CHAPTER 2

Co-ordinatewise Product $\mathcal{A} \times \mathcal{B}$

In this chapter, we study the stability of some Banach algebra properties in $\mathcal{A} \times \mathcal{B}$ with respect to the co-ordinatewise operations. Such kind of Banach algebras have been studied by several people \footnote{24, 42, 43, 53}. Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras. It is proved that the Gelfand space (resp., the Shilov boundary) of $\mathcal{A} \times \mathcal{B}$ is homeomorphic to the topological sum of the Gelfand spaces (resp., the Shilov boundaries) of $\mathcal{A}$ and $\mathcal{B}$. Based on this result, we prove that the properties like regularity, UUNP, $UC^*\text{-}NP$, MHBP, SSEP, QDZ, TAN etc. can be carry forwarded from $\mathcal{A}$ and $\mathcal{B}$ to $\mathcal{A} \times \mathcal{B}$, and vice-versa. On the other hand, the property TDZ holds true only one way.

The results proved in this chapter are published in \footnote{24}.

2.1. Definition of Co-ordinatewise Product

We start with the definition of co-ordinatewise operation on the cartesian product $\mathcal{A} \times \mathcal{B}$ of two algebras $\mathcal{A}$ and $\mathcal{B}$. Then we define different norms on $\mathcal{A} \times \mathcal{B}$. We show that they all are equivalent to the maximum norm $\| \cdot \|_\infty$.

**Definition 2.1.1.** Let $\mathcal{A}$ and $\mathcal{B}$ be algebras. Let $\mathcal{A} \times \mathcal{B}$ denote the cartesian product of $\mathcal{A}$ and $\mathcal{B}$. Define the co-ordinatewise product on $\mathcal{A} \times \mathcal{B}$ as

$$(a,b)(c,d) = (ac,bd) \quad ((a,b), (c,d) \in \mathcal{A} \times \mathcal{B}).$$

Then $\mathcal{A} \times \mathcal{B}$ is an algebra with the co-ordinatewise product.
If \((A, \| \cdot \|_A)\) and \((B, \| \cdot \|_B)\) are Banach algebras, then \(A \times B\) is a Banach algebra with respect to the maximum norm
\[
\|(a, b)\|_\infty := \max\{\|a\|_A, \|b\|_B\} \quad ((a, b) \in A \times B).
\]

Proposition 2.1.2. Let \(A\) and \(B\) be algebras. Then

(i) \(A \cong A \times \{0\}\) and \(B \cong \{0\} \times B\) are closed ideals of \(A \times B\).

(ii) \(A \times B\) is commutative if and only if \(A\) and \(B\) are commutative.

(iii) \(A \times B\) is unital if and only if \(A\) and \(B\) are unital.

Proof. (i) and (ii) are trivial.

(iii). Suppose \(A \times B\) is unital. Let \((e, f)\) be the identity for \(A \times B\). Let \(a \in A\). Then \((a, 0) = (a, 0)(e, f) = (ae, 0)\) and \((a, 0) = (e, f)(a, 0) = (ea, 0)\). Thus \(ae = ea = a\). Hence, \(e\) is the identity for \(A\). Similarly, it follows that \(f\) is the identity for \(B\).

Conversely, suppose that \(A\) and \(B\) are unital. Let \(e\) and \(f\) be the identities for \(A\) and \(B\), respectively. Then \((a, b)(e, f) = (a, b) = (e, f)(a, b)\) for all \((a, b) \in A \times B\). Thus \((e, f)\) is the identity for \(A \times B\). Hence, \(A \times B\) is unital. \(\square\)

Proposition 2.1.3. Let \(A\) and \(B\) be unital algebras and \((a, b) \in A \times B\). Then

(i) \((a, b) \in (A \times B)^{-1}\) iff \(a \in A^{-1}\) and \(b \in B^{-1}\);

(ii) \((a, b) \in (A \times B)_{-1}\) iff \(a \in A_{-1}\) and \(b \in B_{-1}\);

(iii) \(\sigma_{A \times B}(a, b) = \sigma_A(a) \cup \sigma_B(b)\);

(iv) \(r_{A \times B}(a, b) = \max\{r_A(a), r_B(b)\}\);

(v) if \(A\) and \(B\) are division algebras, then \(A \times B\) may not be the same.

Proof. (i). Let \(e\) and \(f\) be the identities for \(A\) and \(B\), respectively. Suppose that \((a, b)\) is invertible in \(A \times B\). Then there exists \((c, d) \in A \times B\) such that \((a, b)(c, d) = (c, d)(a, b) = (e, f)\). Which implies \(ac = ca = e\) and \(bd = db = f\). Thus \(a \in A^{-1}\) and \(b \in B^{-1}\). The converse follows by similar arguments.
(ii). Suppose that \((a,b) \in (A \times B)^{-1}\). Then there exists \((c,d) \in A \times B\) such that \((a,b) \circ (c,d) = 0\). This implies \(a + c - ac = 0\) and \(b + d - bd = 0\). i.e., \(a \circ c = 0\) and \(b \circ d = 0\). Thus \(a \in A^{-1}\) and \(b \in B^{-1}\). The converse follows by similar arguments.

(iii). This follows from (i).

(iv). This follows from (iii).

(v). Take \(A = B = \mathbb{C}\). Then it is obvious that \(A\) and \(B\) are division algebras whereas \(A \times B = \mathbb{C}^2\) is not a division algebra. \(\square\)

**Theorem 2.1.4.** Let \(X\) and \(Y\) be normed linear spaces, \(1 \leq p \leq \infty\) and \((x,y) \in X \times Y\). Define
\[
\|(x,y)\|_p := \begin{cases} 
(\|x\|^p + \|y\|^p)^{1/p}, & (1 \leq p < \infty); \\
\max\{\|x\|, \|y\|\}, & (p = \infty).
\end{cases}
\]
Then \(\cdot\|_p\) is a norm on \(X \times Y\).

The next result is a slight modification of [41, Theorem 3.4.2].

**Theorem 2.1.5.** [41, Theorem 3.4.2] Let \((X, \| \cdot \|)\) be a norm linear space and let \(1 \leq p \leq \infty\). Let \(\theta, \eta : [0,1] \rightarrow [0,1]\) be onto continuous functions such that \(\theta(t) + \eta(t) = 1\) \((0 \leq t \leq 1)\). For \((x,y) \in X \times X\), define
\[
\|(x,y)\|_p := \begin{cases} 
\left( \int_0^1 \|\theta(t)x + \eta(t)y\|^p dt \right)^{\frac{1}{p}}, & (1 \leq p < \infty); \\
\sup \{ \|\theta(t)x + \eta(t)y\| : t \in [0,1] \} & (p = \infty).
\end{cases}
\]
Then each \(\| \cdot \|_p\) is a norm on \(X \times X\).

**Proof.** Let \(1 \leq p < \infty\). Let \(x, y \in X\). It is clear from definition that
\[
\|(x,y)\|_p \geq 0.\] If \(\|(x,y)\|_p = 0\), then \(\int_0^1 \|\theta(t)x + \eta(t)y\|^p dt = 0\). This implies
\[ \|\theta(t)x + \eta(t)y\|^p = 0 \quad (0 \leq t \leq 1). \] Therefore, \( \theta(t)(x - y) + y = 0 \). Since \( \theta \) is onto, we get \( x = y = 0 \). It is clear that \( \|\alpha(x, y)\|_{[p]} = |\alpha|\|(x, y)\|_{[p]} \) \( (\alpha \in \mathbb{C}) \).

Next, let \( x_1, x_2, y_1, y_2 \in X \). Then, using Minkowski’s inequality, we get

\[
\|(x_1, y_1) + (x_2, y_2)\|_{[p]} = 
\left( \int_0^1 \|\theta(t)(x_1 + x_2) + \eta(t)(y_1 + y_2)\|^p dt \right)^{\frac{1}{p}} 
\leq \left( \int_0^1 \|\theta(t)x_1 + \eta(t)y_1\|^p dt \right)^{\frac{1}{p}} + \left( \int_0^1 \|\theta(t)x_2 + \eta(t)y_2\|^p dt \right)^{\frac{1}{p}} = 
\|(x_1, y_1)\|_{[p]} + \|(x_2, y_2)\|_{[p]}.
\]

Thus \( \|\cdot\|_{[p]} \) is a norm on \( X \times X \). It is clear that \( \|\cdot\|_{[\infty]} \) is a norm.

The next result is also a modification of [41, Theorem 3.2.2].

**Theorem 2.1.6.** [41, Theorem 3.2.2] Let \( (X, \|\cdot\|) \) and \( (Y, \|\cdot\|) \) be normed spaces and \( 1 \leq p \leq \infty \). Let \( \theta, \eta : [0, 1] \rightarrow [0, 1] \) be onto continuous functions such that \( \theta(t) + \eta(t) = 1 \quad (0 \leq t \leq 1) \). For \( (x, y) \in X \times Y \), define

\[
\|(x, y)\|_{(p)} := \begin{cases} 
\left( \int_0^1 [\theta(t)\|x\| + \eta(t)\|y\|]^p dt \right)^{\frac{1}{p}}, & (1 \leq p < \infty); \\
\sup\{\theta(t)\|x\| + \eta(t)\|y\| : t \in [0, 1]\}, & (p = \infty). 
\end{cases}
\]

Then \( \|\cdot\|_{(p)} \) is a norm on \( X \times Y \).

**Proof.** This proof is similar to the proof of Theorem 2.1.5. \( \square \)
The following are different forms of Jensen’s inequality.

**Lemma 2.1.7.** Let $X$ and $Y$ be norm linear spaces and $1 \leq p < q \leq \infty$. Then

(i) \[ \| (x, y) \|_q \leq \| (x, y) \|_p \quad ((x, y) \in X \times Y); \]

(ii) \[ \| (x, y) \|_{[p]} \leq \| (x, y) \|_{[q]} \quad ((x, y) \in X \times X); \]

(iii) \[ \| (x, y) \|_{(p)} \leq \| (x, y) \|_{(q)} \quad ((x, y) \in X \times Y). \]

The following theorem says that there is nothing special about the norm $\| \cdot \|_\infty$ taken in Definition 2.1.1 as each of the norms $\| \cdot \|_p$, $\| \cdot \|_{(p)}$ and $\| \cdot \|_{[p]}$ is equivalent to the norm $\| \cdot \|_\infty$.

