CHAPTER 1

CHAOS AND LASERS

In this chapter we describe chaos in various dynamical systems, with special reference to lasers. We give the conditions necessary for the occurrence of chaos, different tools for characterizing chaos and various routes to chaos exhibited by dynamical systems. Hopf bifurcation and period doubling route to chaos are also discussed taking Rossler system and van der Pol oscillator as examples.
1.1 WHAT IS CHAOS?

“Chaos is the science of complexity of change”

Chaos is there in the unpredictable weather, in the share price fluctuations in stock market and even in the variation of commodity prices. We can see chaos in ecology, economics, epidemiology, in the beating of the heart and in the electrical signals from the brain. Discovery of chaos changed the world’s understanding of the foundations of Physics and led to new findings on the frontiers of lasers, fluid mechanics, chemical reactions, neural networks and biological rhythms. Chaos based studies of EEG and ECG signals lead to new developments in brain research and cardiology. It provided a novel way of thinking meaningfully about many phenomena such as turbulence which otherwise were considered as a matter of utter confusion [1].

The word chaos means a state of disorder. Henri Poincare, a prominent mathematician and theoretical astronomer was the first person to glimpse the possibility of chaos. He developed a powerful geometric approach to various scientific problems. That approach flowered into the modern subject of dynamics with its applications spreading over vast scientific areas [2]. Interest in nonlinear dynamics especially chaos gained pace after 1963, when Lorenz published his pioneering numerical work on a simple convection model and discussed its implications for weather prediction [3].

A physical system whose state changes with time is termed as a dynamical system. The changes occur due to influence of forces that act on the system.
The evolution of physical systems depends on the nature of the forces acting on them and also on their initial condition. If the forces acting on a system are nonlinear we call the system as a nonlinear dynamical system. For certain values of the parameters that control the system, the evolution of many of the nonlinear dynamical systems become unpredictable [4]. A system whose temporal or spatial evolution seems random, but is completely deterministic, i.e obeys some definite evolutionary equation, is called a chaotic system. Its central characteristic is that the system does not repeat its past behavior. For chaotic systems even a small difference in the initial conditions leads to an exponentially growing error and the system dynamics becomes unpredictable after a very short time. This is known as sensitivity to initial conditions. As Poincare said “...it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon” [2]. If the prediction becomes impossible a chaotic system can resemble a stochastic system. But for a chaotic system the irregularity arises from its intrinsic dynamics, not from any unpredictable external influences.

We can define chaos as “aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions.” [5]. The most interesting thing about chaos is that in spite of its highly complex nature it happens in systems which are not complex and are even very simple.
1.2 CONDITIONS NECESSARY FOR CHAOS

Dynamical systems can be broadly classified into continuous time dynamical system and discrete time dynamical system. A continuous time dynamical system can be represented by a set of equations of the form

\[
\frac{d \vec{X}}{dt} = \vec{F}(\vec{X}, \mu) \tag{1.1}
\]

where

\[ \vec{X} = (x_1, x_2, \ldots, x_n) \quad \text{and} \quad \vec{F} = (F_1, F_2, \ldots, F_n) \]

\((x_1, x_2, \ldots, x_n)\) are the dynamical variables and \((F_1, F_2, \ldots, F_n)\) are the source functions. \(\mu\) is a set of control parameters that can be varied.

We can write the necessary conditions for the occurrence of chaos in such a system as

1) It must be nonlinear
2) The order or dimension of the system must be greater than or equal to 3.

Discrete time dynamical system can be represented by maps of the form

\[
\vec{X}_{n+1} = F_\mu(\vec{X}_n) \tag{1.2}
\]

where \(n\) refers to discrete values of time.

Linear maps do not show chaotic behavior. Nonlinear maps are of two types namely invertible and noninvertible. Invertible maps can exhibit chaos only if the dimension is greater than one. Noninvertible maps can exhibit chaos even if the dimension is one.
1.3 TOOLS FOR THE STUDY OF CHAOS

Three powerful mathematical tools used in the study of dynamical systems are phase space, Poincare section and power spectra. They give a qualitative and global understanding of chaos.

