Chapter III

Double-Diffusive Instabilities in Newtonian Porous Fluid Layer with Soret and Dufour Effects
3.1 Introduction

The origin and the interesting phenomena of two-component natural convection, viz. double-diffusive convection, is stated at length in chapter II. In chapter II, the onset criteria and the influences of cross-diffusion on the onset of double-diffusive convection in a nonporous Newtonian horizontal fluid layer heated and salted from above has been studied extensively. But in reality a large number of fluid layers are porous in nature, e.g. sandstone, limestone, wood, the human lung etc. So in this chapter we confine our study in natural double-diffusive convection in a porous fluid layer.

The study of the onset of double-diffusive convection in porous medium due to temperature and concentration gradients has been emerged as an interesting field of study because of its numerous fundamental and industrial applications specially the study of the behaviour of fluids in the crust of the earth, in geology, geophysics, metallurgy, material sciences and petroleum engineering. The appearance of convective motion is characterized by the critical value of the Darcy-Rayleigh number. The enormous volume of work of this field is being well documented in the book by Neild and Bejan (2006).

The problem of the onset of convective instability in a saturated porous medium under Darcy law was first analyzed by Horton, Rogers (1945) and Lapwood (1948) in a Newtonian fluid model heated uniformly from below. They obtained, for the stationary convection under the principle of exchange of stabilities, critical value of the Darcy-Rayleigh number characterizing the appearance of convective motion, as $4\pi^2$. This limit determines the particular point at which the convective instability may start to occur in a porous fluid layer. Thereafter so many theoretical results were elucidated in the works of Katto and Masuoka (1967), Combarnous and Bories (1975), Cheng (1978). Combarnous and Bories (1975) noted that the thermo-convective motion in a stationary mode become oscillatory with increasing value of the Darcy-Rayleigh number beyond six or seven times larger values of its critical one. The occurrence of oscillatory instabilities will be expected at the threshold of the stationary convection in a horizontal porous layer.

The double-diffusive instability, were observed first by Melvin Stern (1960) and the mechanism of instability at the onset of thermohaline convection in a porous
medium had been described by Nield (1968). Thermohaline convection in porous medium and its pattern of instabilities are important for study and have wider applications in chemical and industrial flows, geological and environmental flows and many others. Horton, Rogers and Lapwood made first the generalization of double-diffusive convection using Boussinesq approximation and later on, that was studied by Nield (1968), Turner (1974) and Turner (1985). In this case the critical value of Darcy-Rayleigh number for monotonic or stationary instability is increased by the increase of solutal Darcy-Rayleigh number. The linear stability analysis was used to investigate the onset of convection in a double-diffusive fluid flow [Taslim and Narusawa (1986) and Malashetty 1993)]. Rudraiah and Swami (1980) investigated the onset of cellular convection in a fluid-saturated porous layer using finite amplitude stability analysis. Rudraiah et al. (1987) studied mixed thermohaline convection in a porous layer, Rudraiah and Siddheswar (1998) studied weak nonlinear stability analysis of a double-diffusive convection in a porous fluid layer with cross-diffusion and Rudraiah et al. (2003) extended this problem to nonlinear stability analysis. Motsa (2008) investigated the effects of both the Soret and Dufour parameters on the onset of double-diffusive convection heated from below.

There are several studies on the onset of double-diffusive convection with and without cross-diffusion effects in a horizontal fluid layer in a saturated porous medium heated and salted uniformly from below. But there is quite a few studies on the onset of double diffusive convection with cross-diffusion effects in a horizontal fluid layer in a saturated porous medium heated and salted uniformly from above. In this study the double-diffusive convection with the effects of Soret and Dufour parameters in a horizontal layer of fluid saturated porous medium, under the framework of Darcy, heated and salted uniformly from above has investigated. Linear stability analysis is made to find the critical value of Darcy-Rayleigh number manifesting the onset of stationary mode of instabilities due to double-diffusive convection. The onset of oscillatory instabilities at the threshold of convection in a horizontal porous fluid layer is also investigated. This mechanism can be of interest in the transport of contaminants in ground water and since the ground water flow is relatively slow and so the Darcy model considered is quite appropriate.

### 3.2 Basic equations of the problem

We consider an infinite horizontal porous layer saturated with fluid of vertical depth $h$ and is confined between two stress-free boundaries maintained at constant temperature and salinity. The hot salty water is above the cold fresh water so that the upper boundary is kept at a temperature $(T + \Delta T)$ and salinity $(C + \Delta C)$ with the lower boundary at temperature $T$ and concentration $C$. It is assumed that the thermal diffusivity $\kappa_T$ is much greater than solutal diffusivity $\kappa_C$.