**Theorem 2.1.8.** Let $X$ and $Y$ be norm linear spaces and $1 \leq p \leq \infty$. Then

(i) \[ \| \cdot \|_{[p]} \cong \| \cdot \|_p \cong \| \cdot \|_\infty \text{ on } X \times X; \]

(ii) \[ \| \cdot \|_{(p)} \cong \| \cdot \|_p \cong \| \cdot \|_\infty \text{ on } X \times Y. \]

**Theorem 2.1.9.** Let $(A, \| \cdot \|_A)$ and $(B, \| \cdot \|_B)$ be normed algebras. Then $A \times B$ has left (right) (bounded) approximate identity if and only if both $A$ and $B$ have left (right) (bounded) approximate identity.

**Proof.** Let $A \times B$ has left approximate identity $((e_\alpha, f_\alpha))$ and $a \in A$. Then

\[ \|(e_\alpha)a - a\|_A = \|(e_\alpha, f_\alpha)(a, 0) - (a, 0)\|_\infty \to 0 \]

as $\alpha \to \infty$. Thus $(e_\alpha)$ is a left approximate identity for $A$. By similar arguments it follows that, $(f_\alpha)$ is a left approximate identity for $B$.

Conversely, suppose that $A$ and $B$ have left approximate identities. Let $(e_\alpha)$ and $(f_\beta)$ be left approximate identities for $A$ and $B$ respectively. Then

\[ \|(e_\alpha, f_\beta)(a, b) - (a, b)\|_\infty = \|(e_\alpha a - a, f_\beta b - b)\|_\infty \leq \|e_\alpha a - a\| + \|f_\beta b - b\|; \]

which converges to 0 for all $(a, b) \in A \times B$ as $\alpha, \beta \to \infty$. Thus $((e_\alpha, f_\beta))$ is a left approximate identity for $A \times B$. Thus, $A \times B$ has left approximate identity.
The proof for the right approximate identity is similar. The proof for the bounded approximate identity follows from the fact that a sequence \((e_\alpha, f_\beta)\) is bounded in \(A \times B\) if and only if the sequences \((e_\alpha)\) and \((f_\beta)\) are bounded in \(A\) and in \(B\), respectively. □

**Theorem 2.1.10.** Let \(A\) and \(B\) be Banach algebras. Then \(A \times B\) is a uniform algebra if and only if \(A\) and \(B\) are uniform algebras.

**Proof.** Suppose that \(A \times B\) is a uniform algebra. Let \(|\cdot|\) be the complete uniform norm on \(A \times B\). Define \(\|a\| = |(a, 0)| \quad (a \in A)\). Then \(\|\cdot\|\) is a complete norm on \(A\) and \(\|a^2\| = |(a^2, 0)| = |(a, 0)^2| = |(a, 0)|^2 = \|a\|^2 \quad (a \in A)\). Thus \(A\) is a uniform algebra. Similarly, it follows that \(B\) is a uniform algebra.

Conversely, suppose that \((A, \|\cdot\|)\) and \((B, \|\cdot\|)\) are uniform algebras. Define \(\|(a, b)\|_\infty = \max\{|a|, |b|\} \quad (a \in A; b \in B)\). Then \(\|\cdot\|_\infty\) is a complete norm on \(A \times B\) and

\[
\|(a, b)^2\|_\infty = \|(a^2, b^2)\|_\infty = \max\{|a^2|, |b^2|\} = \max\{|a|^2, |b|^2\} = \max\{|a|, |b|\}^2 = \|(a, b)\|_\infty^2.
\]

Thus \(A \times B\) is a uniform algebra. □
IMPORTANT NOTATION

Here, we list out some important notations which will be frequently used in the thesis.

(1). Let \( \varphi \in \Delta(A) \) and \( S \) be a non-empty set. Define \( \varphi^\circ : A \times S \rightarrow \mathbb{C} \) and \( \varphi_\circ : S \times A \rightarrow \mathbb{C} \) as follows. For \( (a, x) \in A \times S \),
\[
\varphi^\circ((a, x)) := \varphi(a), \quad \text{and} \quad \varphi_\circ((x, a)) := \varphi(a).
\]

(2). Let \( \mathcal{I} \) be an ideal in \( A \). Let \( \varphi \in \Delta(\mathcal{I}) \) and \( u \in \mathcal{I} \) such that \( \varphi(u) = 1 \). Define \( \varphi^+, \varphi^-, \varphi^{+i}, \varphi^{-i} : A \times \mathcal{I} \rightarrow \mathbb{C} \) as follows. For \( (a, x) \in A \times \mathcal{I} \),
\[
\begin{align*}
(i) \quad & \varphi^+((a, x)) := \varphi(au) + \varphi(x), \\
(ii) \quad & \varphi^-((a, x)) := \varphi(au) - \varphi(x), \\
(iii) \quad & \varphi^{+i}((a, x)) := \varphi(au) + i\varphi(x), \\
(iv) \quad & \varphi^{-i}((a, x)) := \varphi(au) - i\varphi(x).
\end{align*}
\]

We should note that \( \varphi^+, \varphi^-, \varphi^{+i}, \varphi^{-i} \) are independent of \( u \). Moreover, if \( \varphi \in \Delta(A) \), then \( \varphi(au) = \varphi(a)\varphi(u) = \varphi(a) \). So that we do not need to choose \( u \in A \) in this case.

(3). Let \( F \subset \Delta(A) \). Define
\[
\begin{align*}
(i) \quad & F^+ := \{ \varphi^+ : \varphi \in F \}, \\
(ii) \quad & F^- := \{ \varphi^- : \varphi \in F \}, \\
(iii) \quad & F^\circ := \{ \varphi^\circ : \varphi \in F \}, \\
(iv) \quad & F_\circ := \{ \varphi_\circ : \varphi \in F \}, \\
(v) \quad & \Delta^\circ(A) := \{ \varphi^\circ : \varphi \in \Delta(A) \}, \\
(vi) \quad & \Delta_\circ(A) := \{ \varphi_\circ : \varphi \in \Delta(A) \}, \\
(vii) \quad & F^{+i} := \{ \varphi^{+i} : \varphi \in F \}, \\
(viii) \quad & F^{-i} := \{ \varphi^{-i} : \varphi \in F \}.
\end{align*}
\]

(4). Define the following functions on \( \Delta(A) \).
\[
\begin{align*}
(i) \quad & f^+ : \Delta(A) \rightarrow \Delta^+(A); \quad f^+(\varphi) = \varphi^+, \\
(ii) \quad & f^- : \Delta(A) \rightarrow \Delta^-(A); \quad f^-(\varphi) = \varphi^-, \\
(iii) \quad & f^\circ : \Delta(A) \rightarrow \Delta^\circ(A); \quad f^\circ(\varphi) = \varphi^\circ, \\
(iv) \quad & f_\circ : \Delta(A) \rightarrow \Delta_\circ(A); \quad f_\circ(\varphi) = \varphi_\circ.
\end{align*}
\]
2.2. Gelfand Space and Shilov Boundary

The set of all non-zero, multiplicative, linear functionals on a commutative Banach algebra $\mathcal{A}$ is a locally compact Hausdorff topological space with respect to the relative weak* topology. It is nothing but the Gelfand space of $\mathcal{A}$. This space helps to represent a commutative Banach algebra as an algebra of continuous functions on it. Because of its importance, we find the Gelfand space of $\mathcal{A} \times \mathcal{B}$ in terms of the Gelfand spaces of $\mathcal{A}$ and $\mathcal{B}$. This result is very important throughout this chapter.

**Definition 2.2.1.** [15, p-33] Let $\{X_\lambda : \lambda \in \Lambda\}$ be an indexed family of disjoint topological spaces. Let $X = \bigcup_{\lambda \in \Lambda} X_\lambda$. The collection

$$\mathcal{T} = \{U \subset X : U \cap X_\lambda \text{ is an open in } X_\lambda \text{ for each } \lambda \in \Lambda\}$$

is a topology on $X$. The topology $\mathcal{T}$ is called the *sum topology* on $X$. The topological space $(X, \mathcal{T})$ is called the *topological sum* of the topological spaces $\{X_\lambda\}_{\lambda \in \Lambda}$ and it is denoted by $X = \biguplus_{\lambda \in \Lambda} X_\lambda$.

**Lemma 2.2.2.** If $X$ and $Y$ are disjoint, locally compact, Hausdorff, topological spaces, then $X \biguplus Y$ is a locally compact, Hausdorff space. Moreover, a function $h : X \biguplus Y \to \mathbb{C}$ is continuous if and only if the functions $h|_X : X \to \mathbb{C}$ and $h|_Y : Y \to \mathbb{C}$ are continuous.

**Lemma 2.2.3.** Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be topological spaces. Let $\mathcal{T}$ be the sum topology on $X \biguplus Y$ defined by $\mathcal{T}_1$ and $\mathcal{T}_2$. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the subspace topologies on $X$ and $Y$ induced by $\mathcal{T}$. Then $\mathcal{T}_X = \mathcal{T}_1$ and $\mathcal{T}_Y = \mathcal{T}_2$.

**Remark 2.2.4.** Let $X$ and $Y$ be topological spaces. Then, by Lemma 2.2.3, it is clear that $X$ and $Y$ are open as well as closed in $X \biguplus Y$. 
The next result was proved in [53] for commutative Banach algebras \( A \) and \( B \) being unital. Because of the identities, their Gelfand spaces are compact and Hausdorff. Therefore the proof becomes much easier. Here we prove it without assuming identities and with different method.

**Theorem 2.2.5.** Let \( A \) and \( B \) be commutative Banach algebras. Then

(i) \( \Delta(A \times B) \cong \Delta^o(A) \cup \Delta_o(B) \).

(ii) \( \partial(A \times B) \cong \partial^o(A) \cup \partial_o(B) \).

