Id}(Q) \) is not so. Consider the ideal \( I = \{0, a, b, c\} \) and the \( l \)-filter \( K = \{d', 1\} \). Observe that \( I \cap K = \phi \), but there is no prime ideal, say \( G \) of \( Q \) for which \( I \subseteq G \) and \( G \cap K = \phi \).

Now, to prove that the statement of Theorem 4.11 is true in the case when \( P \) is a meet semilattice, for which we need the following result.

**Lemma 4.13.** A meet semilattice \( P \) is distributive if and only if \( \text{Id}(P) \) is distributive.

**Proof:** Suppose that \( P \) is a distributive meet semilattice. In order to show that \( \text{Id}(P) \) is distributive it is sufficient to show that, for \( I, J, L \in \text{Id}(P) \), \( I \cap (J \vee L) \subseteq (I \cap J) \vee (I \cap L) \), as the converse inclusion is always true. In fact, in view of Lemma 2.8, we show that \( I \cap C_n(J, L) \subseteq C_n(I \cap J, I \cap L) \) for each \( n \in \mathbb{N} \), by induction, where \( \mathbb{N} \) denotes the set of natural numbers.

(1) Let \( n = 1 \) and \( x \in I \cap C_1(J, L) \). Then \( x \in I \) and \( x \in (a, b)^{ul} \) for some \( a, b \in J \cup L \). Without loss of generality, assume that \( a \in J \) and \( b \in L \). We have \( x \wedge a \in I \cap J \) and \( x \wedge b \in I \cap L \) and so \( (x \wedge a, x \wedge b)^{ul} \subseteq C_1(I \cap J, I \cap L) \). By distributivity of \( P \), we have \( \{x, (a, b)^{ul}\}^l = (x \wedge a, x \wedge b)^{ul} \subseteq C_1(I \cap J, I \cap L) \). Since \( x \in (a, b)^{ul} \), we get that \( x \in C_1(I \cap J, I \cap L) \) and thus
the statement is true at \( n = 1 \).

(2) Suppose \( I \cap C_n(J, L) \subseteq C_n(I \cap J, I \cap L) \) is true for some \( n \) and suppose \( x \in I \cap C_{n+1}(J, L) \). We have \( x \in I \) and \( x \in (a, b)^{ul} \) for some \( a, b \in C_n(J, L) \). Since each \( C_n(J, L) \) is a down set, we have that \( x \land a, x \land b \in I \cap C_n(J, L) \). By induction hypothesis, \( x \land a, x \land b \in C_n(I \cap J, I \cap L) \) and hence \( (x \land a, x \land b)^{ul} \subseteq C_{n+1}(I \cap J, I \cap L) \). By distributivity of \( P \), we have \( x \in \{x,(a,b)^{u}\}^l = (x \land a, x \land b)^{ul} \subseteq C_{n+1}(I \cap J, I \cap L) \), as required. Therefore the statement is true for each \( n \in \mathbb{N} \).

Converse is obvious.

By Theorem 4.11 and Lemma 4.13, the following Corollary follows.

**Corollary 4.14.** Let \( P \) be a distributive meet semilattice. Let \( I \) be an ideal and \( K \) be a filter of \( P \) such that \( I \cap K = \emptyset \). Then there exists a prime ideal \( G \) of \( P \) such that \( I \subseteq G \) and \( G \cap K = \emptyset \).

**Corollary 4.15.** Let \( P \) be a finite poset such that \( Id(P) \) is distributive, and let \( I \) be an ideal of \( P \) and \( x \in P - I \). Then there exists a prime ideal \( G \) such that \( I \subseteq G \) and \( x \notin G \).

**Proof:** Put \( K = [x] \) and the proof follows from Theorem 4.11.

**Corollary 4.16.** Let \( P \) be a finite poset such that \( Id(P) \) is
distributive, \( x, y \in P \) and \( x \neq y \). Then there exists a prime ideal containing exactly one of \( x \) and \( y \).

**Proof:** Observe that either \( [x] \cap [y] = \emptyset \) or \( [y] \cap [x] = \emptyset \) and by Theorem 4.11, there exists a prime ideal \( G \) of \( P \) such that \( x \in G \) or \( y \in G \) respectively.

**Corollary 4.17.** Let \( P \) be a finite poset such that \( \text{Id}(P) \) is distributive. Then every ideal of \( P \) is the intersection of all prime ideals containing it.