The porous medium with specific porosity parameter $K$ under the assumptions of homogeneous and isotropic medium and the validity of Darcy’s law is considered. The momentum equation may be written under the framework of Darcy model for simple flow as

$$u_i = -\frac{K}{\mu} \left( \frac{\partial p}{\partial x_i} - \rho g \delta_{i3} \right). \quad (3.1)$$

The density decreases with increase in temperature but it slightly increases with increase in concentration of the salt so that the thermal expansion coefficient $\alpha > 0$ and the solutal expansion coefficient $\beta > 0$.

The equation of state is then given by
\[ \rho = \rho_s [1 - \alpha(T - T_0) + \beta(C - C_0)]. \]  

(3.2)

We assume that the medium is homogeneous and isotropic and Darcy’s law is valid and the Boussinesq approximation is applicable. The appropriate governing equations are represented as follows:

\[ \frac{\partial u_i}{\partial x_i} = 0, \]  

(3.3)

\[ u_i = -\frac{K}{\mu} \left[ \frac{\partial p}{\partial x_i} - \rho_0 [1 - \alpha(T - T_0) + \beta(C - C_0)] g \delta_{i3} \right], \]  

(3.4)

\[ \sigma \frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \kappa_f \nabla^2 T + \beta_f \nabla^2 C, \]  

(3.5)

\[ \phi \frac{\partial C}{\partial t} + u_i \frac{\partial C}{\partial x_i} = \kappa_s \nabla^2 C + \beta_s \nabla^2 T, \]  

(3.6)

where \( i = 1, 2, 3 \) and \( K \) permeability of the porous media, \( \mu \) fluid viscosity, \( \sigma \) ratio of specific heat capacities of solid and fluid, \( \phi \) porosity of the porous media, \( \kappa_f \) thermal diffusivity, \( \kappa_s \) solutal diffusivity, \( \alpha \) coefficient of thermal expansion, \( \beta \) coefficient of solutal expansion, \( \beta_f \) Dufour coefficient and \( \beta_s \) Soret coefficient.

The boundary conditions are specified as

\( T = T_0 \) and \( C = C_0 \) at \( z = 0 \) and \( T = T_0 + \Delta T \) and \( C = C_0 + \Delta C \) at \( z = h \).  

(3.7)

The rising water blob absorbs heat much more quickly than it absorbs salt. Hence, it becomes lighter than the surrounding fluids and continues to rise. If a fluid blob drops by \( \Delta z \) the faster heat loss can make it heavier than its surroundings fluids and sinking continues. Thus, the fluid system can be unstable. The physical sketch of the problem is depicted in the Fig3.1.

### 3.3 Linear stability analysis

The basic state of the system is perturbed with infinitesimal disturbances and linear perturbation quantities are taken into consideration for the stability analysis of the system. The motion is then decomposed into a steady state of no motion and the perturbation quantities as follows:
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\[ u_i = 0 + u_i'(x,t), \]
\[ p = p_b(z) + p'(x,t), \]
\[ T = T_b(z) + T'(x,t), \]
\[ C = C_b(z) + C'(x,t). \]

(3.8)

The basic state satisfies the following equations

\[ \frac{\partial p_b}{\partial z} = \rho_0\{1 - \alpha(T_b - T_0) + \beta(C_b - C_0)\}g, \]

(3.9)

\[ \kappa_T\frac{d^2T_b}{dz^2} + \beta_T\frac{d^2C_b}{dz^2} = 0, \]

(3.10)

\[ \kappa_S\frac{d^2C_b}{dz^2} + \beta_S\frac{d^2T_b}{dz^2} = 0. \]

(3.11)

From the equations (3.10) and (3.11) we have

\[ T_b = T_0 + \frac{\Delta T}{h}z = T_0 + \Gamma_1z \quad \text{and} \quad C_b = C_0 + \frac{\Delta C}{h}z = C_0 + \Gamma_2z. \]

(3.12)

where \( \Gamma_1 = \frac{dT_b}{dz} > 0 \) and \( \Gamma_2 = \frac{dC_b}{dz} > 0. \)

Using the perturbation quantities (3.8) in the equations (3.3) to (3.6), considering only the linear perturbation terms and using the equations (3.9) to (3.12) the perturbed equations are

\[ \frac{\partial u_i'}{\partial x_i} = 0, \]

