<table>
<thead>
<tr>
<th>No.</th>
<th>Name of Membership Function (MF)</th>
<th>Functional Form</th>
</tr>
</thead>
</table>
| 1   | Zadeh’s Z-shaped MF (ZMF)       | \( \mu(x) = \begin{cases} 
1 - 2\left( \frac{x-a}{c-a} \right)^2 & \text{if } a < x \leq \frac{a+c}{2} \\
2\left( \frac{x-a}{c-a} \right)^2 & \text{if } \frac{a+c}{2} < x \leq c \\
1 & \text{if } x \leq a \\
0 & \text{otherwise} 
\end{cases} \) |
| 2   | Triangular-shaped MF (TRIMF)    | \( \mu(x) = \begin{cases} 
\frac{x-a}{b-a} & \text{if } a \leq x \leq b \\
\frac{c-x}{c-b} & \text{if } b \leq x \leq c \\
0 & \text{otherwise} 
\end{cases} \) |
| 3   | Trapezoidal MF (TRAPMF)        | \( \mu(x) = \begin{cases} 
\frac{x-a}{b-a} & \text{if } a \leq x \leq b \\
1 & \text{if } b \leq x \leq c \\
\frac{d-x}{d-c} & \text{if } c \leq x \leq d \\
0 & \text{otherwise} 
\end{cases} \) |
| 4   | Zadeh’s S-Shaped MF (SMF)      | \( \mu(x) = \begin{cases} 
2\left( \frac{x-c}{c-a} \right)^2 & \text{if } a < x \leq \frac{c+a}{2} \\
1 - 2\left( \frac{x-c}{c-a} \right)^2 & \text{if } \frac{c+a}{2} < x \leq c \\
1 & \text{if } x \geq c \\
0 & \text{otherwise} 
\end{cases} \) |
| 5   | Zadeh’s Bell-shaped π MF       | \( \mu(x) = \begin{cases} 
S(x;c-b,c-b/2,c) & \text{if } x \leq c \\
1 - S(x;c,c+\frac{b}{2},c+b) & \text{if } x \geq c 
\end{cases} \) |
| 6   | Generalized π MF (PIMF)        | \( \mu(x) = \begin{cases} 
S(x;a_1,c_1) & \text{if } x \leq c_1 \\
1 & \text{if } c_1 \leq x \leq c_2 \\
Z(x;a_2,c_2) & \text{if } x \geq c_2 
\end{cases} \) |
| 7   | Gaussian MF (GAUSSMF)          | \( \mu(x) = e^{-\frac{(x-c)^2}{2\sigma^2}} \) |
| 8   | Generalized Bell-shaped MF (GBELLMF) | \( \mu(x) = \begin{cases} 
0.5 & \text{if } x = c & \text{& } b = 0 \\
0 & \text{if } x = c & \text{& } b < 0 \\
\frac{1}{1+\left( \frac{x-c}{b-\sigma} \right)^{25}} & \text{otherwise} 
\end{cases} \) |
| 9   | Sigmoidally-shaped MF (SIGMF)  | \( \mu(x) = \frac{1}{1+e^{ax-c}} \) |
| 10  | LR-Representation of Fuzzy Sets | \( \mu(x) = \begin{cases} 
L\left( \frac{m-x}{\sigma} \right) & \text{if } x \leq m \\
R\left( \frac{x-m}{\sigma} \right) & \text{if } x \geq m 
\end{cases} \) |
| 11  | Function of COS                | \( \mu(x) = \cos^2\left( \frac{x-\mu}{\sigma} \right) \) if \( z \in [\mu - \sigma, \mu + \sigma] \) |
APPENDIX II

SEMI-CIRCULAR MEMBERSHIP FUNCTIONS

The quality of the fuzzy approximation depends on the quality of the rules, which in turn depends on the nature of membership functions in terms of its shape, spread, and position. A membership function (MF) is a curve that defines how each point in the input space is mapped to a membership value (or degree of membership) in the interval [0,1] instead from the two-element set \{0,1\}. The function itself can be any arbitrary curve whose shape can be defined by a function that suits us from the point of view of simplicity, convenience, speed, and efficiency [44].