1.3.1 Phase space

The phase space of a dynamical system is a mathematical space having orthogonal coordinate directions which represent each of the variables needed to specify the instantaneous state of the system. The state of a particle moving in one dimension is specified by its position \( x \) and velocity \( \dot{x} \). Thus its phase space is a plane. For a particle moving in three dimensions the phase space will be six dimensional. The phase space variables need not be mechanical coordinates like position and velocity. They may be concentration of reactants in a chemical reaction or output intensity or gain as in the case of laser etc. The state of a dynamical system is represented by a point in the phase space. As the system evolves in time, it constitutes a trajectory in the phase space [2, 4, 5].

1.3.2 Poincare section

Poincare section provides a means for simplifying the complicated structure of attractors. Generally for an \( n \)-dimensional flow, the Poincare section will be an \((n-1)\) dimensional hypersurface transverse to the flow. In order to construct the Poincare section for a three dimensional attractor, choose a
plane transverse to the direction of motion of the trajectories. Put a point on this plane every time the trajectory crosses it. This plane then constitutes the Poincare section for the attractor. It should be noted that while constructing the Poincare section, motion of the trajectories only in one direction has to be considered. The time interval between successive intersections need not be equal. The nonlinear three dimensional differential equations are replaced by nonlinear algebraic two dimensional difference equations that are simple and easier to handle. At the same time it retains the essential qualitative features of the phase flow. This reduction in dimension provides greater simplification. The observation of the distribution of points on a computer-generated Poincare section [2, 4, 5] will be useful for the study of chaos.

1.3.3 Power spectrum

The spectral analysis of a signal is an important tool in the study of the temporal behavior of a system. Power spectrum gives the relative strengths of various frequency components of a time series. It is calculated by taking the Fourier transform of the auto-correlation function of the time series. Auto-correlation function measures the correlation between observations at different distances (times) apart. These functions exhibit the loss of information along the trajectory. The power spectrum is a function in the frequency domain. The power spectrum for a chaotic motion will be of continuous and broad band nature [2, 4].

1.4 QUANTITATIVE MEASURES OF CHAOS

The motion of typical nonlinear systems undergoes characteristic changes as certain control parameters are varied continuously. These changes are
identified by the changes of the system attractors or phase space structures and stability properties. In addition to these we use some quantitative criteria to differentiate between chaotic and regular motions. The most important criteria used for this purpose are
1) Lyapunov exponents
2) Correlation function
3) Attractor dimensions

1.4.1 Lyapunov exponents

Named after A.M.Lyapunov, a Russian mathematician, Lyapunov exponents describe the rate of divergence or convergence of nearby trajectories onto the attractor in different directions in phase space. It gives a measure of the sensitive dependence upon initial conditions which is a characteristic of chaotic system. Consider a system evolving from two slightly different initial conditions $x$ and $x+\varepsilon$. After $n$ iterations the divergence of the two trajectories may be represented as [2, 4, 5]

$$ e(n) \approx \varepsilon e^{\lambda n} \quad (1.3) $$

where $\lambda$, the Lyapunov exponent gives the average rate of divergence. If $\lambda$ is negative, the two nearby trajectories converge and the evolution is not chaotic. If $\lambda$ is positive, nearby trajectories diverge as the evolution is sensitive to initial conditions and hence chaotic. We can similarly define Lyapunov exponents for continuous systems.

Consider an $n$-dimensional system represented by the equation

$$ \dot{X} = F(X) \quad (1.4) $$
\[ x_1 = F_1(x_1, x_2, \ldots, x_n) \]
\[ x_2 = F_2(x_1, x_2, \ldots, x_n) \]
\[
\vdots
\]
\[ x_n = F_n(x_1, x_2, \ldots, x_n) \]

\( X(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) represent the trajectory of the system (1.4).

Consider two nearby trajectories in the n-dimensional phase space starting from slightly different initial conditions \( X_0 \) and \( X'_0 = X_0 + \delta X_0 \) respectively. Their time evolution will give the vectors \( X(t) \) and \( X(t) + \delta X(t) \). In order to find the time evolution of \( \delta X \), we linearize equation (1.4) so as to get

\[
\dot{\delta X} = M(\dot{X}(t)) \cdot \delta(\dot{X})
\]

where \( M = \frac{\partial F}{\partial X_{X=x_0}} \) is the Jacobian matrix of \( F \).