(3.13)

\[ u_i' = -\frac{K}{\mu}\left\{\frac{\partial p'}{\partial x_i} + \rho_0g(\alpha T' - \beta C')\delta_{ij}\right\}, \]

(3.14)

\[ \sigma\frac{\partial T'}{\partial t} + \Gamma_1w' = \kappa_T\nabla^2T' + \beta_T\nabla^2C', \]

(3.15)

\[ \phi\frac{\partial C'}{\partial t} + \Gamma_2w' = \kappa_S\nabla^2C' + \beta_S\nabla^2T'. \]

(3.16)

To express the perturbation equations in terms of \( w', T' \) and \( C' \) by eliminating the unknown pressure term, using certain mathematical manipulations, we get the perturbation equations as

\[ \nabla^2w' = -\frac{K}{\mu}\{\rho_0g\alpha\nabla^2T' - \rho_0g\beta\nabla^2C'}\}, \]

(3.17)
\[ \sigma \frac{\partial T'}{\partial t} + \Gamma_1 w' = \kappa_T \nabla^2 T' + \beta_T \nabla^2 C', \quad (3.18) \]

\[ \phi \frac{\partial C'}{\partial t} + \Gamma_2 w' = \kappa_S \nabla^2 C' + \beta_S \nabla^2 T', \quad (3.19) \]

where \( \nabla^2_T = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \), is the planar Laplacian.

The boundary conditions are expressed as

\[ w' = T' = C' = 0 \quad \text{at the free surfaces } z=h \text{ and } z=0. \]

The last two conditions are resulting from zero stress. As \( w' = 0 \) for all \( x \) & \( y \) on the boundaries, then \( (\partial u'/\partial z) = (\partial v'/\partial z) = 0 \) at the boundaries. From the equation of continuity

\[ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0 \]

and differentiating with respect to \( z \) and using the conditions \( (\partial u'/\partial z) = (\partial v'/\partial z) = 0 \) we have \( (\partial^2 w'/\partial z^2) = 0 \) at the free surfaces.

Finally, the boundary conditions are given below as

\[ w' = T' = C' = \frac{\partial^2 w'}{\partial z^2} = 0 \quad \text{at } z=0 \text{ and } z=h. \quad (3.20) \]

Using the following transformations we introduce the dimensionless variables

\[ (x, y, z) \rightarrow (x^*, y^*, z^*, h), t \rightarrow (h^2 \sigma / \kappa_T) t^*, w' \rightarrow (\kappa_T / h) w^*, T' \rightarrow \Delta T. \theta^*, C' \rightarrow \Delta C.C^* \]

and dropping “*” to the co-ordinates \( (x, y, z) \) and time \( t \) for brevity, the equations (3.17) to (3.19) become

\[ \nabla^2 w^* = -(K_T \alpha \Delta Th)/(\nu \kappa_T) \nabla^2 \theta^* + (K_T \beta \Delta Ch)/(\nu \kappa_T) \nabla^2 C^*, \quad (3.21) \]

\[ \frac{\partial \theta^*}{\partial t} + w^* = \nabla^2 \theta^* + (\beta_T \Delta C)/(\kappa_T \Delta T) \nabla^2 C^*, \quad (3.22) \]

\[ (\phi / \sigma) \frac{\partial C^*}{\partial t} + w^* = (\kappa_S / \kappa_T) \nabla^2 C^* + (\beta_S \Delta T)/(\kappa_T \Delta C) \nabla^2 \theta^*. \quad (3.23) \]

The boundary conditions are

\[ w^* = C^* = \theta^* = \frac{\partial^2 w'}{\partial z^2} = 0 \quad \text{for } z = 0 \text{ & } z = 1. \quad (3.24) \]
3.4 Normal mode analysis

Under the normal mode analysis the time-dependent periodic disturbances in a horizontal plane may be taken the following form

\[(w^*, \theta^*, C^*) = (W(z), \theta(z), C(z))e^{i(a_x x + a_y y) + st}\]  

(3.25)

where ‘i’ is the imaginary number, \(a_x\) and \(a_y\) are dimensionless wave numbers in the \(x-y\) plane. The temporal rate of change of disturbances \(s\) can be decomposed as \(s = s_r + i\omega\). If \(s_r\) is positive for any value of the wave number, the system is unstable. While the system with \(s_r < 0\) is always stable. When \(s = 0\) and so \(s_r = \omega = 0\), then the system is marginally stable under the principle of exchange of stabilities. With \(s_r = 0\) and \(\omega \neq 0\), the overstability of periodic motion is possible at the marginal state and oscillatory motion appears with the dimensionless frequency \(\omega\).