Narrow membership functions have high Information Granularity (IG) while broader ones will have low IG [108]. The more linguistic terms in the rule base (and the higher their granularity), the more adjustment can be made to the piecewise characteristics of the fuzzy systems. At the same time, very narrow membership functions (which characterize fuzzy sets of higher granularity) are not suitable for handling noise in the data.

Fuzzy sets with straight lines offer a simplified notation, which can be compared to internal notations in fuzzy systems or the LR-notation by Dubois and Prade [30]. But smooth shapes of the membership functions are often preferred over the sharp shapes since they give smooth transitions in the activation process from one local fuzzy model to another, and as a result produce better (more appropriate) models [141].

Recent studies on construction of differentiable membership functions by Grauel and Ludwig [42], smooth membership functions for mathematical analysis of fuzzy controllers by Grauel and Mackenberg [43] and polynomial membership functions for smooth first order Takagi-Sugeno systems by Runkler and Bezdek [120] highlight the importance of membership functions in the design of fuzzy systems.

A uniform representation of the membership functions is desirable, for
computational efficiency, efficient use of memory, and for performance analysis. This uniform representation can be achieved by employing membership functions with uniform shape and parametric functional definition. This Appendix deals with a new type of membership function called Semi-Circular Membership Functions (SCMF) for use with the fuzzy linguistic values. Some other, simple variants of the new membership function SCMF are also been introduced. These membership functions are used in the controller of the benchmarking control problem of balancing a simple inverted pendulum and the simulation results are presented.

Semi-Circular Membership Function (SCMF) is derived from the equation of a circle. The derivation is as given below. Consider the equation of a circle with center at (c,0) and radius r.

\[(x - c)^2 + (y - 0)^2 = r^2\]  
\[\Rightarrow y = \sqrt{r^2 - (x - c)^2}\]  

(247)  
(248)

The value of y in (248) will range from -r to r. To make the value of y range in [-1,1], the right side of (248) is divided by r.

\[y = \frac{\sqrt{r^2 - (x - c)^2}}{r}\]  
\[\Rightarrow y = \sqrt{1 - \frac{(x - c)^2}{r^2}}\]  

(249)  
(250)

Note that by taking the modulus of the terms in (250), the values can be normalized to fall in the range [0,1].

Thus the Semi-Circular Membership Function (SCMF) can be described using two parameters namely the center (c) and the radius (r) and the function takes the form given by (251). The figure 162(a) shows the corresponding plot.

\[ F(x) = \begin{cases} \sqrt{1 - \frac{(c - x)^2}{r^2}} & \text{if } -r < x - c < r \\ 0 & \text{otherwise.} \end{cases} \] (251)

Obviously, this function has the advantage of simplicity. It is symmetric and continuous. It's inverse is easily computable and hence defuzzification can easily be carried out.

Another advantage of this function is the ability of transformation from one co-ordinate system to another. As example, to convert from x-y plane to polar co-ordinate system, the formulae \( x = r \cos(\theta) \) and \( y = r \sin(\theta) \) can be applied.

Further, other variants of this membership functions like asymmetric SCMF, drum shaped SCMF and generalized SCMF are described below.