The Lyapunov exponent of the system can be defined as

\[
\lambda(X_0, \delta X) = \lim_{t \to \infty} \frac{1}{t} \log \left( \frac{d(X_0, t)}{d(X_0, 0)} \right) \tag{1.5}
\]

where \( d(X_0, t) \) is a measure of the distance between the trajectories \( X(t) \) and \( X'(t) \).
1.4.2 Correlation function

For a map \( x_{n+1} = f(x_n) \) the correlation function \( C(m) \) is defined as [6]

\[
C(m) = \lim_{N \to \infty} \left( \frac{1}{N} \sum_{n=0}^{N} x_n x_{n+m} - \bar{x}^2 \right)
\]  

(1.6)

where \( \bar{x} \) is the mean value of the \( x_n \) given by

\[
\bar{x} = \lim_{N \to \infty} \left( \frac{1}{N} \sum_{n=0}^{N} x_n \right)
\]  

(1.7)

Correlation function gives a measure of the extent to which iterates \( m \) steps apart are correlated in their evolution. Decaying correlation is a characteristic of the chaotic motion. Or we can say that correlation function measures the degree of randomness of a chaotic system.

1.4.3 Attractor dimensions

An attractor is a certain subspace of the full phase space into which all the trajectories settle down eventually. For regular nonchaotic systems attractor could be a point (of zero dimension) or a curve- a limit cycle- (of dimension one) or a torus (of dimension two). But when it comes to a chaotic system the task of finding dimension becomes tedious. A number of dimensions can be used to describe the characteristic features of a chaotic attractor. Most important of them are discussed below.
Consider a straight line of length $L$. Suppose this line can be covered by $N(\varepsilon)$ one-dimensional boxes of side $\varepsilon$, then we can write

$$N(\varepsilon) = L \left(\frac{1}{\varepsilon}\right)^1$$

Similarly for a two-dimensional square of side $L$, the number of boxes required are

$$N(\varepsilon) = L^2 \left(\frac{1}{\varepsilon}\right)^2$$

For a three-dimensional cube the exponent is 3.

Generally for a $d$-dimensional figure we can write

$$N(\varepsilon) = L^d \left(\frac{1}{\varepsilon}\right)^d \quad (1.8)$$

Taking logarithms on both sides

$$d = \frac{\log N(\varepsilon)}{\log L + \log \left(\frac{1}{\varepsilon}\right)} \quad (1.9)$$

In the limit of small $\varepsilon$, we can write

$$D_F = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log \left(\frac{1}{\varepsilon}\right)} \quad (1.10)$$

which is the capacity dimension or fractal dimension. There are attractors, called strange attractors for which the dimension will be noninteger. However when the phase space dimension becomes greater than two, the task of computing $D_F$ becomes very difficult.
ii) Correlation dimension ($D_C$) [4,5]

The Correlation dimension is defined by

$$D_C = \lim_{r \to 0} \frac{\log C(r)}{\log r}$$  \hspace{1cm} (1.11)

where $C(r)$ is the correlation function discussed earlier.

Correlation dimension can be calculated using a well known algorithm developed by Grassberger and Procaccia. As correlation dimension takes into account the density of data points on the attractor, its estimation requires a large number of data points. For a point attractor $D_C = 0$ and for a limit cycle attractor $D_C = 1$. A chaotic attractor is characterized by a noninteger $D_C$ value.

Based on the above discussions we can summarize the important characteristic features of a chaotic system as

1) At least one positive Lyapunov exponent.
2) Continuous broad band power spectrum.
3) Decaying autocorrelation function.
4) A noninteger fractal and correlation dimension.

1.5 ROUTES TO CHAOS

For every dynamical system there will be a set of control parameters whose variation can produce sudden changes in the system dynamics. These sudden qualitative changes which occur at a critical value of the control parameter are called bifurcations. Bifurcations may lead the system to chaos. Often
there will be some intermediate states through which the system evolves before coming to the chaotic state. Usually one type of motion losses stability at a particular value of the control parameter at the same time giving rise to a new type of stable motion. This process continues further to give new and often more complicated type of motions. We can identify various routes to chaos taken by dynamical systems. The most common are

i) The period doubling route
ii) The quasiperiodic route
iii) Intermittency route.