**Proof.** (i). It is clear that \( \Delta^o(A) \) and \( \Delta_o(B) \) are subsets of \( \Delta(A \times B) \). Conversely, suppose \( \tilde{\psi} \in \Delta(A \times B) \). Then there exists \( (c, d) \in A \times B \) such that \( \tilde{\psi}(c, d) = 1 \). Then \( \tilde{\psi}((c, 0))\tilde{\psi}((0, d)) = 0 \) and \( \tilde{\psi}((c, 0)) + \tilde{\psi}((0, d)) = 1 \). Hence one of them has to be 0 and another one has to be 1. Suppose \( \tilde{\psi}((0, d)) = 0 \) and \( \tilde{\psi}((c, 0)) = 1 \). Define \( \varphi(a) = \tilde{\psi}((a, 0)) \) \( (a \in A) \). Then \( \varphi \in \Delta^o(A) \) and so \( \varphi^o \in \Delta^o(A) \). Also, for \( (a, b) \in A \times B \),

\[
\tilde{\psi}((a, b)) = \tilde{\psi}((a, b))\tilde{\psi}((c, d)) = \tilde{\psi}((ac, bd)) = \tilde{\psi}((ac, 0)) + \tilde{\psi}((0, bd)) = \tilde{\psi}((a, 0))\tilde{\psi}((c, 0)) + \tilde{\psi}((0, b))\tilde{\psi}((0, d)) = \tilde{\psi}((a, 0)) = \varphi^o((a, b)).
\]

Thus \( \tilde{\psi} = \varphi^o \in \Delta^o(A) \). Similarly, if \( \tilde{\psi}((c, 0)) = 0 \) and \( \tilde{\psi}((0, d)) = 1 \), then \( \tilde{\psi} = \varphi_o \) for some \( \varphi \in \Delta(B) \). Hence \( \Delta(A \times B) \subset \Delta^o(A) \cup \Delta_o(B) \). Thus \( \Delta(A \times B) \) and \( \Delta^o(A) \cup \Delta_o(B) \) are same as sets.

Now, let \( T_s \) and \( T_g \) be the sum topology and the Gelfand topology on \( \Delta^o(A) \cup \Delta_o(B) \), respectively. Let \( (a, b) \in A \times B \). Then

\[
(a, b)(\varphi^*) = \begin{cases} 
\hat{a}(\varphi) & \text{if } \varphi^* \in \Delta^o(A) \\
\hat{b}(\varphi) & \text{if } \varphi^* \in \Delta_o(B).
\end{cases}
\]

Since \( \hat{a} \) and \( \hat{b} \) are continuous on \( \Delta(A) \) and \( \Delta(B) \), respectively, we get \( (a, b) \) is continuous on \( \Delta^o(A) \cup \Delta_o(B) \) with respect to \( T_s \) due to Lemma 2.2.2. But the Gelfand topology \( T_g \) is the weakest topology, which makes every function \( (a, b) \) continuous on \( \Delta^o(A) \cup \Delta_o(B) \). Hence \( T_g \subset T_s \).
For the reverse inclusion, we first show that $\Delta^\diamond(\mathcal{A})$ and $\Delta^\diamond(\mathcal{B})$ are open in $(\Delta^\diamond(\mathcal{A}) \cup \Delta^\diamond(\mathcal{B}), T_g)$. Let $\{\varphi^\diamond_\alpha\} \subset \Delta^\diamond(\mathcal{A})$ be a net such that $\varphi^\diamond_\alpha \rightarrow \varphi^\star$ in $(\Delta^\diamond(\mathcal{A}) \cup \Delta^\diamond(\mathcal{B}), T_g)$. Let $b \in \mathcal{B}$. Then $\varphi^\diamond_\alpha((0, b)) = 0$ for each $\alpha$. So that $\varphi^\star((0, b)) = \lim_{\alpha \rightarrow \infty} \varphi^\diamond_\alpha((0, b)) = 0$. Thus $\varphi^\star = 0$ on $\{0\} \times \mathcal{B}$ and so $\varphi^\star$ has to be nonzero on $\mathcal{A} \times \{0\}$. Thus $\varphi^\star \in \Delta^\diamond(\mathcal{A})$. This implies $\Delta^\diamond(\mathcal{A})$ is closed. i.e., $\Delta^\diamond(\mathcal{B})$ is open in $(\Delta^\diamond(\mathcal{A}) \cup \Delta^\diamond(\mathcal{B}), T_g)$. By similar arguments it follows that $\Delta^\diamond(\mathcal{A})$ is open in $(\Delta^\diamond(\mathcal{A}) \cup \Delta^\diamond(\mathcal{B}), T_g)$.

Now, let $T_A$ and $T_B$ be the Gelfand topologies on $\Delta^\diamond(\mathcal{A})$ and $\Delta^\diamond(\mathcal{B})$, respectively. Next we claim that $T_A \subset T_g|_{\Delta^\diamond(\mathcal{A})}$ and $T_B \subset T_g|_{\Delta^\diamond(\mathcal{B})}$. Let $a \in \mathcal{A}$. Then $(a, 0)$ is continuous on $(\Delta^\diamond(\mathcal{A}) \cup \Delta^\diamond(\mathcal{B}), T_g)$. Hence $(a, 0)|_{\Delta^\diamond(\mathcal{A})}$ is continuous on $(\Delta^\diamond(\mathcal{A}), T_g|_{\Delta^\diamond(\mathcal{A})})$. Since the Gelfand topology $T_A$ on $\Delta^\diamond(\mathcal{A})$ is the smallest topology such that every $(a, 0)$ is continuous on $\Delta^\diamond(\mathcal{A})$, $T_A \subset T_g|_{\Delta^\diamond(\mathcal{A})}$. Similarly, $T_B \subset T_g|_{\Delta^\diamond(\mathcal{B})}$. This proves our claim.

Finally, let $W \in T_s$. Then $W \cap \Delta^\diamond(\mathcal{A}) \in T_A$ because $T_s$ is the sum topology on $\Delta^\diamond(\mathcal{A}) \cup \Delta^\diamond(\mathcal{B})$. Since $T_A \subset T_g|_{\Delta^\diamond(\mathcal{A})}$ and $\Delta^\diamond(\mathcal{A})$ is open in $(\Delta^\diamond(\mathcal{A}) \cup \Delta^\diamond(\mathcal{B}), T_g)$, $W \cap \Delta^\diamond(\mathcal{A})$ is open in $(\Delta^\diamond(\mathcal{A}) \cup \Delta^\diamond(\mathcal{B}), T_g)$. Similarly, $W \cap \Delta^\diamond(\mathcal{B})$ is open in $(\Delta^\diamond(\mathcal{A}) \cup \Delta^\diamond(\mathcal{B}), T_g)$. Hence $W = [W \cap \Delta^\diamond(\mathcal{A})] \cup [W \cap \Delta^\diamond(\mathcal{B})]$ is open in $(\Delta^\diamond(\mathcal{A}) \cup \Delta^\diamond(\mathcal{B}), T_g)$. This proves $T_s \subset T_g$. Thus $\Delta(\mathcal{A} \times \mathcal{B}) \cong \Delta^\diamond(\mathcal{A}) \cup \Delta^\diamond(\mathcal{B})$.

(ii). Let $F_A = \{\varphi \in \Delta(\mathcal{A}) : \varphi^\diamond \in \partial(\mathcal{A} \times \mathcal{B})\}$. Let $a \in \mathcal{A} \setminus \{0\}$. Since $|(a, 0)^\wedge|$ assumes its maximum on $\partial(\mathcal{A} \times \mathcal{B})$, there exists $\tilde{\psi} \in \partial(\mathcal{A} \times \mathcal{B})$ such that $\|(a, 0)^\wedge\|_{\infty} = |(a, 0)^\wedge(\tilde{\psi})|$. Then we must have $\tilde{\psi} = \psi^\diamond$ for some $\psi \in \Delta(\mathcal{A})$. Hence, $\psi \in F_A$. Now, $\|\tilde{a}\|_{\infty} = \|(a, 0)^\wedge\|_{\infty} = |(a, 0)^\wedge(\tilde{\psi})| = |\tilde{a}(\psi)|$. Thus $|\tilde{a}|$ attains its maximum on $F_A$. Therefore, $F_A$ is a boundary of $\mathcal{A}$. Since $\partial \mathcal{A}$ is the smallest closed boundary for $\mathcal{A}$, $\partial \mathcal{A} \subset F_A$. Hence $\partial^\diamond(\mathcal{A}) \subset \partial(\mathcal{A} \times \mathcal{B})$. Similarly, $\partial^\diamond(\mathcal{B}) \subset \partial(\mathcal{A} \times \mathcal{B})$. Thus $\partial^\diamond(\mathcal{A}) \cup \partial^\diamond(\mathcal{B}) \subset \partial(\mathcal{A} \times \mathcal{B})$.

For the reverse inclusion, let $(a, b) \in \mathcal{A} \times \mathcal{B}$. Then there exist $\varphi^\diamond \in \partial^\diamond(\mathcal{A})$ and $\psi^\diamond \in \partial^\diamond(\mathcal{B})$ such that $\|\tilde{a}\|_{\infty} = |(a, 0)^\wedge(\varphi^\diamond)|$ and $\|\tilde{b}\|_{\infty} = |(0, b)^\wedge(\psi^\diamond)|$. 
Then
\[
\|(a, b)^\wedge\|_\infty = \max\{\|\hat{a}\|_\infty, \|\hat{b}\|_\infty\} = \max\{(\|(a, 0)^\wedge\|_\infty, \|(0, b)^\wedge\|_\infty)\}.
\]
Thus, \(\mathcal{D}(A) \cup \mathcal{D}(B)\) is a boundary for \(A \times B\). Since \(\partial(A \times B)\) is the smallest closed boundary for \(A \times B\), \(\partial(A \times B) \subset \mathcal{D}(A) \cup \mathcal{D}(B)\).

**Definition 2.2.6.** Let \(A\) and \(B\) are Banach \(*\)-algebras. Define \((a, b)^* = (a^*, b^*)\) on \(A \times B\), then \(A \times B\) is a Banach \(*\)-algebra with this involution.

**Theorem 2.2.7.** Let \(A\) and \(B\) be commutative Banach \(*\)-algebras. Then \(\Delta^h(A \times B) \cong \Delta^{h_0}(A) \cup \Delta^h(B)\), where \(\Delta^{h_0}(A) = \{ \varphi^0 : \varphi \in \Delta^h(A)\}\) and \(\Delta^h(B) = \{ \varphi_0 : \varphi \in \Delta^h(B)\}\).