Defining the non-dimensional parameters, the Lewis number \(Le = \kappa_T / \kappa_S\), the Darcy-Rayleigh number \(Ra_D = (Kg\alpha h\Delta T)/(\nu\kappa_T)\), the solutal Darcy-Rayleigh number \(Ra_{D,S} = (Kg\beta h\Delta C)/(\nu\kappa_S)\), the Soret number \(S_r = (\beta_S \Delta T)/(\kappa_T \Delta C)\), the Dufour number \(D_f = (\beta_T \Delta C)/(\kappa_S \Delta T)\) and then using (3.25) in the equations (3.21) to (3.23) the equations transform to

\[ (D^2 - a^2)W = a^2 Ra_D \theta - \frac{1}{Le} a^2 Ra_{D,S} C, \]  

(3.26)

\[ W = (D^2 - a^2 - s)\theta + D_f (D^2 - a^2)C, \]  

(3.27)

\[ W = \left\{ \frac{1}{Le} (D^2 - a^2) - \frac{\phi}{\sigma} s \right\} C + S_r (D^2 - a^2)\theta. \]  

(3.28)

and the boundary conditions (3.24) become

\[ W = \theta = C = 0 \text{ and } D^2 W = 0 \text{ at } z = 0 \& z = 1 \]  

(3.29)

where \(D(d/dz)\) denotes the differential operator. Equations (3.26) to (3.28) are combined to get a single equation, characterizing the instability of the system, as

\[ \{a^2 Ra_D - (D^2 - a^2)(D^2 - a^2 - s)} \left\{ (1 - Le D_f)(D^2 - a^2) - Le \frac{\phi}{\sigma} s \right\} \theta \]  

\[ = \left\{ a^2 \frac{1}{Le} Ra_{D,S} + D_f (D^2 - a^2)^2 \right\} \left\{ Le(1 - S_r)(D^2 - a^2) - Le s \right\} \theta \]  

(3.30)
If $S_r \to 0$ and $D_f \to 0$ then the foregoing problem approaches the classical Darcy-Rayleigh problem with Newtonian fluid as studied by Horton, Rogers (1945) and Lapwood (1948).

The boundary conditions are reduced as

$$\theta = D^2 \theta = 0 \text{ at } z = 0, 1.$$ (3.31)

Both boundaries are isothermal and viscous stress-free in Darcy’s law. The solution of the governing equation (3.30) with boundary conditions (3.31) can be regarded as the usual eigenvalue problem involving eigenparameters $Ra_D, Le, S_r, D_f, a$ and $s$ as the governing parameters. In order to solve the present differential equation (3.30) with boundary conditions (3.31), the required solution is assumed as $\theta = \theta_0 \sin(n \pi z)$ where $\theta_0$ denotes the integration constant subject to the boundary conditions and $n$ is an integer.

Substituting this form of solution in equation (3.30) results in the following characteristic equation

$$\{a^2 Ra_D - (n^2 \pi^2 + a^2)(n^2 \pi^2 + a^2 + s)\} \left\{ (1 - Le D_f) (n^2 \pi^2 + a^2) + Le \frac{\phi}{\sigma} s \right\} = \left\{ a^2 \frac{1}{Le} Ra_{D,S} + D_f (n^2 \pi^2 + a^2)^2 \right\} \left\{ Le (1 - S_r) (n^2 \pi^2 + a^2) + Le s \right\}.$$ (3.32)

This algebraic equation can be rearranged into the form of second degree polynomials of $s$, with the consideration $(n^2 \pi^2 + a^2) = J$, as

$$s^2 + \frac{1}{J \Phi} \{ (1 + \Phi) J^2 - a^2 (Le Ra_D - Ra_{D,S}) \} s$$

$$+ \frac{1}{\Phi} \{ (1 - Le S_r D_f) J^2 - a^2 [(1 - D_f Le) Ra_D - (1 - S_r) Ra_{D,S}] \} = 0.$$ (3.33)