1.5.1 Period doubling route

In period doubling route, the period of the oscillation doubles as the control parameter is varied [1, 4, 7]. After a sequence of such period doublings the system becomes aperiodic or chaotic. Logistic map is a simple model system where we can see this period doubling route to chaos.

Logistic map is a discrete dynamical system represented by the equation

\[ X_{n+1} = \lambda X_n (1 - X_n) \]  \hspace{1cm} (1.12)

where \( 0 \leq \lambda \leq 4 \) and \( 0 \leq X \leq 1 \)

Bifurcation diagram for a logistic map showing period doubling phenomena is given below [1, 4]
1.5.2 Quasiperiodic route

The quasiperiodic route to chaos is also termed as Ruelle- Newhouse - Takens scenario. The basic mechanism taking place here is the phenomenon called Hopf bifurcation. In Hopf bifurcation a stable fixed point loses stability at a particular value of the control parameter and evolves into a limit cycle. The system undergoes one more Hopf bifurcation as the control parameter is varied further to form a two frequency periodic orbit. If the two frequencies of this orbit are not commensurate we call it as a quasiperiodic orbit. This quasiperiodic orbit then bifurcates into chaotic motion as the
control parameter is varied. Quasiperiodic route to chaos is observed in maps and in continuous time dynamical systems [4].

1.5.3 Intermittency route

This route is also known as Pomeau-Manneville scenario. Time series appears as nearly periodic interrupted by occasional irregular bursts. The time between bursts is statistically distributed like a random variable even though the system is completely deterministic. As the control parameter is varied the bursts become more and more frequent and finally the system becomes fully chaotic at a particular value of the control parameter. This intermittent behavior was first observed by Pomeau and Manneville in Lorenz equations [4, 5].

1.6 TWO SIMPLE MODEL SYSTEMS

In this section we describe two simple model systems namely the Rossler system and the van der Pol oscillator.

1.6.1 CHAOS IN A ROSSLER SYSTEM

The Rossler system is represented by the following set of equations [8, 9]

\[
\begin{align*}
x & = -y - z \\
y & = x + ay \\
z & = b + z(x - c)
\end{align*}
\]
where a,b,c are parameters. The parameter c is treated as the control parameter.

This system has only one quadratic nonlinearity xz. These equations represent a continuous time dynamical system that exhibits chaotic dynamics as the control parameter is varied. The famous simply folded band attractor for the Rossler system is shown in figure 1.2. An orbit within the attractor follows an outward spiral close to the X-Y plane around an unstable fixed point. Once the graph spirals out enough, it shows a rise and twist in the z dimension because of the influence of a second fixed point. Even though each variable is oscillating within a fixed range of values, the oscillations are found to be chaotic.

Figure 1.2: Simply folded band attractor of the Rossler system
To study the dynamics of the Rossler system we fix the parameters $a=b=0.1$ and vary $c$. The attractors corresponding to various $c$ values are given in figure 1.3.

**Figure 1.3:** Attractors of the Rossler system for various $c$ values. a) $c=4$; b) $c=6$; c) $c=8.5$; d) $c=9$

We can see the structure of attractor changing as the $c$ value is varied. The attractor is a single loop periodic orbit at low $c$ value. It suddenly undergoes period doubling at a bifurcation value of $c$ changing into a double loop. The period doubling continues infinitely at higher $c$ values and at last there is chaos. The route taken by the Rossler system is called the period doubling route to chaos. Figure 1.4 is the bifurcation diagram where the local maxima of the $x$-variable are plotted against the bifurcation parameter $c$. Similar
period doubling route to chaos is exhibited by semiconductor laser [10] and Nd:YAG laser [11]. Also Rossler system is a very good candidate for studying different types of synchronization such as phase and lag [12].