A. Asymmetric SCMF

It can be described by using three parameters namely the center (c), the left radius (R) and the right radius (r). The function is as shown by (252) and figure 162(b) shows the corresponding plot.
\[ F(x) = \begin{cases} \sqrt{1 - \frac{(c - x)^2}{r^2}} & \text{if } 0 \leq x - c < r \\ \sqrt{1 - \frac{(c - x)^2}{R^2}} & \text{if } -R < x - c < 0 \\ 0 & \text{otherwise.} \end{cases} \] (252)

B. Drum shaped SCMF

Drum shaped membership function can be described by using three parameters namely radius \((r)\), left-center \((c_1)\), right-center \((c_2)\), and the function takes the form as given by (253). Figure 162(c) shows the plot of the membership function.

\[ F(x) = \begin{cases} \sqrt{1 - \frac{(c_1 - x)^2}{r^2}} & \text{if } c_1 - r < x < c_1 \\ \sqrt{1 - \frac{(c_2 - x)^2}{r^2}} & \text{if } c_2 < x < c_2 + r \\ 1 & \text{if } c_1 \leq x \leq c_2 \\ 0 & \text{otherwise} \end{cases} \] (253)

C. Generalized SCMF

Generalized semi-circular membership function can be described by using four parameters namely left radius \((R)\), right radius \((r)\), left-center \((c_1)\), right-center \((c_2)\) and the function takes the form as given by (254). Figure 162(d) shows the plot of the membership function.

\[ F(x) = \begin{cases} \sqrt{1 - \frac{(c_1 - x)^2}{R^2}} & \text{if } c_1 - R < x < c_1 \\ \sqrt{1 - \frac{(c_2 - x)^2}{r^2}} & \text{if } c_2 < x < c_2 + r \\ 1 & \text{if } c_1 \leq x \leq c_2 \\ 0 & \text{otherwise} \end{cases} \] (254)
In this section, certain properties of membership functions, such as ‘granularity’ of fuzzy sets, percentage of data points for various range of membership values and cardinality of \( \alpha \) - cuts are compared. 10,000 equidistant sample points in the range \([-50, 50]\) are taken for each type of membership functions. The parameters considered in the simulation are: a TRIMF with \( a = -50, b = 0 & c = 50 \), a SCMF with \( c = 0 & r = 50 \), a GAUSSMF with \( c = 0, \sigma = 14 \), a GBELLMF with \( a = 12, b = 2, c = 0 \), a PIMF with \( a_1 = -50, c_1 = -0.001, a_2 = 50, c_2 = 0,001 \).

The number of data points having membership values less than 0.5 is 50.2\% for TRIMF, while in the case of SCMF it is only 13.4\%. It is very clear from the figure 163. This means 86.6\% of data points concentrate on the center of the SCMF with membership value more than 0.5, whereas it is only 49.8\% in the case of TRIMF, which explains the usefulness of this function. Table XXI shows the percentage of data points in various ranges of membership values.

The granularity of fuzzy sets reflects how “specific” they are. In other words, the term granularity deals with a size of an information granule being represented (embraced) by the concept conveyed therein. Very narrow intervals are definitely
more specific than broader ones. In the limit, if the set covers the entire universe of discourse, then its granularity level attains a minimum. The specificity of a fuzzy set \( A \) is defined as an integral [108] given by the (255).

\[
S(A) = \int_0^{\alpha_{max}} \frac{1}{1 + \exp\left(-\frac{1}{\text{card}(A_\alpha) - 0.5}\right)} d\alpha
\]  

(255)

Some parameters of fuzzy sets are related to specificity, for example, the cardinality of each \( \alpha \)-cut, \( A_\alpha \), is \( A_\alpha = 2\alpha(1 - \alpha) \) in the case of TRIMF and it is \( 2r\sqrt{1 - \alpha^2} \) in the case of SCMF. Figure 164 shows the relation between specificity and spread.

An interesting design question concerns the choice of information granules of some specificity. An analysis of the characteristics of the fuzzy controller reveals that the design concerns are affected by the granularity of the fuzzy sets in the control rules. The more linguistic terms (and the higher their granularity), the more adjustment can be made to the piecewise characteristics of the fuzzy controller.

There is another essential design practice of the fuzzy controller that links information granules with level of noise (disturbances) occurring in the system under control. The intention is to make the granules meaningful enough so that they help to ‘absorb’ existing disturbances. It is quite apparent that vary narrow membership functions which characterize fuzzy sets of higher specificity (higher granularity) may not be capable of handling noise.

