CHAPTER VI

TRANSIENT THERMOELASTIC WAVES IN AN AELOTROPIC MEDIUM
WITH A CYLINDRICAL HOLE

6.1 INTRODUCTION

In this chapter the displacement, temperature, and stress due to a thermal shock in a homogeneous, transversely isotropic elastic medium with a cylindrical hole have been investigated in the context of generalised theories of thermoelasticity developed by Green and Lindsay [50], Lord and Shulman [20], respectively. The Laplace transform on time has been used to obtain the solutions and the small time approximations have been considered because of short duration of "second sound" effects.

6.2 FORMULATION OF THE PROBLEM

We consider an infinitely extended homogeneous transversely isotropic thermoelastic medium initially at temperature $T_0$ having an infinite cylindrical hole of radius $a$. We choose the origin of cylindrical coordinate system $(r, \theta, z)$ at the axis of the cylindrical hole. We consider the case of radial symmetry so that the non-zero displacement $u(r,t)$ and temperature $T(r,t)$ satisfy the differential equations [50].
\[ C_{11} \left[ u_{,rr} + \left( r^{-1} u \right)_{,r} \right] - \beta_1 (\Theta + \kappa \dot{\Theta})_{,r} = \rho \ddot{u}, \quad (6.2.1) \]

\[ K(\Theta_{,rr} + r^{-1} \Theta_{,r}) - \rho C_e (\dot{\Theta} + \kappa \ddot{\Theta}) = \xi_0 \beta_1 (\dot{u}_{,r} + r^{-1} u), \quad (6.2.2) \]

where \( \beta_1 = (C_{11} + C_{12}) \kappa_1 + C_{13} \kappa_3 \). \( C_{ij} \) are isothermal parameters; \( K, \rho, C_e, \kappa \) and \( \kappa_0 \) be the thermal conductivity, density of the medium, specific heat at constant strain and the thermal relaxation times respectively. The comma notation and the superposed dot represent the space and time derivatives respectively. The thermal relaxation times \( \kappa \) and \( \kappa_0 \) in equations (6.2.1) and (6.2.2) satisfy the inequalities [50],

\[ \kappa \geq \kappa_0 \geq 0. \quad (6.2.3) \]

Introducing the dimensionless quantities

\[
\begin{align*}
\tau' &= \omega^* \tau / \sqrt{\rho}, \\
t' &= \omega^* t, \\
u' &= \rho \omega^* v, \\
\xi_0/\tau_0 &= T_0, \\
\zeta &= T_0 \beta_1 / \rho C_e C_{11}, \\
w^* &= C_{11} C_e / K,
\end{align*}
\]

where \( v_p^2 = c_{11} / \rho \), is the velocity of the longitudinal waves.

Using (6.2.4) in (6.2.1) and (6.2.2), we obtain (on suppressing the dashes)

\[ u_{,rr} + \tau^{-1} u_{,r} - \tau^{-2} u - \ddot{u} = \zeta, \quad (6.2.5) \]
The boundary of the cylindrical hole is given by
\[ r = w_0 a / v_p = \eta. \] (6.2.7)

The initial boundary conditions are
\[ u = 0 = Z, \text{ at } t = 0, \ r \geq \eta, \ r \to \infty \] (6.2.8)
\[ \frac{\partial u}{\partial t} = 0, \text{ at } r = 0 \]
and
\[ S_{rr}(\eta, t) = 0, \ Z(\eta, t) = f(t), \] (6.2.9)
where
\[ S_{rr} = u_r + c_{11} r^{-1} c_{12} r^{-1} u - (Z + \alpha \dot{Z}), \] (6.2.10)
is the dimensionless form of the stress and \( f(t) \) is a sufficiently well behaved function of time.