**Proof.** Let \(A\) and \(B\) be commutative Banach \(*\)-algebras. It is clear that \(\Delta^{h_0}(A) \subset \Delta^h(A \times B)\) and \(\Delta^h(B) \subset \Delta^h(A \times B)\). For the reverse inclusion, let \(\tilde{\psi} \in \Delta^h(A \times B)\). Then \(\tilde{\psi} \in \Delta^0(A)\) or \(\tilde{\psi} \in \Delta_0(B)\). Without loss of generality, assume that \(\tilde{\psi} \in \Delta^0(A)\). Then \(\tilde{\psi} = \varphi^0\) for some \(\varphi \in \Delta(A)\). Let \(a \in A\). Then
\[
\varphi^*(a) = \varphi(a^*) = \varphi^0((a^*, 0)) = \varphi^0((a, 0)^*) = \tilde{\psi}((a, 0)^*)
\]
\[
= \tilde{\psi}^*((a, 0)) = \tilde{\psi}((a, 0)) = \varphi^0((a, 0)) = \varphi(a)
\]
Thus \(\varphi \in \Delta^h(A)\). Therefore \(\tilde{\psi} = \varphi^0 \in \Delta^{h_0}(A)\). Similarly, if \(\tilde{\psi} \in \Delta_0(B)\), then \(\tilde{\psi} \in \Delta^h(B)\). Thus \(\Delta^h(A \times B) \subset \Delta^{h_0}(A) \cup \Delta^h(B)\). Hence, finally we get, \(\Delta^h(A \times B) \cong \Delta^{h_0}(A) \cup \Delta^h(B)\). \(\square\)

**Theorem 2.2.8.** Let \(A\) and \(B\) be commutative Banach algebras. Then \(A \times B\) is semisimple if and only if \(A\) and \(B\) are semisimple.

**Proof.** Suppose that \(A \times B\) is a semisimple algebra. Let \(a \in A\) such that \(\varphi(a) = 0\) \((\varphi \in \Delta(A))\). Then \(\varphi^0((a, 0)) = \varphi(a) = 0\) \((\varphi^0 \in \Delta^0(A)\)) and \(\varphi_0((a, 0)) = \varphi(0) = 0\) \((\varphi_0 \in \Delta_0(B)\)). Thus \(\varphi^*((a, 0)) = 0\) for all \(\varphi^* \in \Delta^0(A) \cup \Delta_0(B)\). Since \(A \times B\) is semisimple, by Theorem 1.2.19, \((a, 0) = (0, 0)\).
i.e., \(a = 0\). Thus \(\Delta(A)\) separates points of \(A\). Hence, again by Theorem 1.2.19, \(A\) is semisimple. By similar arguments, it follows that \(B\) is semisimple.

Conversely, suppose that \(A\) and \(B\) are semisimple. Let \((a, b) \in A \times B\) such that \(\varphi^*((a, b)) = 0\) (\(\varphi^* \in \Delta(A \times B)\)). Then \(\varphi(a) = \varphi^*((a, b)) = 0\) (\(\varphi \in \Delta(A)\)) and \(\psi(b) = \psi_0((a, b)) = 0\) (\(\psi \in \Delta(B)\)). Since \(A\) and \(B\) are semisimple, we get \(a = b = 0\). Thus \(\Delta(A \times B)\) separates points of \(A \times B\). So, by Theorem 1.2.19, \(A \times B\) is semisimple. □

**Corollary 2.2.9.** Let \(A\) and \(B\) be Banach \(*\)-algebras. Then

(i) \(A \times B\) is \(*\)-semisimple if and only if both \(A\) and \(B\) are \(*\)-semisimple;

(ii) \(A \times B\) is Hermitian if and only if both \(A\) and \(B\) are Hermitian.

**Proof.** (i). Suppose that \(A \times B\) is a \(*\)-semisimple, Banach \(*\)-algebra. So, by Theorem 1.2.30, \(A \times B\) admits a \(C^*\)-norm, say \(|\cdot|\). Define

\[
|a|_A := |(a, 0)| \quad (a \in A) \quad \text{and} \quad |b|_B := |(0, b)| \quad (b \in B).
\]

Then \(|\cdot|_A\) and \(|\cdot|_B\) are \(C^*\)-norms on \(A\) and \(B\), respectively. Hence, again by Theorem 1.2.30, \(A\) and \(B\) are \(*\)-semisimple.

Conversely, suppose that \(A\) and \(B\) are \(*\)-semisimple, Banach \(*\)-algebras. Then, by Theorem 1.2.30, both \(A\) and \(B\) admit \(C^*\)-norms, say \(|\cdot|_A\) and \(|\cdot|_B\), respectively. Define \(|(a, b)|_\infty := \max\{|a|_A, |b|_B\} \quad ((a, b) \in A \times B)\). Then \(|\cdot|_\infty\) is a \(C^*\)-norm on \(A \times B\). Thus \(A \times B\) is \(*\)-semisimple due to Theorem 1.2.30.

(ii). This follows from Definition 1.2.24 and Proposition 2.1.3(iii). □

### 2.3. Uniqueness and Separation Properties

The concept of uniqueness is very important in the Banach algebra theory. B. E. Johnson proved in 1967 [34] that any two complete norms on semisimple algebras give same topology. In this section, we study the uniqueness properties of \(A \times B\). We also study various types of separation properties in \(A \times B\).
Theorem 2.3.1. Let $A$ and $B$ be semisimple, commutative Banach algebras. Then $A \times B$ has UUNP if and only if both $A$ and $B$ have UUNP.

Proof. Suppose that $A \times B$ has UUNP. Let $F \subset \Delta(A)$ be a closed set of uniqueness for $A$. Let $F^o = \{ \varphi^o : \varphi \in F \}$. Then, by the definition of sum topology, $F^o \cup \Delta_o(B)$ is a closed subset of $\Delta^o(A) \cup \Delta_o(B)$. Moreover, it is also a set of uniqueness for $A \times B$. Since $A \times B$ has UUNP, by Theorem 1.3.3, $\partial A \cup \partial B \subset F^o \cup \Delta_o(B)$. Since $\Delta^o(A)$ and $\Delta_o(B)$ are disjoint, we must have $\partial A \subset F^o$. Hence $\partial A$ is the smallest closed set of uniqueness for $A$. Hence, again by Theorem 1.3.3, $A$ has UUNP. By similar arguments, it follows that $B$ has UUNP.

Conversely, suppose that $A$ and $B$ have UUNP. Let $F = \Delta^o(A) \cup \Delta_o(B)$ be a closed sets of uniqueness for $A \times B$. Let $F_A = \{ \varphi \in \Delta(A) : \varphi^o \in F \}$ and $F_B = \{ \varphi \in \Delta(B) : \varphi^o \in F \}$. Then $F_A$ and $F_B$ are closed set of uniqueness for $A$ and $B$, respectively, and $F = F_A^o \cup F_B^o$. Since $A$ and $B$ have UUNP, by Theorem 1.3.3, $\partial A \subset F_A$ and $\partial B \subset F_B$. Hence, by Theorem 2.2.5(ii), we get

$$\partial(A \times B) \cong \partial A \cup \partial B \subset F_A^o \cup F_B^o = F.$$ 

Thus, by Theorem 1.3.3, $A \times B$ has UUNP.

\[ \square \]

Theorem 2.3.2. Let $A$ and $B$ be $*$-semisimple, Banach $*$-algebras.

(i) If $A \times B$ has $UC^*NP$, then both $A$ and $B$ have $UC^*NP$;

(ii) Suppose that $A$ and $B$ are commutative. If both $A$ and $B$ have $UC^*NP$, then $A \times B$ has $UC^*NP$.

Proof. (i). Let $A \times B$ have $UC^*NP$. Let $| \cdot |_A$ and $| \cdot |_B$ be the largest $C^*$-norms on $A$ and $B$, respectively. Define

$$|(a, b)| = \max\{|a|_A, |b|_B\} \quad ((a, b) \in A \times B).$$

2.3. Uniqueness and Separation Properties 35
2.3. Uniqueness and Separation Properties

Then \( | \cdot | \) is a \( C^* \)-norm on \( A \times B \). Now, let \( \| | \cdot \|_A \) be any \( C^* \)-norm on \( A \). Define \( \| | (a, b) \| \) = \( \max \{ \| |a|_A, |b|_B \} \) \( (a \in A, b \in B) \). Then \( \| | \cdot \| \) is also a \( C^* \)-norm on \( A \times B \). Hence, by the hypothesis, \( | \cdot | = \| | \cdot \|_A \) on \( A \times B \). Now, \( \| | a \|_A = \max \{ \| |a|_A, |0|_B \} = \| |(a, 0)\| = |a|_A \) \( (a \in A) \).

Thus \( A \) has \( UC^*\text{NP} \). By similar argument, it follows that \( B \) has \( UC^*\text{NP} \).

(ii) Suppose that \( A \) and \( B \) are commutative and both have \( UC^*\text{NP} \). Let \( \tilde{F} \) be a proper closed subset of \( \Delta^h(A) \cup \Delta^h(B) \). Let \( F_A = \{ \varphi \in \Delta^h(A) : \varphi^o \in \tilde{F} \} \) and \( F_B = \{ \varphi \in \Delta^h(B) : \varphi^o \in \tilde{F} \} \). Then \( F_A \) and \( F_B \) are closed subsets of \( \Delta^h(A) \) and \( \Delta^h(B) \), respectively and one of them has to be a proper subset. Suppose that \( F_A \) is a proper closed subset of \( \Delta^h(A) \). Since \( A \) has \( UC^*\text{NP} \), by Theorem 1.3.4, there exists a nonzero element \( a \in A \) such that \( \hat{a}|_{F_A} = 0 \). Then \( (a, 0)^\wedge|_{\tilde{F}} = \hat{a}|_{F_A} = 0 \). Similarly, if \( F_B \) is a proper closed subset of \( \Delta^h(B) \), then there exists a nonzero element \( b \in B \) such that \( \hat{b}|_{F_B} = 0 \). Then \( (0, b)^\wedge|_{\tilde{F}} = \hat{b}|_{F_B} = 0 \). Thus in each case, we get a nonzero element in \( A \times B \) whose Gelfand transform is zero on \( \tilde{F} \). Therefore, by Theorem 1.3.4, \( A \times B \) has \( UC^*\text{NP} \).