Figure 1.4: Bifurcation diagram for the Rossler system

1.6.2 THE VAN DER POL OSCILLATOR

It is a type of non conservative oscillator with a nonlinear damping. It evolves in time according to the pairs of first order differential equations [4-6]

\[ \begin{align*}
  x &= y \\
  y &= b(1 - x^2)y - x
\end{align*} \]  

(1.14)
where b is the damping coefficient. Origin is found to be the equilibrium point for this system. Eigen values of the system are given by

\[
\lambda_{\pm} = \frac{1}{2} \left[ b \pm \sqrt{b^2 - 4} \right]
\]

(1.15)

Corresponding to different regions of the parameter b we can have different types of equilibrium states which are given in table 1.1.

<table>
<thead>
<tr>
<th>Range of b</th>
<th>Nature of eigenvalues</th>
<th>Type of attractor/repellor</th>
</tr>
</thead>
<tbody>
<tr>
<td>-∞&lt;b&lt;-2</td>
<td>( \lambda_{\pm} &lt; 0, \lambda_+ \neq \lambda_- )</td>
<td>Stable node</td>
</tr>
<tr>
<td>b=-2</td>
<td>( \lambda_+ = \lambda_- &lt; 0 )</td>
<td>Stable star</td>
</tr>
<tr>
<td>-2&lt;b&lt;0</td>
<td>( \lambda_{\pm} = \alpha \pm i\beta, \alpha &lt; 0 )</td>
<td>Stable focus</td>
</tr>
<tr>
<td>b=0</td>
<td>( \lambda_{\pm} = \pm i\beta )</td>
<td>Center</td>
</tr>
<tr>
<td>0&lt;b&lt;2</td>
<td>( \lambda_{\pm} = \alpha \pm i\beta, \alpha &gt; 0 )</td>
<td>Unstable focus</td>
</tr>
<tr>
<td>b=2</td>
<td>( \lambda_+ = \lambda_- &gt; 0 )</td>
<td>Unstable star</td>
</tr>
<tr>
<td>2&lt;b&lt;∞</td>
<td>( \lambda_{\pm} &gt; 0, \lambda_+ \neq \lambda_- )</td>
<td>Unstable node</td>
</tr>
</tbody>
</table>

Table 1.1: Dynamics of the van der Pol oscillator for different b values.

The attractor is an unstable equilibrium point for 0<b<∞. At b=0, the real parts of the eigen values \( \lambda_{\pm} \) changes from negative to positive values as the parameter b is increased through zero. This causes a bifurcation from a stable focus to a stable limit cycle. A limit cycle is an isolated closed stable orbit that can exist in two dimensional dynamical systems. Every trajectory beginning sufficiently near a limit cycle approaches it either for t→∞ or t→-∞. A limit cycle is said to be stable if all the nearby trajectories approach it as t→∞. If the trajectories deviate from it as t→∞ (or approach it as t→-∞),
the limit cycle is said to be unstable. A stable limit cycle is a period T attractor or a periodic attractor. Such a bifurcation is called a Hopf bifurcation. i.e as a control parameter is varied a stable equilibrium point losses stability at a critical value and evolves into a limit cycle. It is characterized by a change of the real part of a pair of complex conjugate eigen values from negative to positive. In the van der Pol oscillator the limit cycle attractor is found to occur due to the dynamical balance between the positive and the negative damping. Figure 1.5 shows the limit cycle attractor for the van der Pol oscillator.

**Figure 1.5:** Phase portrait of the van der Pol oscillator for b=0.4 exhibiting limit cycle motion.

When driven by an external periodic forcing the oscillator exhibits interesting dynamical behaviors. A driven van der Pol oscillator is represented by the equation [4]
\begin{equation}
\begin{align*}
  x &= y \\
  y &= b(1 - x^2)y - x + f \cos \omega t
\end{align*}
\end{equation}

Here \( b \) is the damping coefficient, \( f \) is amplitude of the driving force and \( \omega \) is the frequency of the external periodic forcing.

Depending on the values of \( \omega \) we can see mode locking, quasiperiodicity and chaos in the oscillator output. The oscillator dynamics can be summarized as in the following table.