6.3 **SOLUTION OF THE PROBLEM**
Applying the Laplace transform defined by
\[ \bar{\phi}(r, p) = \int_0^\infty \phi(r, t) e^{-pt} \, dt, \] (6.3.1)
with respect to time, to equations (5.2.5) and (6.2.6) and using (6.2.7), we obtain
\[ \left[ \{ D(D+r^{-1}) \}^2 - (m_1^2+m_2^2)D(D+r^{-1})+m_1^2m_2^2 \right] \ddot{u} = 0, \] (6.3.2)
\[
\left\{ (D+r^{-1})D \right\}^2 - \left( m_1^2 + m_2^2 \right) D(D+r^{-1}) + m_1^2 m_2^2 \right \} \tilde{Z} = 0, \tag{6.3.3}
\]

where \( D = d/dr \), and \( m_1^2, m_2^2 \) are the roots of the equation
\[
m^4 - p \left( \lambda_1 + \lambda_2 p \right) m^2 + \xi_0 p^4 = 0 \tag{6.3.5}
\]

where
\[
\lambda_1 = 1 + \xi, \quad \lambda_2 = 1 + \xi + \xi_0, \quad \xi_0 = \xi + p^{-1} \tag{6.3.5}
\]

Solving the equations (6.3.2) and (5.3.3) and on Using (6.2.7), we get
\[
\bar{u}(r,p) = A_1 K_i(m_1 r) + A_2 K_i(m_2 r), \tag{6.3.6}
\]
\[
\tilde{Z}(r,p) = B_1 K_0(m_1 r) + B_2 K_0(m_2 r), \tag{6.3.7}
\]

where \( K_i(m_1 r) \) and \( K_0(m_1 r) \) are the modified Bessel's functions of order one and zero respectively.

\( A_i \) and \( B_i \) are related by the equations
\[
B_i = \left( p^2 - m_i^2 \right) A_i / m_i, \quad i = 1, 2. \tag{6.3.8}
\]

The transformed boundary conditions become
\[
\tilde{S}_{rr}(\eta, p) = D \tilde{u} + C_{12} C^{-1}_{11} \quad r^{-1} \quad \tilde{u} - \xi p \tilde{Z} = 0, \quad \tilde{Z}(r, p) = \tilde{f}(p), \tag{6.3.9}
\]

where
\[
\tilde{f}(p) = \int_0^\infty f(t) e^{-pt} dt, \quad \xi = (\xi + p^{-1}). \tag{6.3.10}
\]

Using equations (6.3.6) and (6.3.7) in (6.3.9) and after simplifying, we obtain
$A_1 = m_2 \tilde{f}(p) \left[ m_2 \left\{ \frac{K_0(m_2 \eta) + b K_1(m_2 \eta)}{\eta} \right\} + c \rho (p^2 - m_2^2) K_0(m_2 \eta) \right] / \Delta$ \hspace{1cm} (6.3.11)

$A_2 = m_2 \tilde{f}(p) \left[ m_1 \left\{ \frac{K_0(m_1 \eta) + b K_1(m_1 \eta)}{\eta} \right\} + c \rho (p^2 - m_1^2) \right]$ \hspace{1cm} (6.3.12)

where

$\Delta = m_2 (p^2 - m_1^2) K_0(m_1 \eta) \left[ m_2 K_0(m_2 \eta) + b K_1(m_2 \eta) / \eta \right] - m_1 (p^2 - m_2^2) K_0(m_2 \eta) \left[ m_1 K_0(m_1 \eta) + b K_1(m_1 \eta) / \eta \right], b = (C_{11} - C_{12}) / C_{11} \cdot$ \hspace{1cm} (6.3.13)

6.4 SMALL TIME APPROXIMATIONS

As the "second sound" effects are short-lived, Green [52], so we restrict ourselves to small time approximations, i.e. we take $p$ large. The roots $m_i$ of equations (6.3.4) are given by