where $a = \sqrt{a_x^2 + a_y^2}$ is the horizontal wave number and $\Phi = Le (\phi / \sigma)$. This is a quadratic equation in $s$, where the coefficients involve the governing parameters are all real. It has two roots, so if complex roots exist that must be in pair otherwise a pair of real roots. We are interested in the marginal stability analysis and in that case $s_r = 0$, where $s_r$ is the real part of the complex roots ($s = s_r \pm i \omega$) and it is apparent that the marginal state ($s_r = 0$) occurs with two cases $\omega = 0$ and $\omega \neq 0$, where $\omega$ is the imaginary part of the complex roots.
When the marginal state \((s_0 = 0)\) is characterized by a stationary pattern of convection for which \(\omega = 0\) so we have \(s = 0\). Then one of the roots of the quadratic equation (3.33) is zero and the condition for that, which gives the Darcy-Rayleigh number for the steady convection, is

\[
Ra_D^{(\text{steady})} = \frac{(1 - Le D_f S_f) J^2}{(1 - Le D_f)} a^2 + \frac{(1 - S_f) Ra_{D,S}}{(1 - Le D_f)}.
\]

(3.34)

Thus the marginal state with cellular convection \((\omega = 0\) and so \(s = 0)\) i.e. the stationary convection is governed by the relation (3.34). The critical value of Darcy-Rayleigh number for stationary convection or monotonic instability is obtained from this equation.

It is apparent that a marginally oscillatory mode \((\omega \neq 0)\) occurs only when there exists a pair of purely imaginary roots of the quadratic equation (3.33) and the instability then sets in as oscillatory convection of growing amplitude known as “overstability”. In the case of oscillatory instability \(s = i\omega\), where \(\omega\) is a non-zero real number, then from equation (3.33) the Rayleigh number is obtained, according to Chandrasekhar (1981), as

\[
a^2 \{(1 - Le D_f)^2 J^2 + \Phi^2 \omega^2\} Ra_D =
\]

\[
(1 - Le D_f) \{(1 - Le S_f D_f) J^2 + a^2 (1 - S_f) Ra_{D,S}\} J^2 + \{(\Phi + Le D_f) + a^2 Ra_{D,S}\} \Phi \omega^2
\]

\[+ i \{(1 - Le D_f) - \Phi (1 - S_f) Le D_f\} J^2 + a^2 \{(1 - Le D_f) - \Phi (1 - S_f)) Ra_{D,S} + \Phi^2 \omega^2\} J \omega\]

(3.35)

From physical considerations, the Darcy-Rayleigh number \((Ra_D)\) is always a real number. Then from the relation (3.35), the imaginary part must be zero, which gives the square of the frequency of the periodic convection as

\[
\Phi^2 \frac{\omega^2}{a^2} = \{\Phi (1 - S_f) Le D_f - (1 - Le D_f)\} \frac{J^2}{a^2} + \{\Phi (1 - S_f) - (1 - Le D_f)\} Ra_{D,S}
\]

(3.36)

and the Darcy-Rayleigh number for oscillatory convection is given by

\[
Ra_D^{(\text{osc})} = \frac{(1 - Le D_f)^2 J^2}{(1 - Le D_f)^2 J^2 + \Phi^2 \omega^2} \left\{\frac{(1 - Le D_f S_f) J^2}{(1 - Le D_f)} a^2 + \frac{(1 - S_f) Ra_{D,S}}{(1 - Le D_f)}\right\}
\]

\[+ \frac{1}{\Phi} \frac{\Phi^2 \omega^2}{(1 - Le D_f)^2 J^2 + \Phi^2 \omega^2} \left\{\Phi + Le D_f\right\} \frac{J^2}{a^2} + Ra_{D,S}\}

\]

(3.37)

The Darcy-Rayleigh number for oscillatory convection is obtained from the relation (3.37), by using (3.36) as
\[ \Phi \text{Ra}_{(osc)} = (\Phi + 1) \frac{J^2}{a^2} + \text{Ra}_{D,S} \]  

(3.38)

Thus the marginal state with oscillatory convection viz. over stability \((\omega \neq 0)\) is governed by the relation (3.38) and the critical value of Darcy-Rayleigh number for oscillatory instability is obtained from this relation.

The condition for oscillatory convection, on the consideration that \(\omega\) is always real, obtains from the relation (3.36) as

\[ \text{Ra}_{D,S} > \frac{(1 - \text{LeD}_f) - \Phi (1 - S_f) \text{LeD}_f}{\Phi (1 - S_f) - (1 - \text{LeD}_f)} \frac{J^2}{a^2}. \]  

(3.39)

Equation (3.39) leads to an important result that for a porous fluid layer heated and salted from above in double-diffusive convection, the oscillatory type of instability is possible for certain fluid mixtures.