Now when TRIMF & SCMF are compared, the specificity of SCMF is lower than that of TRIMF. Hence SCMFs have higher capability of handling noise than
TRIMF. The figure 164 shows the specificity of SCMF & TRIMF.

E. Simulation Results

The inverted pendulum model [145] is used for simulation with the fuzzy rules represented using the SCMF type membership function. The parameter values used in the simulations are: $l = 0.5$ m, $m = 0.1$ Kg and $M = 1$ Kg. Five labels are used to linguistically define $\theta$, $\dot{\theta}$ and $f$: Negative Big (NB), Negative Small (NS), Zero (ZO), Positive Small (PS), Positive Big (PB). Table XXII shows values corresponding to the linguistic labels namely NB, NS, ZO, PS, PB for the variables $\theta$, $\dot{\theta}$ used in the antecedent part and force ($f$) used in the consequent part. Figure 165 shows the plot of the membership functions of the variables $\theta$ and $\dot{\theta}$. Table XXIII shows the eleven rules for the controller. [The rules are taken from [78], after modifications]

Several experiments were carried out by varying the initial pole-angle. Figure 166 shows the plot of the pole angle, $\theta$ for initial state of $\theta = 11^\circ$. 
APPENDIX II. SEMI-CIRCULAR MEMBERSHIP FUNCTIONS ...

Fig. 165. Membership function plots of fuzzy labels used for $\theta$ & $\dot{\theta}$

TABLE XXII
MEMBERSHIP FUNCTION PARAMETERS FOR $\theta$ & $\dot{\theta}$

<table>
<thead>
<tr>
<th></th>
<th>$\theta$</th>
<th>$\dot{\theta}$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C</td>
<td>LS</td>
<td>RS</td>
</tr>
<tr>
<td>NB</td>
<td>-12</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>NS</td>
<td>-6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>ZO</td>
<td>0</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>PS</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>PB</td>
<td>12</td>
<td>6</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

C- Center, LS- Left Spread, RS- Right Spread

TABLE XXIII
RULES USED IN THE FUZZY CONTROLLER

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$\dot{\theta}$</th>
<th>$\theta$</th>
<th>$\dot{\theta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NB</td>
<td>NB</td>
<td>NB</td>
<td></td>
</tr>
<tr>
<td>NS</td>
<td>NS</td>
<td>NS</td>
<td></td>
</tr>
<tr>
<td>ZO</td>
<td>NB</td>
<td>NS</td>
<td>ZO</td>
</tr>
<tr>
<td>PS</td>
<td>PS</td>
<td>PS</td>
<td></td>
</tr>
<tr>
<td>PB</td>
<td>PB</td>
<td>PB</td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX II. SEMI-CIRCULAR MEMBERSHIP FUNCTIONS ...

In this Appendix, a new type of membership function namely, Semi-Circular Membership Function (SCMF), which could be used conveniently in the design of fuzzy systems is introduced. This membership function has many useful properties. In many application situations, it is better representation than the popular TRIMF. It has simple transform properties from one co-ordinate system to another. Moreover the specificity of SCMF is lower than TRIMF and hence SCMF has higher capability of handling noise than TRIMF. This membership function is continuous and has obvious advantages over conventional membership functions like triangular, trapezoidal and Gamma MFs.

Fig. 166. Plot of the $\theta$

F. Summary
REFERENCES


1998.


REFERENCES


[49] L. Holmblad, I.I. Ostergaard, "Control of a cement kiln by fuzzy logic," in
REFERENCES


2000.


REFERENCES


[84] Chien-Hua Lee, "New results for the bounds of the solution for the con-
REFERENCES...


REFERENCES


[121] E. Sanchez, T. Shibata, L. Zadeh, *Genetic algorithms and fuzzy logic*
REFERENCES


[157] H. Ying, "Constructing Nonlinear Variable Gain Controllers via the Takagi-