$m_i = p v_i^{-1} + \phi_i + o(p^{-1}), i = 1, 2, \hspace{1cm} (6.4.1)$

where

$v_{1,2} = \sqrt{2} \left[ \lambda_{2} \pm (\lambda_{2}^{-2} - 4 \kappa_{0} \kappa_{0})^{\frac{3}{2}} \right]^{-\frac{3}{2}}, \hspace{1cm} (6.4.2)$

$\phi_{1,2} = \frac{\lambda_1 \lambda_2}{(\lambda_{2}^{-2} - 4 \kappa_{0} \kappa_{0})^{\frac{3}{2}}} \left[ \lambda_2 \pm (\lambda_{2}^{-2} - 4 \kappa_{0} \kappa_{0})^{\frac{3}{2}} \right]^{\frac{3}{2}} \hspace{1cm} (6.4.3)$
and $v_1, v_2$ be the speeds of the elastic and the thermal waves \[44\], respectively. The modified Bessel's function $K_n(m_1 r)$ has the asymptotic expansion \[88\]

$$
K_n(m_1 r) \approx \left( \frac{\pi}{2m_1 r} \right)^{1/2} \exp(-m_1 r) \left[ 1 + \frac{(4n^2-1^2)}{(8m_1 r)} \right] + \frac{(4n^2-1^2)(4n^2-3^2)}{(2(8m_1 r)^2) + \ldots}.
$$

(6.4.4)

Using expansion (6.4.1) and (6.4.4) in (6.3.11) - (6.3.13) and then in equations (6.3.6) - (6.3.8), we obtain

$$
\bar{u}(r,p) = \left( \frac{\eta}{r} \right)^{1/2} \left\{ \frac{v^2}{v_1 v_2} \right\} \exp[-m_1 (r-\eta)] [v_2(v_1^2-v_2^2)p^{-1} + \{3v_1 v_2 (v_1^2-v_2^2)/8r - \gamma^* \}_{1}^{v_1^2} p^{-2} - \frac{\kappa(v_2^2-1)}{v_1 v_2} \exp[-m_2 (r-\eta)]
$$

$$
[v_1(v_2^2-v_1^2)p^{-1} + \{3v_1 v_2 (v_2^2-v_1^2)/8r - \gamma^* \}_{2}^{v_2^2} p^{-2}] \right\}
$$

(6.4.5)

$$
\bar{z}(r,p) = \left( \frac{\eta}{r} \right)^{1/2} (v_1 v_2)^{-1} \left\{ \frac{v_1^2}{v_2^2} \right\} \exp[-m_1 (r-\eta)] [v_2(v_1^2-v_2^2)(v_1^2-1) + \{v_1(v_1^2-v_2^2)(v_1^2-1) \gamma^*_{1} - \Gamma(v_1^2-1) - v_1 [\phi(v_1^2+1)+(v_1^2-1)/8r] - v_2(v_2^2+v_1^2)(v_2^2-1) + \{v_2(v_2^2+v_1^2)(v_2^2-1) \gamma^*_{2} - \Gamma(v_2^2-1) - v_2 [\phi(v_2^2+1)+(v_2^2-1)/8r] - v_1(v_2^2+v_1^2)(v_2^2-1) \gamma^*_{1} + \gamma^*_{2} \}_{1}^{v_2^2} p^{-1} \right\}
$$

(6.4.6)
and the transformed stress is given by

\[
\tilde{S}_{\varpi}(x, p) \sim \left(\frac{\eta}{x}\right)(v_1 v_2)^{-1}(v_2^2 - 1)\exp[-m_1(x - \eta)] \left[v_1^{-1}(v_1^2 - v_2^2)\right]^{\frac{1}{2}}
\]

\[
(2-v_1^2) p + (v_1^{-1} v_2 (v_1^2 - v_2^2)(2-v_1^2)) + \{v_1^{-1} v_2 (v_1^2 - v_2^2) \lambda_1^*
\]

\[
+ \lambda'_1\right\} p^{-1} - \left(\frac{\eta}{x}\right)^{\frac{1}{2}} (v_1 v_2)^{-1}(v_1^2 - 1)\exp[-m_2(x - \eta)]
\]

\[
\left[\lambda_1^{-1} v_1 v_2^{-1} (v_2^2 - v_1^2)(2-v_1^2) p + (v_1 v_2 (v_2^2 - v_1^2)(2-v_1^2)) + \right.
\]

\[
\{v_1 v_2^{-1} (v_2^2 - v_1^2) \lambda_1^* + \lambda'_2\} p^{-1}\right]\]