<table>
<thead>
<tr>
<th>( f )</th>
<th>Range of ( \omega )</th>
<th>Output dynamics</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0-1.5</td>
<td>Mode locking and quasiperiodicity</td>
</tr>
<tr>
<td>2.5</td>
<td>0-6</td>
<td>Mode locking, large periodic oscillations and quasiperiodicity</td>
</tr>
<tr>
<td>5</td>
<td>2.424-2.502</td>
<td>Periodic windows, period doubling bifurcations and chaos</td>
</tr>
</tbody>
</table>

Table 1.2: Dynamics of the driven van der Pol oscillator for various ranges of \( f \) and \( \omega \)

Bifurcation diagram for the driven van der Pol oscillator showing the period doubling route to chaos is given in figure 1.6.
Figure 1.6: Period doubling route to chaos in driven van der Pol oscillator

1.7 LASERS

Lasers are devices that generate, or amplify coherent radiation at frequencies in the infrared, visible or ultraviolet regions of the electromagnetic spectrum. Invention of Ruby laser by Maiman in 1960 had a great impact on the scientific community. Processes such as signal generation, amplification, transmission and detection at much higher frequencies are made possible. Unique properties of lasers such as ultrashort pulsewidth, high power and short wavelength helped scientists and engineers to perform a wide variety of new and unexpected functions [13].
A laser consists of a gain medium placed inside a highly reflective optical cavity and a source to supply energy to the gain medium. The gain medium is a material which can amplify light by stimulated emission. The optical cavity consists of two mirrors. They are arranged such that the light bounces back and forth between the mirrors each time passing through the gain medium. One of the two mirrors—the output coupler—is made partially transparent so as to couple out the laser beam. Light of a specific wavelength is amplified through stimulated emission as it passes thorough the gain medium. Part of the light that is between the mirrors, after sufficient amplification comes out as laser beam through the output coupler. The process of supplying energy required for amplification is called pumping. Usually electric discharge or flash lamp is used as pumping sources. The output of laser may be of two types—continuous constant amplitude output and pulsed output [14].

1.8 DIFFERENT TYPES OF LASERS

1.8.1 Gas lasers
The Helium-Neon laser was the first gas laser to be operated successfully. It was fabricated by Ali Javan and his co-workers [15]. It emits at a variety of wavelengths with the well-known red light at 6328 Å. It is commonly used for educational purposes because of its low cost. Carbon dioxide lasers can emit hundreds of kilowatts at 9.6 μm and 10.6 μm and are often used in industry for cutting and welding. Argon-ion lasers have emission at wavelengths 351 nm- 528.7 nm. Metal ion lasers are gas lasers that can generate light at deep ultraviolet wavelengths. Helium-Silver (HeAg) and Neon-Copper (NeCu) lasers come under this category. These lasers have
found applications in fluorescence suppressed Raman spectroscopy because of their narrow oscillation linewidth.

1.8.2 Chemical lasers
Chemical lasers obtain their energy from chemical reactions. They can produce power in the megawatt level and are used in industry for cutting and drilling and also for military purposes. Examples are Hydrogen fluoride laser and Deuterium fluoride laser.

1.8.3 Excimer laser
Excimer lasers are powered by chemical reaction involving excimer (excited dimer) which is a short lived dimeric or heterodimeric molecule formed from two atoms one of which is in an excited electronic state. They produce ultraviolet light and are used in semiconductor photolithography and in eye surgery.

1.8.4 Solid state lasers
Solid state laser materials have a crystalline host doped with ions that provide the required energy states. Neodymium is a common dopant in various solid state laser crystals which include yttrium orthovanadate (Nd:YVO₄), yttrium lithium fluoride (Nd:YLF) and yttrium aluminium garnet (Nd:YAG). These lasers can produce high powers in the infrared region. They find applications in cutting, welding, marking of metals, in spectroscopy and for pumping dye lasers. They can be frequency doubled, tripled or quadrupled to produce light at desired wavelengths. Solid state lasers where light is guided due to total internal reflection in an optical fiber are called fiber lasers.
1.8.5 Semiconductor lasers

Stimulated emission in a semiconductor pn junction is the basis of semiconductor lasers. The laser emission is not as monochromatic as that from a gas laser as it occurs between two bands of energies. Because of the direct conversion of electric current to light energy, semiconductor laser is very efficient. External cavity semiconductor lasers can generate high power outputs with good beam quality, narrow linewidth radiation with tunable wavelength and ultra short laser pulses.