It reveals that in double-diffusive system the steady convection strongly depends on the cross-diffusive terms while the oscillatory convection remains independent of the cross-diffusive parameters but depends on the porosity parameter \(\Phi\).

### 3.5 Results and discussion

We are mainly interested to study the instability at marginally steady and oscillatory modes. The characteristic equation (3.33) may be represented as

\[ s^2 + A_1 s + A_2 = 0 \]  

(3.40)

where

\[ A_1 = \frac{1}{J \Phi} \{(1 + \Phi) J^2 - a^2 (\text{LeRa}_D - \text{Ra}_{D,S})\}, \]

\[ A_2 = \frac{1}{\Phi} \{(1 - \text{LeS}_f D_f) J^2 - a^2 \{(1 - D_f \text{Le}) \text{Ra}_D - (1 - S_f) \text{Ra}_{D,S}\}\}. \]

For linear stability of the system the necessary and sufficient condition, according to Routh-Hurwitz criteria (for negative real parts of the roots of the characteristics equation), are \(A_1 > 0\) and \(A_2 > 0\). If any one of the inequalities is not true then the instability sets in. The double-diffusive system with cross-diffusive effects, the stability of the system depends upon the coefficients of the equation (3.40) containing a large number of dimensionless parameters. The task of discussing stability of the system or the instabilities set in can be simplified by fixing some of the parameters. Accordingly some of the parameters are fixed by us and it is observed that for certain
values of the governing parameters, the instability occurs in the above system. We are interested to find the critical value of the Darcy-Rayleigh number at the onset of instability with cross-diffusive effects and the type of the instability at the onset.

When both the Soret and Dufour effects are present in the double-diffusive system, the marginal state characterized the stationary pattern of convection is given by the relation (3.34). It is observed from the relation (3.34) that the stationary or monotonic instability strongly depends on the cross-diffusion terms viz. Soret and Dufour co-efficients. Minimizing firstly over $n$ that is at the threshold of convection $n=1$ and then a further minimization over the square of wave number ($a^2$), we have the critical value of the Darcy-Rayleigh number for the onset of stationary convection, the boundary for monotonic or stationary instability, is obtained from (3.34) as

$$Ra_{D_s}^{(steady)} = \left(\frac{1-LeD_f}{1-LeD_f}\right) 4\pi^2 + \left(\frac{1-S_r}{1-LeD_f}\right) Ra_{D,S}. \tag{3.41}$$

This result is identical with the result obtained by Kuznetsov and Nield (2010) on the consideration of regular fluid instead of nanofluid. It is interesting to note that when $S_r = 1$, the critical value of Darcy-Rayleigh number for steady double-diffusive convection becomes $Ra_{D_s}^{(steady)} = 4\pi^2$, the critical value of the Darcy-Rayleigh number for thermal convection in porous media.

The condition for oscillatory instability in this double-diffusive system as obtained from the relation (3.39), by considering the minimum value of $J^2/a^2 = 4\pi^2$, satisfying the Solutal Darcy-Rayleigh number is

$$Ra_{D,S} > \frac{(1-LeD_f)-\Phi(1-S_r)LeD_f}{\Phi(1-S_r)-(1-LeD_f)} 4\pi^2. \tag{3.42}$$

If the condition (3.42) is satisfied, the marginal state is characterized by the oscillatory pattern of convection ($\omega \neq 0$) which is given by the relation (3.38) and then the critical value of the Darcy-Rayleigh number for the oscillatory instability at the threshold of convection is given by

$$\Phi Ra_{D_s}^{(osc)} = (\Phi+1)4\pi^2 + Ra_{D,S}. \tag{3.43}$$

For a special type of porous medium when $\Phi = 1$, the value of the critical value of the Darcy-Rayleigh number ($Ra_D$) for oscillatory convection at the marginal state becomes $Ra_D^{osc} = 8\pi^2 + Ra_{D,S}$. In that case the boundary lines for the monotonic
instability and the oscillatory instability are parallel in the plane \((Ra_p, Ra_{D,S})\) and so the steady convection is always followed by oscillatory convection. When \(\Phi \neq 1\), the boundary lines for monotonic and oscillatory convection are intersecting. The points of intersection is given by

\[
Ra_p = \frac{\Phi(1-S_r)-(1-LeD_f)S_r}{\Phi(1-S_r)-(1-LeD_f)} \pi^2
\]

and

\[
Ra_{D,S} = \frac{(1-LeD_f)-\Phi(1-S_r)LeD_f}{\Phi(1-S_r)-(1-LeD_f)} \pi^2.
\]