(6.4.7)

where

\[
\lambda_1^* = \Phi_1 v_1 + \Phi_2 v_2 - v_2^2 /\lambda(v_2^2 - 1) - 2\Phi_2 v_2^3/(v_2^2 - 1) - bv_2/\eta
\]

(6.4.8)

\[
\lambda_2^* = \Phi_1 v_1 + \Phi_2 v_2 - v_2^2 /\lambda(v_1^2 - 1) - 2\Phi_1 v_1^3/(v_1^2 - 1) - bv_1/\eta
\]

(6.4.9)

\[
\lambda_1' = \Gamma_1 (v_1^2 - 2)v_1^{-1} + v_2 (v_2^2 - v_2^2) \Phi_1 (v_2^2 + 2) + (8b - 2 + v_1^2)/\theta \xi
\]

(6.4.10)

\[
\lambda_2' = \Gamma_2 (v_2^2 - 2)v_2^{-1} + v_2 (v_2^2 - v_2^2) \Phi_2 (v_1^2 + 2) + (8b - 2 + v_2^2)/\theta \xi
\]

\[
\Gamma_1 = v_2 (v_1^2 - v_2^2)(v_1 + v_2)/8\eta - 2\Phi_1 v_1 v_2^3 + \Phi_2 v_2^3 (v_1^2 + v_2^2)
\]

(6.4.11)

\[
\Gamma_2 = v_1 (v_2^2 - v_2^2)(v_1 + v_2)/8\eta - 2\Phi_2 v_1 v_2^3 + \Phi_1 v_1^3 (v_1^2 + v_2^2)
\]
For a unit step in temperature at the boundary of the surface of the cylindrical hole, \( f(t) = H(t) \) so that \( f(p) = p^{-1} \).

Inverting the Laplace transform of the equations (6.4.5) - (6.4.7), we obtain

\[
\begin{align*}
\mathcal{L}^{-1}\{u(r,t)\} &= \left(\frac{\eta}{R+\eta}\right)^{\frac{\kappa}{2}} \frac{1}{v_1 v_2} \left[ (v_2^2-1) \exp(-\phi_1 R) \{ v_2 (v_1^2-v_2^2) H(t-R/v_1) \right. \\
&\quad \left. + \left[ 3v_1 v_2 (v_1^2-v_2^2)/8(R+\eta)-\Gamma_1+v_2 (v_1^2-v_2^2) \lambda_1^* \right] (t-R/v_1) \right) \\
&\quad \left. \cdot H(t-R/v_1) \right] \cdot (v_2^2-1) \exp(-\phi_2 R) \{ v_1 (v_2^2-v_1^2) H(t-R/v_2) \right. \\
&\quad \left. + \left[ 3v_1 v_2 (v_2^2-v_1^2)/8(R+\eta)-\Gamma_2+v_1 (v_2^2-v_1^2) \lambda_2^* \right] (t-R/v_2)H(t-R/v_2) \right]
\end{align*}
\]