Next we show that the properties weak regularity, regularity, and complete regularity are stable with respect to the co-ordinatewise product of Banach algebras.

**Theorem 2.3.3.** Let \( A \) and \( B \) be semisimple, commutative Banach algebras. Then \( A \times B \) is weakly regular if and only if both \( A \) and \( B \) are weakly regular.

**Proof.** Suppose that \( A \times B \) is weakly regular. Let \( F \) be a proper closed subset of \( \Delta(A) \). Then \( F^o \cup \Delta^o(B) \) is a proper closed subset of \( \Delta^o(A) \cup \Delta^o(B) \). Hence, by the hypothesis, there exists a nonzero element \( (a, b) \in A \times B \) such that \( (a, b)^\wedge = 0 \) on \( F^o \cup \Delta^o(B) \). Now let \( \varphi \in \Delta(B) \). Then \( \varphi^o \in \Delta^o(B) \). Therefore \( \varphi(b) = \varphi_o((a, b)) = (a, b)^\wedge(\varphi_o) = 0 \). Since \( B \) is semisimple, we get
2.3. Uniqueness and Separation Properties

$b = 0$. Therefore, we must have $a \neq 0$. On the other hand if $\varphi \in F$, then $\varphi^o \in F^o$ and so that $\varphi(a) = \varphi^o((a,b)) = (a,b)^*(\varphi^o) = 0$. Hence $\hat{a}|_F = 0$. Thus $\mathcal{A}$ is weakly regular. By similar argument, it follows that $\mathcal{B}$ is weakly regular.

Conversely, suppose that $\mathcal{A}$ and $\mathcal{B}$ are weakly regular. Let $\tilde{F}$ be a proper closed subsets of $\Delta^o(\mathcal{A}) \cup \Delta_o(\mathcal{B})$. Let $F_A = \{ \varphi \in \Delta(\mathcal{A}) : \varphi^o \in \tilde{F} \}$ and $F_B = \{ \varphi \in \Delta(\mathcal{B}) : \varphi^o \in \tilde{F} \}$. Then $F_A$ and $F_B$ are closed subsets of $\Delta(\mathcal{A})$ and $\Delta(\mathcal{B})$, respectively such that $\tilde{F} = F_A^o \cup F_B^o$. Since $\tilde{F}$ is a proper subset of $\Delta^o(\mathcal{A}) \cup \Delta_o(\mathcal{B})$, either $F_A$ is a proper subset of $\Delta(\mathcal{A})$ or $F_B$ is a proper subset of $\Delta(\mathcal{B})$. Suppose that $F_A$ is a proper closed subset of $\Delta(\mathcal{A})$. Then, by the hypothesis, there exists a nonzero element $a \in \mathcal{A}$ such that $\hat{a}|_{F_A} = 0$. Then $(a,0)^\wedge|_{\tilde{F}} = \hat{a}|_{F_A} = 0$. Similarly, if $F_B$ is a proper closed subset of $\Delta(\mathcal{B})$, then there exists a nonzero element $b \in \mathcal{B}$ such that $\hat{b}|_{F_B} = 0$. Then $(0,b)^\wedge|_{\tilde{F}} = \hat{b}|_{F_B} = 0$. Thus, in both cases, we can find a nonzero element in $\mathcal{A} \times \mathcal{B}$ whose Gelfand transform is zero on $\tilde{F}$. Thus $\mathcal{A} \times \mathcal{B}$ is weakly regular. □

Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras. Since $\mathcal{A}$ is a closed ideal of $\mathcal{A} \times \mathcal{B}$ and $(\mathcal{A} \times \mathcal{B})/\mathcal{A} \cong \mathcal{B}$, by Theorem 1.3.15, it follows that $\mathcal{A} \times \mathcal{B}$ is regular if and only if $\mathcal{A}$ and $\mathcal{B}$ are regular. However, we prove it here without using these deep theorems.

**Theorem 2.3.4.** Let $\mathcal{A}$ and $\mathcal{B}$ be semisimple, commutative Banach algebras. Then $\mathcal{A} \times \mathcal{B}$ is regular if and only if both $\mathcal{A}$ and $\mathcal{B}$ are regular.

**Proof.** Suppose that $\mathcal{A} \times \mathcal{B}$ is regular. Let $F$ be a closed subset of $\Delta(\mathcal{A})$ and $\psi \in \Delta(\mathcal{A}) \setminus F$. Then $F^o$ is closed in $\Delta^o(\mathcal{A}) \cup \Delta_o(\mathcal{B})$ and $\psi^o \notin F^o$. Hence, by the hypothesis, there exists an element $(a,b) \in \mathcal{A} \times \mathcal{B}$ such that $(a,b)^\wedge|_{F^o} = 0$ and $(a,b)^\wedge(\psi^o) = 1$. Then $\hat{a}|_F = (a,b)^\wedge|_{F^o} = 0$ and $\hat{a}(\psi) = (a,b)^\wedge(\psi^o) = 1$. Thus, we get an element $a \in \mathcal{A}$ such that $\hat{a}|_F = 0$ and $\hat{a}(\psi) = 1$. Hence, $\mathcal{A}$ is regular. By similar arguments, it follows that $\mathcal{B}$ is regular.
Conversely, suppose that $\mathcal{A}$ and $\mathcal{B}$ are regular. Let $\tilde{F}$ be a closed subset of $\Delta^\circ(\mathcal{A}) \cup \Delta^\circ(\mathcal{B})$ and $\tilde{\psi} \in (\Delta^\circ(\mathcal{A}) \cup \Delta^\circ(\mathcal{B})) \setminus \tilde{F}$. Let $F_A = \{ \varphi \in \Delta(\mathcal{A}) : \varphi^\circ \in \tilde{F} \}$ and $F_B = \{ \varphi \in \Delta(\mathcal{B}) : \varphi^\circ \in \tilde{F} \}$. Then $F_A$ and $F_B$ are closed subsets of $\Delta(\mathcal{A})$ and $\Delta(\mathcal{B})$, respectively such that $\tilde{F} = F_A^\circ \cup F_B^\circ$. Also, observe that either $\tilde{\psi} = \psi^\circ$ for some $\psi \in \Delta(\mathcal{A}) \setminus F_A$ or $\tilde{\psi} = \psi_0$ for some $\psi \in \Delta(\mathcal{B}) \setminus F_B$.

Suppose that $\tilde{\psi} = \psi^\circ$ for some $\psi \in \Delta(\mathcal{A}) \setminus F_A$. Then, by the hypothesis, there exists $a \in \mathcal{A}$ such that $\hat{a}|_{F_A} = 0$ and $\hat{a}(\psi) = 1$. Then $(a, 0)^\wedge|_{\tilde{F}} = \hat{a}|_{F_A} = 0$ and $(a, 0)^\wedge(\tilde{\psi}) = (a, 0)^\wedge(\psi^\circ) = \hat{a}(\psi) = 1$. Similarly, if $\tilde{\psi} = \psi_0$ for some $\psi \in \Delta(\mathcal{B}) \setminus F_B$. Then there exists $b \in \mathcal{B}$ such that $\hat{b}|_{F_B} = 0$ and $\hat{b}(\psi) = 1$. Then $(0, b)^\wedge|_{\tilde{F}} = \hat{b}|_{F_B} = 0$ and $(0, b)^\wedge(\psi_0) = \hat{b}(\psi) = 1$. Thus, in both the cases, we can find a nonzero element in $\mathcal{A} \times \mathcal{B}$ whose Gelfand transform is zero on $\tilde{F}$ and one at $\tilde{\psi}$. So, $\mathcal{A} \times \mathcal{B}$ is regular. \[\square\]

**Theorem 2.3.5.** Let $\mathcal{A}$ and $\mathcal{B}$ be semisimple, commutative Banach algebras. Then $\mathcal{A} \times \mathcal{B}$ is CR if and only if both $\mathcal{A}$ and $\mathcal{B}$ are CR.

**Proof.** Suppose that $\mathcal{A} \times \mathcal{B}$ is completely regular. Let $K$ and $F$ be compact and closed subsets of $\Delta(\mathcal{A})$, respectively such that $K \cap F = \emptyset$. Then $K^\circ$ and $F^\circ$ are compact and closed subsets of $\Delta^\circ(\mathcal{A}) \cup \Delta^\circ(\mathcal{B})$ such that $K^\circ \cap F^\circ = \emptyset$. So, by the hypothesis, there exists $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that $(a, b)^\wedge|_{K^\circ} = 1$ and $(a, b)^\wedge|_{F^\circ} = 0$. Then $\hat{a}|_K = (a, b)^\wedge|_{K^\circ} = 1$ and $\hat{a}|_F = (a, b)^\wedge|_{F^\circ} = 0$. Thus $\mathcal{A}$ is completely regular. Similarly, it follows that $\mathcal{B}$ is completely regular.

Conversely, suppose that $\mathcal{A}$ and $\mathcal{B}$ are completely regular. Let $\tilde{K}$ and $\tilde{F}$ be compact and closed subsets of $\Delta^\circ(\mathcal{A}) \cup \Delta^\circ(\mathcal{B})$, respectively such that $\tilde{K} \cap \tilde{F} = \emptyset$. Let $F_A = \{ \varphi \in \Delta(\mathcal{A}) : \varphi^\circ \in \tilde{F} \}$ and $F_B = \{ \varphi \in \Delta(\mathcal{B}) : \varphi^\circ \in \tilde{F} \}$. Then $F_A$ and $F_B$ are closed subsets of $\Delta(\mathcal{A})$ and $\Delta(\mathcal{B})$, respectively such that $\tilde{F} = F_A^\circ \cup F_B^\circ$. Similarly, the sets $K_A$ and $K_B$ corresponding to $\tilde{K}$ are compact subsets of $\Delta(\mathcal{A})$ and $\Delta(\mathcal{B})$, respectively such that $K_A^\circ \cup K_B^\circ = \tilde{K}$. Then $K_A \cap F_A = \emptyset$. Since $\mathcal{A}$ is CR, there exists $a \in \mathcal{A}$ such that $\hat{a}|_{K_A} = 1$
and $\hat{a}|_{F_A} = 0$. Similarly, $K_B \cap F_B = \phi$. Since $B$ is CR, there exists $b \in B$ such that $\hat{b}|_{K_B} = 1$ and $\hat{b}|_{F_B} = 0$. Since $K_A^\diamond \cup K_B^\diamond = \tilde{K}$ and $F_A^\diamond \cup F_B^\diamond = \tilde{F}$, we get $(a, b)^\diamond|_{\tilde{K}} = 1$ and $(a, b)^\diamond|_{\tilde{F}} = 0$. Thus $A \times B$ is completely regular. □

**Theorem 2.3.6.** Let $A$ and $B$ be semisimple, commutative Banach algebras.

(i) $A \times B$ is BR if and only if both $A$ and $B$ are BR;

(ii) $A \times B$ is BCR if and only if both $A$ and $B$ are BCR.