(3.44)

It is to be noted that the Dufour effects are prominent in gases. In liquids, its value is negligible of an order smaller than the Soret term. Now we shall study the influences of Soret effect on the double-diffusive convection in the absence of Dufour effects or with its constant value and in both the cases we observe the same result. The critical value of the Darcy-Rayleigh number for stationary convection and the condition for oscillatory convection with the frequency of the periodic convection depends on the Soret co-efficient when the Dufour parameter is held constant, while the value of critical Darcy-Rayleigh number for oscillatory convection does not depend on the Soret or Dufour co-efficient. Now in the presence of solute \((Ra_{D,S} \neq 0)\), the critical value of the Darcy-Rayleigh number for stationary convection decreases when the Soret number increases but less than 1, which is observed in Fig 3.2 and for Soret number greater than equal to 1 the oscillatory convection does not observed, which follows from relation (3.36) with the consideration that \(\omega\) is always real. As the Soret number increases the region of stability decreases in the \(Ra_p - Ra_{D,S}\) plane, which is depicted in Fig 3.3. So the Soret effect serves to advance the onset of convection. Thus the Soret effect destabilizes the flow.

Next we shall study the influences of Dufour effects on the double-diffusive convection. In this case also we observe the same result in the absence of Soret parameter or with its fixed value. It is observed in the Fig 3.4 that the critical value of the Darcy-Rayleigh number for stationary convection will increase when the Dufour number, provided \(D_f < (1/Le)\) i.e. the value of Dufour number depending on the Lewis number, increases. As the Dufour number increases, the region of stability
increases in $Ra_D \sim Ra_{D,S}$ plane, which is illustrated in Fig.3.5. So the Dufour effect serves to delay the onset of convection. Hence the Dufour effect tends to stabilize the flow.

The critical value of the Darcy-Rayleigh number for the oscillatory convection at the marginal state, as in the relation (3.43) depends on the porosity parameter $\Phi$ and Fig.3.6 exhibits that the oscillatory convection prevails earlier with higher value of $\Phi$. Hence the porosity parameter has tendency to destabilize the fluid flow. Though the Darcy-Rayleigh number for oscillatory convection does not depend on the Soret and Dufour number, but the condition for oscillatory convection (42) and frequency of oscillatory convection (36) depends on the cross-diffusion terms. It is observed that for Soret number greater than equal to one the oscillatory convection does not occur. It is also to be noted that the frequency of oscillatory convection decreases for increasing Soret number while increases with increasing Dufour number as depicted in Fig.3.7 and Fig.3.8 respectively.

In the absence of cross-diffusive effects that is the Soret and Dufour parameter, the critical value of the Darcy-Rayleigh number for steady convection becomes $Ra_{D,cr}^{(steady)} = 4\pi^2 + Ra_{D,S}$. Thus in the absence of cross-diffusion parameters the critical value of the Darcy-Rayleigh number increases with increasing Solutal Darcy-Rayleigh number and so the solutal Darcy-Rayleigh number tends to stabilize the system. The critical value of Darcy-Rayleigh number for oscillatory convection, from the relation (3.43), is $\Phi Ra_{D,cr}^{(osc)} = (\Phi + 1)4\pi^2 + Ra_{D,S}$. For all practical values of the parameters, the porosity parameter must be $\Phi = (\phi / \sigma)Le > 1$ (Straughan, 2011) and in that case the boundary lines for stationary and oscillatory convection can never be parallel. In this case the points of intersection of the boundary lines for stationary and oscillatory convection are

$$Ra_D = \frac{\Phi}{\Phi - 1} 4\pi^2 \quad \text{and} \quad Ra_{D,S} = \frac{1}{\Phi - 1} 4\pi^2.$$ \hspace{1cm} (3.45)

The oscillatory convection is not possible for $\Phi \leq 1$ as follows from the relation (3.45) and in those cases stationary convection prevails long after the onset of instability. When $(Ra_D - Ra_{D,S})$ exceeds $(4\pi^2)$ long thin salt fingers are observed and if the salinity gradient is large, then experiments as well as calculations show that a
Double-Diffusive Instabilities in Newtonian Porous Fluid …

deep layer salt fingers become unstable and breaks down into a series of convecting layers with fingers confined to the interfaces. It is observed from the relation (45) that as \( \Phi \) increases beyond 1, the oscillatory convection prevails earlier but always preceded by the stationary convection for all values of \( \Phi \) greater than 1 and for correspondingly small values of the Solutal Darcy-Rayleigh number in the absence of cross-diffusion terms (e.g.\( \Phi = 100, Ra_{p,s} = 0.3984 \)) (Griffiths, 1981).