(6.4.11)

\[
\begin{align*}
\mathcal{L}^{-1}\{Z(r,t)\} &= \left(\frac{\eta}{R+\eta}\right)^{\frac{\kappa}{2}} \frac{1}{v_1 v_2} \left[ v_1^{-1}(v_2^2-1) \exp(-\phi_1 R) \{ v_2 (v_1^2-v_2^2)(v_1^2-1) \\
&\quad \left. + [v_2 (v_1^2-v_2^2)(v_1^2-1) \lambda_1^* - \Gamma_1(v_1^2-1) - v_1 v_2 (v_1^2-v_2^2) \right. \\
&\quad \left. \{ \phi_1(v_1^2+1)+(v_1^2-1)/8(R+\eta) \} \right] H(t-R/v_1) - v_2^{-1}(v_1^2-1) \exp(-\phi_2 R) \right. \\
&\quad \left. \{ v_1 (v_2^2-v_1^2)(v_2^2-1) \delta(t-R/v_1) + [v_1 (v_2^2-v_1^2)(v_2^2-1) \\
&\quad \left. - \Gamma_2(v_1^2-1)-v_1 v_2 (v_2^2-v_1^2) \{ \phi_2(v_2^2+1)+(v_2^2-1)/8(R+\eta) \} \right] \\
&\quad \left. H(t-R/v_2) \right] \right.
\end{align*}
\]

(6.4.12)
where $\Gamma_1$, $\Gamma_2$, $\lambda_1^*$, $\lambda_2^*$, $\lambda_1'$ and $\lambda_2'$ are given by the equations (6.4.3) - (6.4.10).

**LORD-SHULMAN THEORY**

If we consider and work out the problem in the context of generalised theory of thermoelasticity developed of Lord-Shulman [20] and advanced by Dhauliwai and Sherief [38] to anisotropic solids, then the displacement, temperature, and stress are obtained as [cf. [44]]

$$u(r,t) = \left(\frac{\eta}{R+\eta}\right)^{1/2} \left[ v_1^{-1} v_2^2 (v_2^2 - v_1^2) \exp(-\phi_1 R) \left\{ (t-R/v_1) H(t-R/v_1) 
+ \left[ 3v_1/8(R+\eta) + \phi_1 v_1 - \phi_2 v_2 - b(v_2^2-1)/v_2 \eta - \Gamma_1/v_2 (v_2^2-v_1^2) \right] 
(t-R/v_1)^2 H(t-R/v_1) \right\} - v_1^{-1} v_2^2 (v_1^2 - v_2^2) \exp(-\phi_2 R) \left\{ (t-R/v_2) H(t-R/v_2) + \left[ 3v_2/8(R+\eta) + \phi_2 v_2 - \phi_1 v_1 - b(v_1^2-1)/v_1 \eta - \Gamma_2/v_1 (v_2^2-v_1^2) \right] (t-R/v_2)^2 H(t-R/v_2) \right\} \right],$$

(6.4.14)
\[ Z(r, t) \approx \left( \frac{\eta}{R + \eta} \right)^{3/2} \left[ v_1^2 v_2^2 \frac{(v_1^2 - v_2^2)(v_1^2 - 1)}{v_1^2(v_1^2 - v_2^2)} \exp(-\Phi_1 R) \left\{ H(t-R/v_1) \right. \right. \\
- \left\{ \Phi_1 v_1 (v_1^2 + 1) + 1/v_1 \left. \left( R + \eta \right) - \Phi_1 v_1 + \Phi_2 v_2 \right\} \right] \\
- \left\{ \frac{\gamma_1}{v_2^2 v_2^2} \right\} (t-R/v_1) \tag{6.4.15} \]