**Proof.** (i). This follows from Theorem 2.3.4 and Lemma 2.1.3(iv).

(ii). This follows from Theorem 2.3.5 and Lemma 2.1.3(iv). □

### 2.4. MHBP, SSEP, SEP, and WSEP

The purpose of this section is to characterize some spectral properties such as MHBP and SSEP of $A \times B$ in terms of $A$ and $B$. If $A \times B$ has SEP (WSEP), then both $A$ and $B$ have SEP (WSEP). We do not know their converse. These are proved for non-commutative $A$ and $B$.

**Theorem 2.4.1.** Let $A$ and $B$ be commutative Banach algebras. Then $A \times B$ has MHBP if and only if both $A$ and $B$ have MHBP.

**Proof.** Suppose that $A \times B$ have MHBP. Let $C$ be a commutative extension of $A$, then $C \times B$ is a commutative extension of $A \times B$. Let $\varphi \in \Delta(A)$. Then $\varphi^\diamond \in \Delta(A \times B)$. Since $A \times B$ has MHBP, $\varphi^\diamond$ can be extended to an element of $\Delta(C \times B) = \Delta^\diamond(C) \cup \Delta^\diamond(B)$. Suppose $\tilde{\psi} \in \Delta^\diamond(C) \cup \Delta^\diamond(B)$ such that $\tilde{\psi} = \varphi^\diamond$ on $A \times B$. Then, there exist $a \in A$ such that $\tilde{\psi}((a, 0)) = \varphi^\diamond((a, 0)) = \varphi(a) \neq 0$. Thus $\psi \in \Delta^\diamond(C)$. Hence $\tilde{\psi} = \varphi^\diamond$ for some $\psi \in \Delta(C)$. Therefore,

$$\varphi(a) = \varphi^\diamond((a, 0)) = \tilde{\psi}((a, 0)) = \psi^\diamond((a, 0)) = \psi(a) \quad (a \in A).$$

Thus, $\psi$ is an extension of $\varphi$ on $C$. Therefore $A$ has MHBP. By similarly arguments, it follows that $B$ has MHBP.
Conversely suppose that $\mathcal{A}$ and $\mathcal{B}$ have MHBP. Let $\mathcal{C}$ be a commutative extension of $\mathcal{A} \times \mathcal{B}$. Then $\mathcal{C}$ is an extension of both $\mathcal{A} \times \{0\}$ as well as of $\{0\} \times \mathcal{B}$. Let $\tilde{\varphi} \in \Delta^\circ(\mathcal{A}) \cup \Delta_\mathcal{B}(\mathcal{B})$. Then $\tilde{\varphi} \in \Delta^\circ(\mathcal{A})$ or $\tilde{\varphi} \in \Delta_\mathcal{B}(\mathcal{B})$. Suppose that $\tilde{\varphi} \in \Delta^\circ(\mathcal{A})$. Then $\tilde{\varphi} = \varphi^\circ$ for some $\varphi \in \Delta(\mathcal{A})$. Since $\mathcal{C}$ is an extension of $\mathcal{A} \times \{0\}$, by the hypothesis, $\varphi^\circ$ can be extended to some element of $\Delta(\mathcal{C})$. Similarly, if $\tilde{\varphi} \in \Delta_\mathcal{B}(\mathcal{B})$, then also it can be extended to some element of $\Delta(\mathcal{C})$. Thus $\mathcal{A} \times \mathcal{B}$ has MHBP.

\begin{proof}
Suppose that $\mathcal{A} \times \mathcal{B}$ has SSEP. Let $\mathcal{C}$ be a commutative extension of $\mathcal{A} \times \mathcal{B}$. Then $\mathcal{C}$ is a commutative extension of both $\mathcal{A} \times \{0\}$ as well as $\{0\} \times \mathcal{B}$. Hence, by Proposition 2.1.3(iii),

$$\sigma_{\mathcal{A}}(a) \cup \{0\} = \sigma_{\mathcal{A} \times \mathcal{B}}((a,0)) = \sigma_{\mathcal{C} \times \mathcal{B}}((a,0)) = \sigma_{\mathcal{C}}(a) \cup \{0\} \quad (a \in \mathcal{A}).$$

Thus $\mathcal{A}$ has SSEP. By similar arguments it follows that $\mathcal{B}$ has SSEP.

Conversely, suppose that $\mathcal{A}$ and $\mathcal{B}$ have SSEP. Let $\mathcal{C}$ be a commutative extension of $\mathcal{A} \times \mathcal{B}$. Then $\mathcal{C}$ is a commutative extension of both $\mathcal{A} \times \{0\}$ and $\{0\} \times \mathcal{B}$. Hence, by Proposition 2.1.3(iii), for $(a,b) \in \mathcal{A} \times \mathcal{B}$, we have

$$\sigma_{\mathcal{A} \times \mathcal{B}}((a,b)) \cup \{0\} = \sigma_{\mathcal{A}}(a) \cup \sigma_{\mathcal{B}}(b) \cup \{0\} = \sigma_{\mathcal{C}}((a,0)) \cup \sigma_{\mathcal{C}}((0,b)) \cup \{0\} = \sigma_{\mathcal{C}}(a) \cup \sigma_{\mathcal{C}}(b) \cup \{0\} = \sigma_{\mathcal{C}}((a,b)) \cup \{0\}.$$

Thus $\mathcal{A} \times \mathcal{B}$ has SSEP. \hfill \qed

The spectral extension property is one of the important properties in Banach algebras [36, p-222]. We do not know the converse of the following result even if $\mathcal{A}$ and $\mathcal{B}$ are commutative.

\begin{theorem}
Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. If $\mathcal{A} \times \mathcal{B}$ has SEP, then $\mathcal{A}$ and $\mathcal{B}$ have SEP.
\end{theorem}

\begin{proof}
Suppose that $\mathcal{A} \times \mathcal{B}$ has SEP. Let $\mathcal{C}$ be a commutative extension of $\mathcal{A} \times \mathcal{B}$. Then $\mathcal{C}$ is a commutative extension of both $\mathcal{A} \times \{0\}$ as well as $\{0\} \times \mathcal{B}$. Hence, by Proposition 2.1.3(iii), for $(a,b) \in \mathcal{A} \times \mathcal{B}$, we have

$$\sigma_{\mathcal{A} \times \mathcal{B}}((a,b)) \cup \{0\} = \sigma_{\mathcal{A}}(a) \cup \sigma_{\mathcal{B}}(b) \cup \{0\} = \sigma_{\mathcal{C}}((a,0)) \cup \sigma_{\mathcal{C}}((0,b)) \cup \{0\} = \sigma_{\mathcal{C}}(a) \cup \sigma_{\mathcal{C}}(b) \cup \{0\} = \sigma_{\mathcal{C}}((a,b)) \cup \{0\}.$$

Thus $\mathcal{A} \times \mathcal{B}$ has SEP. \hfill \qed

The spectral extension property is one of the important properties in Banach algebras [36, p-222]. We do not know the converse of the following result even if $\mathcal{A}$ and $\mathcal{B}$ are commutative.

\begin{theorem}
Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. If $\mathcal{A} \times \mathcal{B}$ has SEP, then $\mathcal{A}$ and $\mathcal{B}$ have SEP.
\end{theorem}
2.5. QDZ, TAN, and TDZ

**Proof.** Let $|\cdot|$ be a norm on $\mathcal{A}$. Define $|(a,b)|_1 = |a| + ||b||$, where $||\cdot||$ is the Banach algebra norm on $\mathcal{B}$. Since $\mathcal{A} \times \mathcal{B}$ has SEP, we have

$$r_\mathcal{A}(a) = r_{\mathcal{A} \times \mathcal{B}}((a,0)) \leq |(a,0)|_1 = |a| \quad (a \in \mathcal{A}).$$

Thus $|\cdot|$ is a spectral norm on $\mathcal{A}$, and so $\mathcal{A}$ has SEP. Similarly, $\mathcal{B}$ has SEP. $\square$

Unfortunately, we do not know the converse of the following result also.

**Theorem 2.4.4.** Let $\mathcal{A}$ and $\mathcal{B}$ be norm algebras. If $\mathcal{A} \times \mathcal{B}$ has WSEP, then $\mathcal{A}$ and $\mathcal{B}$ have WSEP.

**Proof.** Let $|\cdot|$ and $\|\cdot\|$ be semisimple norms on $\mathcal{A}$ and $\mathcal{B}$, respectively. Define

$$|(a,b)|_1 = |a| + \|b\| \quad ((a,b) \in \mathcal{A} \times \mathcal{B}).$$

Then $|\cdot|_1$ is a norm on $\mathcal{A} \times \mathcal{B}$. Since $|\cdot|$ and $\|\cdot\|$ are semisimple norms on $\mathcal{A}$ and $\mathcal{B}$, $(\mathcal{A},|\cdot|_1) \times (\mathcal{B},\|\cdot\|)$ is semisimple. Therefore, by the hypothesis $|\cdot|_1$ is spectral on $\mathcal{A} \times \mathcal{B}$. Hence,

$$|a| = |(a,0)|_1 \leq r_{\mathcal{A} \times \mathcal{B}}((a,0)) = r_{\mathcal{A}}(a).$$

Thus, $|\cdot|$ is a spectral norm on $\mathcal{A}$. By similar argument it follows that $\|\cdot\|$ is a spectral norm on $\mathcal{B}$. $\square$

2.5. QDZ, TAN, and TDZ

In this section, we are going to characterize three more properties of $\mathcal{A} \times \mathcal{B}$; namely, QDZ, TAN and TDZ. The first two properties were introduced and studied by M. J. Meyer [44].