It is not possible to conclude analytically the type of instability at the onset when both the cross-diffusion terms are present from the relations (3.34) and (3.38). We investigate the nature of instability at the onset numerically with the help of graphs. It is observed from the Fig3.9 that the entire curve representing the oscillatory convection lies below the curve representing the steady convection for \( S_r = 0.0, D_f = 0.0001, \Phi = 30 \) and thus \( Ra_{p,s}^{(osc)} < Ra_{p,s}^{(steady)} \). Hence for these values of the Soret, Dufour and porosity parameter the onset of instability is manifested by oscillatory convection. But in Fig3.10 with \( S_r = 0.08, D_f = 0.0001 \) and for all values of \( \Phi (=10,20,30) \), it is observed that the entire curve representing the steady convection lies below the curves representing the oscillatory convection for all values of \( \Phi \), so the steady convection prevails at the onset. While Fig3.11 depicts that with \( S_r = 0.08, D_f = 0.0005 \) and for \( \Phi (=20,30) \) the onset is manifested by the oscillatory convection and Fig3.12 illustrates that for \( S_r = 0.6, D_f = 0.0005 \) with all values of \( \Phi (=10,20,30) \), the steady convection prevails at the onset. For clarity, Table 3.1 and Table 3.2 is presented showing the critical value of the Darcy-Rayleigh number, representing the onset of convection, for some typical values of \( S_r, D_f, \Phi \).

Thus analyzing the graphs in the figures Fig3.9 to Fig3.12 and the critical values of the Rayleigh number in Tables 3.1 and 3.2, it reveals that the increasing Dufour number with higher value of the porosity parameter provokes oscillatory instability at the onset whereas the increasing Soret number instigates stationary instability at the onset. Hence the type of instability at the onset is influenced by the Soret and Dufour effects along with the porosity parameter.
Fig 3.1 Physical sketch of the problem representing an infinite horizontal saturated porous fluid layer with linear distribution of temperature and solute.

Fig 3.2 The variations of Darcy-Rayleigh number with the square of the wave number for different values of the Soret number in stationary convection.
Fig 3.3 The effects of Soret number on the boundary of monotonic or stationary instability in the $R_{a_D} \sim R_{a_{D,S}}$ plane.

Fig 3.4 The variations of Darcy-Rayleigh number with the square of the wave number for different values of the Dufour number in stationary convection.
Fig3.5 The effects of Dufour number on the boundary of monotonic or stationary instability in the $Ra_D \sim Ra_{D,S}$ plane.

Fig3.6 The variations of Darcy-Rayleigh number with the square of the wave number for different values of the porosity parameter $\Phi$ in oscillatory convection.
**Fig 3.7** The variations of the square of frequency of the periodic convection with the square of the wave no. for different values of the Soret number.

**Fig 3.8** The variations of the square of frequency of the periodic convection with the square of the wave number for different values of the Dufour number.
Fig. 3.9 The variations of Rayleigh number with square of wave number in steady and oscillatory convection with $Le = 100, Ra_{D,S} = 1.0, S_R = 0.0, D_f = 0.0001$.

Fig. 3.10 The variations of Rayleigh number with square of wave number in steady and oscillatory convection with $Le = 100, Ra_{D,S} = 1.0, S_R = 0.08, D_f = 0.0001$. 
Fig 3.11 The variations of Rayleigh number with square of wave number in steady and oscillatory convection with $Le = 100, Ra_{D,s} = 1.0, S_r = 0.08, D_f = 0.0005$.

Fig 3.12 The variations of Rayleigh number with square of wave number in steady and oscillatory convection with $Le = 100, Ra_{D,s} = 1.0, S_r = 0.6, D_f = 0.0005$. 
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**Table 3.1** The critical values of the Darcy-Rayleigh number, representing onset of convection, for some typical values of the Soret number, Dufour number and porosity parameter.

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<th>$Ra_{D,S}$</th>
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<th>$D_f$</th>
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**Table 3.2** The critical values of the Darcy-Rayleigh number, representing onset of convection, with cross-diffusion parameters.

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