**Proof:** We show that \( \bigcap \{ J \in \text{Id}(P) \mid I \subseteq J, \ J \text{ is prime} \} \subseteq I \), as the converse inclusion is trivial. Consider an element \( x \in P \) such that \( x \in \bigcap \{ J \in \text{Id}(P) \mid I \subseteq J, \ J \text{ is prime} \} \).

If \( x \notin I \), then \( I \cap [x] = \emptyset \). By Theorem 4.11, there exists a prime ideal \( G \) such that \( I \subseteq G \) and \( x \notin G \), a contradiction.

We state the following corollaries in meet semilattices.

**Corollary 4.18.** Let \( P \) be a distributive meet semilattice and let \( I \) be an ideal of \( P \) and \( x \in P - I \). Then there exists a prime ideal \( G \) such that \( I \subseteq G \) and \( x \notin G \).

**Corollary 4.19.** Let \( P \) be a distributive meet semilattice and let \( x, y \in P, \ x \neq y \). Then there exists a prime ideal containing exactly one of \( x \) and \( y \).

**Corollary 4.20.** Let \( P \) be a distributive meet semilattice.
Then every ideal of \( P \) is the intersection of all prime ideals containing it.

We conclude this section by giving a characterization of distributivity in posets in terms of semiprime ideals. In fact, it is a generalization of the following result due to Rav [1989].

**Lemma 4.21.** (Rav [1989]) A lattice \( L \) is distributive if and only if for every ideal \( I \) and filter \( F \) of \( L \) for which \( I \cap F = \phi \) there exists an ideal \( J \) and a filter \( G \) of \( L \) such that \( I \subseteq J, \ F \subseteq G, J \cap G = \phi \), and either \( J \) or \( G \) is semiprime.

Here we generalize the above Lemma for posets.

**Theorem 4.22.** Let \( P \) be a poset. The following statements are equivalent.

(i) \( P \) is a distributive.

(ii) For every principal ideal \( I \) and a principal filter \( K \) of \( P \) for which \( I \cap K = \phi \), there exists an ideal \( G \) and a filter \( F \) of \( P \) such that \( I \subseteq G, K \subseteq F, G \cap F = \phi \) and either \( G \) or \( F \) is semiprime.

**Proof:** (i) \( \Rightarrow \) (ii) Follows from Theorem 2.26.

(ii) \( \Leftrightarrow \) (i) Suppose that statement (ii) holds about \( P \). To show that \( P \) is distributive suppose that \( a \in \{ x, (y, z)\}^l \) and \( b \in \{ (x, y)^l, (x, z)^l \}^u \), for \( a, b, x, y, z \in P \). We have to show that \( a \leq b \). If possible, suppose that \( a \nless b \). We have \( \{ x, (y, z)^l \}^u \subseteq [a], \{ (x, y)^l, (x, z)^l \}^u \subseteq (b) \) and \([a] \cap (b) = \phi \).
Therefore, by assumption there exists an ideal $G$ and a filter $F$ of $P$ such that \( (b) \subseteq G, [a) \subseteq F, G \cap F = \phi \) and either $G$ or $F$ is semiprime. Observe that $G$ is not semiprime. If $G$ is semiprime, then as \( (x, y)^l \subseteq G, (x, z)^l \subseteq G \), we would have \( \{x, (y, z)^u\}^l \subseteq G \), and hence $a \in G$, which is a contradiction to the fact that $a \in F$ and $G \cap F = \phi$. Consequently, $G$ is not semiprime, and so $F$ is semiprime.

Now, since $x \in F$, we have $(x, y)^u \subseteq F$. Also, we have $(y, z)^u \subseteq F$ and so by semiprimeness of $F$, \( \{y, (x, z)^l\}^u \subseteq F \).

We claim that \( \{(x, y)^l, (x, z)^l\}^u \subseteq F \). Assume that $c \in \{(x, y)^l, (x, z)^l\}^u$. Now $c \in (x, z)^{lu}$ and \( \{y, (x, z)^l\}^u \subseteq F \), we have $(y, c)^u \subseteq F$. Also $x \in F$, therefore $(x, c)^u \subseteq F$ and since $F$ is semiprime, we get \( \{c, (x, y)^l\}^u \subseteq F \). But $c \in (x, y)^{lu}$, hence $c \in F$. Now, $b \in \{(x, y)^l, (x, z)^l\}^u$, therefore $b \in F$, a contradiction to the fact that $b \in G$ and $F \cap G = \phi$. Therefore, we must have $a \leq b$ and the proof is complete. \( \bullet \)