**Theorem 2.5.1.** Let $\mathcal{A}$ and $\mathcal{B}$ be commutative Banach algebras. Then $\mathcal{A} \times \mathcal{B}$ has QDZ property if and only if both $\mathcal{A}$ and $\mathcal{B}$ have QDZ property.

**Proof.** Suppose that $\mathcal{A} \times \mathcal{B}$ have QDZ property. Then there exists an open subset $\tilde{G}$ of $\Delta(\mathcal{A} \times \mathcal{B})$ satisfying the following properties.
(i) $\partial^\circ(A) \uplus \partial_\circ(B) \subset \widehat{G}$.

(ii) For every open subset $\widehat{U}$ of $\widehat{G}$, there exists $(a, b) \in A \times B$ and a non-empty open subset $\widehat{V}$ of $\widehat{U}$ such that

$$(a, b)(\varphi^*) = \begin{cases} 0 & (\varphi^* \in \widehat{U}^c); \\ 1 & (\varphi^* \in \widehat{V}). \end{cases}$$

Let $G_A = \{ \varphi \in \Delta(A) : \varphi^\circ \in \widehat{G} \}$ and $G_B = \{ \varphi \in \Delta(B) : \varphi_\circ \in \widehat{G} \}$. Then $\widehat{G} = G^\circ_A \cup G^\circ_B$. First we show that $G_A$ is open in $\Delta(A)$. Let $\varphi \in G_A$. Then $\varphi^\circ \in \widehat{G} \cap \Delta^\circ(A)$. Since $\widehat{G}$ and $\Delta^\circ(A)$ are open in $\Delta^\circ(A) \uplus \Delta_\circ(B)$, $\widehat{G} \cap \Delta^\circ(A)$ is also open in $\Delta^\circ(A) \uplus \Delta_\circ(B)$. Hence there exists an open set $W^\circ$ of $\widehat{G} \cap \Delta^\circ(A)$ such that $\varphi^\circ \in W^\circ \subset \widehat{G} \cap \Delta^\circ(A)$. Then it is clear that $W \subset \Delta(A)$ is open such that $\varphi \in W \subset G_A$. Thus $G_A$ is an open subset of $\Delta(A)$. Also from (i) above, we get $\partial A \subset G_A$. Now, let $U \subset G_A$ be open. Then $U^\circ$ will be open in $\widehat{G}$. Hence, by (ii) above, there exist $(a, b) \in A \times B$ and a nonempty open set $V^\circ \subset U^\circ$ such that

$$(a, b)(\varphi^\circ) = \begin{cases} 0 & (\varphi^\circ \in U^\circ c); \\ 1 & (\varphi^\circ \in V^\circ). \end{cases}$$

Then, we have $\widehat{a}(\varphi) = (a, b)(\varphi^\circ) = 0$ if $\varphi \in U$ and $\widehat{a}(\varphi) = (a, b)(\varphi^\circ) = 1$ if $\varphi \in V$. Hence $A$ has QDZ property. Similarly $B$ has QDZ property.

Conversely, suppose that $A$ and $B$ have QDZ property. Then there exist open subsets $G_A \subset \Delta(A)$ and $G_B \subset \Delta(B)$ as stated in the definition of QDZ. Let $\widehat{G} = G^\circ_A \cup G^\circ_B$. Then $\widehat{G}$ is open in $\Delta^\circ(A) \uplus \Delta_\circ(B)$ and

$\partial(A \times B) = \partial^\circ(A) \uplus \partial_\circ(B) \subset \overline{G^\circ_A} \cup \overline{G^\circ_B} = \overline{G_A \cup G_B} = \overline{G}$.

Let $U \subset G$ be open. Then $U_A = U \cap G_A$ and $U_B = U \cap G_B$ are open in $G_A$ and $G_B$, respectively. Hence, there exist $a \in A$ and $b \in B$ such that $\widehat{a} = 0$ on $U^c_A$, $\widehat{a} = 1$ on some non-empty open subset $V_A$ of $U_A$, $\widehat{b} = 0$ on $U^c_B$ and $\widehat{b} = 1$ on some non-empty open subset $V_B$ of $U_B$. Then $(a, b)(\varphi) = 0$ on $U^c = U^c_A \cup U^c_B$ and $(a, b)(\varphi) = 1$ on $V_A \cup V_B \subset U$. So $A \times B$ has QDZ property. □
Corollary 2.5.2. Let \( A \) and \( B \) be commutative Banach algebras. Then \( A \times B \) has BQDZ property if and only if both \( A \) and \( B \) have BQDZ property.

Proof. This follows from Theorem 2.5.1 and Proposition 2.1.3(iv).

Theorem 2.5.3. Let \( A \) and \( B \) be commutative Banach algebras. Then \( A \times B \) has TAN property if and only if both \( A \) and \( B \) have TAN property.

Proof. Suppose that \( A \times B \) has TAN property. Then there exists a dense subset \( \tilde{D} \) of \( \partial^e A \cup \partial^e B \) such that \( \ker(\varphi^*) (\varphi^* \in \tilde{D}) \) admits a separating net. Let \( D_A = \{ \varphi \in \partial A : \varphi^e \in \tilde{D} \} \) and \( D_B = \{ \varphi \in \partial B : \varphi^e \in \tilde{D} \} \). Then \( D_A^e \cup D_B^e = \tilde{D} \). We show that \( D_A \cup D_B \) is dense in \( \partial A \). Let \( \varphi \in D_A \). Then \( \varphi^e \in \tilde{D} \). Since \( \tilde{D} \) is dense in \( \partial^e A \cup \partial^e B \), there exists a net \( (\varphi^*_\lambda)_{\lambda \in \Lambda} \) in \( \partial^e A \cup \partial^e B \) such that \( \varphi^*_\lambda \rightarrow \varphi^e \). Define \( \varphi_\lambda(a) = \varphi^*_\lambda((a,0)) (a \in A) \). Then, it is clear that \( (\varphi^*_\lambda)_{\lambda \in \Lambda} \) is a net in \( \partial A \) and \( \varphi_\lambda \rightarrow \varphi \). Which proves our claim. Let \( \varphi \in D_B \). Then \( \varphi^e \in \tilde{D} \). Hence \( \ker(\varphi^e) \) admits a separating net say \( (a_\lambda, b_\lambda) \). Then \( (a_\lambda) \) is a separating net for \( \ker(\varphi) \). Thus \( A \) has TAN property. By similar arguments it follows that \( B \) has TAN property.

Conversely, suppose \( A \) and \( B \) have TAN property. Then there exist dense subsets \( D_A \subset \partial A \) and \( D_B \subset \partial B \) such that \( \ker(\varphi) (\varphi \in D_A \cup D_B) \) admits a separating net. Then \( \tilde{D} = D_A^e \cup D_B^e \) is a dense subset of \( \partial(A \times B) \) such that \( \ker(\varphi^*) (\varphi^* \in \tilde{D}) \) admits a separating net. Hence \( A \times B \) has TAN property.

Theorem 2.5.4. Let \( A \) and \( B \) be commutative Banach algebras. If \( A \) and \( B \) have TDZ property, then \( A \times B \) has TDZ property.

Proof. Suppose that \( A \) and \( B \) have TDZ property. Let \( (a, b) \in A \times B \) be non-zero. Suppose that \( a \neq 0 \). Since \( A \) has TDZ property, there exists a sequence \( (a_n) \) in \( A \) such that \( \|a_n\| = 1 \) \( (n \in \mathbb{N}) \) and \( a_n a \rightarrow 0 \) as \( n \rightarrow \infty \). Then \( ((a_n,0)) \) is a sequence in \( A \times B \) such that \( \|(a_n,0)\|_\infty = \|a_n\| = 1, (n \in \mathbb{N}) \) and \( (a_n,0)(a,b) = (a_n a, 0) \rightarrow (0,0) \) as \( n \rightarrow \infty \). This implies \( (a,b) \in A \times B \) is a
TDZ. Next, suppose that \( a = 0 \). Since \((a, b) \neq 0\), we must have \( b \neq 0\). Since \( B \) has TDZ property, there exists a sequence \((b_n)\) in \( B \) such that \( \|b_n\| = 1 \) and \( b_n b \to 0 \) as \( n \to \infty \). Then \((0, b_n)\) is a sequence in \( A \times B \) such that \( \|(0, b_n)\|_\infty = \|b_n\| = 1 \) \((n \in \mathbb{N})\) and \((0, b_n)(a, b) = (0, b_n b) \to (0, 0)\) as \( n \to \infty \). Thus \((a, b)\) is a TDZ in \( A \times B \). Hence, \( A \times B \) has TDZ property. \( \square \)

**Remark 2.5.5.** The converse of Theorem 2.5.4 is not true. Let \( A = \mathbb{C} \) and \( B \) be any Banach space with trivial multiplication (i.e, \( ab = 0 \) \((a, b \in B))\). Then \( B \) is a Banach algebra. Let \((\alpha, b) \in \mathbb{C} \times B\) be non-zero. Suppose that \( b = 0 \). In this case, take \( c_n = c/\|c\| \) \((n \in \mathbb{N})\), where \( c \in B \) is non-zero. Then \((0, c_n)\) is a sequence in \( \mathbb{C} \times B \) such that \( \|(0, c_n)\|_\infty = \|c_n\| = 1 \) \((n \in \mathbb{N})\) and \((0, c_n)(\alpha, b) \to (0, 0)\) as \( n \to \infty \). Thus, \((\alpha, b)\) is a TDZ in \( \mathbb{C} \times B \).

Now, if possible, suppose that \( 0 \neq \alpha \in \mathbb{C} \) is a TDZ. Then there exists a sequence \((\alpha_n)\) in \( \mathbb{C} \) such that \( |\alpha_n| = 1 \) \((n \in \mathbb{N})\) and \( \alpha_n \alpha \to 0 \) as \( n \to \infty \). Then

\[
1 = \lim_{n \to \infty} |\alpha_n| = \lim_{n \to \infty} \frac{|\alpha_n \alpha|}{|\alpha|} = \frac{1}{|\alpha|} \lim_{n \to \infty} |\alpha_n \alpha| = 0.
\]

This is a contradiction. Hence \( \alpha \) can not be a TDZ. Thus \( \mathbb{C} \) does not have TDZ property.