1.8.6 Dye lasers

Dye lasers use organic dye as the lasing medium. They are highly tunable and capable of producing very short-duration pulses in the femtosecond range. Dye lasers are used dermatologically to make skin tone more even.

1.9 CHAOS IN LASERS

Instabilities in laser action were apparent since early days of laser technology. Semi classical model of the laser known as the Maxwell-Bloch equations [2] describe the time dependence of the electric field E, the mean polarization P of the atoms and the amount of population inversion D.

\[
\frac{dE}{dt} = -\kappa E + \kappa P \\
\frac{dP}{dt} = \gamma_j E D - \gamma_j P \tag{1.17} \\
\frac{dD}{dt} = \gamma_2 (\lambda + 1) - \gamma_2 D - \gamma_2 \lambda EP
\]
where $\kappa$ is the decay rate in the laser cavity due to beam transition, $\gamma_1$ is the decay rate of the atomic polarization, $\gamma_2$ is the decay rate of the population inversion and $\lambda$ is a pumping energy parameter.

Even though numerical simulation of Maxwell-Bloch equations may exhibit chaos, many conventional lasers do not operate within a parameter range where chaos occurs. In them the polarization and population inversion decay quickly to their steady state values which in turn reduces the dimension of the system to one. This will make the system out of the chaotic behavior. But the chaotic behavior can be induced by modifying the laser configurations i.e. by tuning the cavity length, varying the laser gain, tilting the mirrors or by adding feedback from external cavity [16]. Lasers in which the frequency is broadened by the characteristics of the laser medium exhibit both periodic and chaotic behavior.

A detailed study of Maxwell-Bloch equations was carried out by Herman Haken [17] who reported that the instabilities seen in the Lorenz model [3] in fluids are identical with that of a single mode laser. This discovery was so important that it paved the way to a lot of research in the area of laser chaos. It started with Arecchi who reported the experimental observation of subharmonic bifurcations, generalized multistability, and chaotic behavior in a Q-switched CO$_2$ laser. They also presented a theoretical model which is found to be in good agreement with the experimental results [18]. In 1983 Gioggia and Abraham found instabilities in a single mode, inhomogeneously broadened Xenon laser exhibiting complicated periodic behavior ultimately reaching a deterministic chaotic behavior. Period doubling, two frequency quasiperiodic and intermittency route to chaos have also been found [19]. Chaos in a solid state laser with periodically modulated pump was reported
by Klische et.al [20]. A comparison of theoretical and experimental studies of unidirectional, single mode, inhomogeneously broadened ring laser was carried out by Tarroja et.al [21].

Semiconductor lasers are of special importance because of their potential applications in areas such as secure communication. Some of the pioneering works in this area include those of Min Tang, Shyh Wang and Kawaguchi [22, 23]. In 1986, studies by Winful and Liu revealed a quasiperiodicity route to chaos for a directly modulated self-pulsating semiconductor laser [24]. Kao et.al investigated a period doubling route to chaos in modulated semiconductor laser [25]. Effect of nonlinear gain on period doubling and chaos in a semiconductor laser was studied by G.P.Agrawal and it was shown that chaos occurred at modulation frequency around 1 GHz, which is of much importance in optical communication system [10]. Synchronization of chaotic lasers is widely studied [26] as it can enhance privacy and security of communication systems [27]. Semiconductor laser with feedback is an excellent model of nonlinear optical system which shows chaotic dynamics. The feedback induced instabilities, chaos and their applications have been widely studied [28-34].

Chaos in solid state lasers is also an area of much interest. Nd:YAG laser with intracavity KTP crystal serves as a good model system for this purpose. An early prediction of instabilities in this laser was made by Arecchi and Ricca [35]. Baer reported chaotic intensity fluctuations in this laser system and analyzed it using a rate equation model [36]. Significant contributions include those of Wu, Mandel, Oka and Kubota, Bracikowski and Rajarshi Roy [37-40]. A reverse period doubling route from chaos to stability in a two mode intracavity doubled Nd:YAG laser was reported in 1999 [11].
Dynamics of Nd:YAG laser with three modes has also been studied [41]. Recently it has been shown that multimode lasers have many advantages over single mode lasers which make them suitable for communication purposes [42-44].
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