### 2.6. Ditkin’s Condition and Tauberian Property

In this last section, we study Ditkin’s condition and Tauberian property in \( A \times B \). Let \( a \in A \) and \( b \in B \). It is clear that \( \text{supp}(a, b)^\wedge = \text{supp}(\hat{a}) \cup \text{supp}(\hat{b}) \). Hence, \( (a, b)^\wedge \in C_c(\Delta(A \times B)) \) if and only if \( \hat{a} \in C_c(\Delta(A)) \) and \( \hat{b} \in C_c(\Delta(B)) \).

**Theorem 2.6.1.** Let \( A \) and \( B \) be commutative Banach algebras. Then \( A \times B \) satisfies Ditkin’s condition if and only if \( A \) and \( B \) satisfy Ditkin’s condition.
Proof. Suppose that \( A \times B \) satisfies Ditkin’s condition. Let \( \varphi \in \Delta(A) \) and \( a \in \ker \varphi \). Then \((a,0) \in \ker(\varphi^\circ)\). Since \( A \times B \) satisfy Ditkin’s condition, there exists a sequence \(((a_n,b_n)) \in A \times B\) such that \((a_n,b_n)^\wedge \in C_c(\Delta(A \times B))\), \(\varphi^\circ \notin \text{supp}(a_n,b_n)^\wedge\) and \((a_n,b_n)(a,0) \rightarrow (a,0)\) as \(n \rightarrow \infty\). Then \((a_n)\) is a sequence in \( A \) such that \(\hat{a}_n \in C_c(\Delta(A))\), \(\varphi \notin \text{supp}(\hat{a}_n)\) and \((a_n a) \rightarrow a\) as \(n \rightarrow \infty\). Therefore, \( A \) satisfies Ditkin’s condition at infinity. Let \(\varphi \notin \Delta(A)\). Then there exists a sequence \(\bar{a}_n \in C_c(\Delta(A))\), \(\varphi \notin \text{supp}(\bar{a}_n)\) and \((a_n a) \rightarrow a\) as \(n \rightarrow \infty\). Therefore, \( A \) satisfies Ditkin’s condition at infinity. Similarly, \( B \) satisfies Ditkin’s condition at every \( \psi \in \Delta(B) \).

Next we show that \( A \) and \( B \) satisfy Ditkin’s condition at infinity. Let \( a \in A \). Since \( A \times B \) satisfy Ditkin’s condition, there exists a sequence \(((a_n,b_n)) \in A \times B\) such that \((a_n,b_n)^\wedge \in C_c(\Delta(A \times B))\) and \((a_n,b_n)(a,0) \rightarrow (a,0)\) as \(n \rightarrow \infty\). Then \((a_n)\) is a sequence in \( A \) such that \(\hat{a}_n \in C_c(\Delta(A))\) and \((a_n a) \rightarrow a\) as \(n \rightarrow \infty\). Therefore, \( A \) satisfies Ditkin’s condition at infinity. Similarly, \( B \) satisfies Ditkin’s condition at infinity.

Conversely, suppose that \( A \) and \( B \) both satisfy Ditkin’s condition. Let \( \tilde{\varphi} \in \Delta^\circ(A) \cup \Delta^\circ(B) \) and \((a,b) \in \ker(\tilde{\varphi})\). Then \( \tilde{\varphi} \in \Delta^\circ(A) \) or \( \tilde{\varphi} \in \Delta^\circ(B) \). Suppose that \( \tilde{\varphi} \in \Delta^\circ(A) \). Then there exists \( \varphi \in \Delta(A) \) such that \( \tilde{\varphi} = \varphi^\circ \). Then \( a \in \ker(\varphi) \). Since \( A \) satisfies Ditkin’s condition at \( \varphi \), there exists a sequence \((a_n)\) in \( A \) such that \(\hat{a}_n \in C_c(\Delta(A))\), \(\varphi \notin \text{supp}(\hat{a}_n)\) and \((a_n a) \rightarrow a\) as \(n \rightarrow \infty\). Since \( B \) satisfy ditkin’s condition at infinity, there exists a sequence \((b_n)\) in \( B \) such that \(\hat{b}_n \in C_c(\Delta(B))\) and \((b_n b) \rightarrow b\) as \(n \rightarrow \infty\). But then \(((a_n,b_n))\) is a sequence in \( A \times B \) such that \((a_n,b_n)^\wedge \in C_c(\Delta(A \times B))\), \(\tilde{\varphi} \notin \text{supp}(a_n,b_n)^\wedge\) and \((a_n,b_n)(a,b) \rightarrow (a,b)\) as \(n \rightarrow \infty\). The proof follows by similar argument if \( \tilde{\varphi} \in \Delta^\circ(B) \). Thus \( A \times B \) satisfies Ditkin’s condition at every \( \tilde{\varphi} \in \Delta^\circ(A) \cup \Delta^\circ(B) \).

Next we show that \( A \times B \) satisfies Ditkin’s condition at infinity. Let \((a,b) \in A \times B \). Since both \( A \) and \( B \) satisfy Ditkin’s condition at infinity, there exist sequences \((a_n) \subset A\) and \((b_n) \subset B\) such that \(\hat{a}_n \in C_c(\Delta(A))\),
\[ \hat{b}_n \in C_c(\Delta(B)), (a_n a) \to a \text{ and } (b_n b) \to b. \] Then \((a_n, b_n)^\wedge \in C_c(\Delta(A \times B))\) and \((a_n, b_n)(a, b) \to (a, b)\). Hence, \(A \times B\) satisfies Ditkin’s condition. \(\square\)

**Theorem 2.6.2.** Let \(A\) and \(B\) be commutative Banach algebras. Then \(A \times B\) is Tauberian if and only if both \(A\) and \(B\) are Tauberian.

**Proof.** Let \(A \times B\) is Tauberian. Let \(a \in A, b \in B\) and \(\epsilon > 0\). Since \(A \times B\) is Tauberian, there exits \((a_0, b_0) \in A \times B\) such that \((a_0, b_0)^\wedge \in C_c(\Delta(A \times B))\) and \(\| (a_0, b_0) - (a, b) \|_\infty < \epsilon\). Then \(\hat{a}_0 \in C_c(\Delta(A)), \hat{b}_0 \in C_c(\Delta(B)), \| a_0 - a \| < \epsilon\) and \(\| b_0 - b \| < \epsilon\). Therefore \(A\) and \(B\) are Tauberian.

Conversely, suppose \(A\) and \(B\) are Tauberian. Let \((a, b) \in A \times B\) and \(\epsilon > 0\). Since \(A\) and \(B\) are Tauberian, there exist \(a_0 \in A, b_0 \in B\) such that \(\hat{a}_0 \in C_c(\Delta(A)), \hat{b}_0 \in C_c(\Delta(B)), \| a_0 - a \| < \epsilon/2\) and \(\| b_0 - b \| < \epsilon/2\). Then \((a_0, b_0)^\wedge \in C_c(\Delta(A \times B))\) and \(\|(a_0, b_0) - (a, b)\|_1 < \epsilon\). Therefore, \(A \times B\) is Tauberian. \(\square\)
Table 2.6.2.1. Summary of Results in Chapter 2

<table>
<thead>
<tr>
<th>No.</th>
<th>Property</th>
<th>$A \times B$</th>
<th>$A &amp; B$</th>
<th>$A \geq B$</th>
<th>Result No.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Commutative</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.1.2</td>
</tr>
<tr>
<td>2</td>
<td>Unital</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.1.2</td>
</tr>
<tr>
<td>3</td>
<td>L.A.I./R.A.I./B.A.I.</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.1.9</td>
</tr>
<tr>
<td>4</td>
<td>Uniform algebra</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.1.10</td>
</tr>
<tr>
<td>5</td>
<td>Semisimplicity</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.2.8</td>
</tr>
<tr>
<td>6</td>
<td>*-semisimplicity</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.2.9</td>
</tr>
<tr>
<td>7</td>
<td>Hermiticity</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.2.9</td>
</tr>
<tr>
<td>8</td>
<td>UUNP</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.3.1</td>
</tr>
<tr>
<td>9</td>
<td>$UC^*NP$</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.3.2</td>
</tr>
<tr>
<td>10</td>
<td>Weakly regular</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.3.3</td>
</tr>
<tr>
<td>11</td>
<td>Regular</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.3.4</td>
</tr>
<tr>
<td>12</td>
<td>BR</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.3.6</td>
</tr>
<tr>
<td>13</td>
<td>CR</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.3.5</td>
</tr>
<tr>
<td>14</td>
<td>BCR</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.3.6</td>
</tr>
<tr>
<td>15</td>
<td>MHBP</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.4.1</td>
</tr>
<tr>
<td>16</td>
<td>SSEP</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.4.2</td>
</tr>
<tr>
<td>17</td>
<td>SEP</td>
<td>✓</td>
<td>Unknown</td>
<td></td>
<td>2.4.3</td>
</tr>
<tr>
<td>18</td>
<td>WSEP</td>
<td>✓</td>
<td>Unknown</td>
<td></td>
<td>2.4.4</td>
</tr>
<tr>
<td>19</td>
<td>QDZ</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.5.1</td>
</tr>
<tr>
<td>20</td>
<td>BQDZ</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.5.2</td>
</tr>
<tr>
<td>21</td>
<td>TAN</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.5.3</td>
</tr>
<tr>
<td>22</td>
<td>TDZ</td>
<td>×</td>
<td>✓</td>
<td></td>
<td>2.5.4</td>
</tr>
<tr>
<td>23</td>
<td>Ditkin’s condition</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.6.1</td>
</tr>
<tr>
<td>24</td>
<td>Tauberian</td>
<td>✓</td>
<td>✓</td>
<td></td>
<td>2.6.2</td>
</tr>
</tbody>
</table>

Remark 2.6.3. We should note the following.

(i) No.2, 3, 6, 7, 17 and 18 are true without assuming commutativity.
(ii) Rest results are true for commutative algebras $A$ and $B$. 