\[ S_{\tau \tau}(r, t) \approx \left( \frac{\eta}{R + \eta} \right)^{3/2} v_1 v_2^2 \left[ v_2^2 \frac{(v_1^2 - v_2^2)}{v_1^2(v_1^2 - v_2^2)} \exp(-\Phi_1 R) \left\{ H(t-R/v_1) \right. \right. \\
+ \left\{ \left[ \Phi_1 (v_1^2 + 2) + (8b - 2 + v_1^2)/(8(R + \eta)) + \gamma_1 (v_1^2 - 2)/v_1 v_2 \right] \\
\left( v_1^2 - v_2^2 \right) - (2-v_1^2) \left\{ \Phi_1 v_1 - \Phi_2 v_2 - b(v_2^2 + 1)/v_2 \right\} \right] (t-R/v_1) \\
H(t-R/v_1) \right. \right\} - v_1 \left( v_2^2 - v_1^2 \right) \exp(-\Phi_2 R) \left\{ \right. \right. H(t-R/v_2) \\
+ [\Phi_2 (v_1^2 + 2) + (8b - 2 + v_2^2)/(8(R + \eta)) + \gamma_2 (v_2^2 - 2) \\
\left. \right] (v_1^2 v_2^2 - (2-v_1^2) \left\{ \Phi_2 v_2 - \Phi_1 v_1 - b(v_1^2 - 1)/v_1 \right\} \right] (t-R/v_2) \tag{6.4.16} \]

where \( \gamma_1 \) and \( \gamma_2 \) are given by the equations (5.4.10) and \( v_1, v_2 \) and \( \phi_1, \phi_2 \) are given by the equations (6.4.2) and (6.4.3) respectively, when \( \lambda_0 \) is replaced by \( \sigma_0 \) [44].
6.5 DISCUSSION AT THE WAVE-FRONT

In the context of Green-Lindsay theory of thermoelasticity the displacement, temperature, and stress all are found to be discontinuous at the wave fronts. The discontinuities at the wavefronts are given by

\[
[u^+ - u^-] = \frac{\eta}{R + \eta} \left( \frac{\nu_2^2 - 1}{\nu_1^2 - 1} \right) \left( \phi_1 v_1 t \right) \exp \left( -\phi_1 \nu_1 t / \nu_1 \right)
\]

\[
[u^+ - u^-] = \frac{\eta}{R + \eta} \left( \frac{\nu_2^2 - 1}{\nu_1^2 - 1} \right) \left( \phi_2 v_2 t \right) \exp \left( -\phi_2 \nu_2 t / \nu_2 \right)
\]

\[
[z^+ - z^-] = \frac{\eta}{R + \eta} \left( \frac{\nu_2^2 - 1}{\nu_1^2 - 1} \right) \left( \phi_1 v_1 t \right) \exp \left( -\phi_1 \nu_1 t / \nu_1 \right) \left\{ \nu_1 \left( \nu_2^2 - \nu_1^2 \right) \right\}
\]

\[
[z^+ - z^-] = \frac{\eta}{R + \eta} \left( \frac{\nu_2^2 - 1}{\nu_1^2 - 1} \right) \left( \phi_2 v_2 t \right) \exp \left( -\phi_2 \nu_2 t / \nu_2 \right) \left\{ \nu_1 \left( \nu_2^2 - \nu_1^2 \right) \right\}
\]

\[
[s^+_{xx} - s^-_{xx}] = \frac{\nu_2^2}{\nu_1^2} \left( \frac{\nu_2^2 - 1}{\nu_1^2 - 1} \right) \left( \phi_1 v_1 t \right) \exp \left( -\phi_1 \nu_1 t / \nu_1 \right) \left\{ \nu_1 \left( \nu_2^2 - \nu_1^2 \right) \right\}
\]

\[
[s^+_{xx} - s^-_{xx}] = \frac{\nu_2^2}{\nu_1^2} \left( \frac{\nu_2^2 - 1}{\nu_1^2 - 1} \right) \left( \phi_2 v_2 t \right) \exp \left( -\phi_2 \nu_2 t / \nu_2 \right) \left\{ \nu_1 \left( \nu_2^2 - \nu_1^2 \right) \right\}
\]
And in the context of Lord - Shulman theory of thermoelasticity the displacement is found to be continuous but the temperature and the stress are found to be discontinuous.

The discontinuities are given by

\[
[z^+ - z^-] = \frac{n}{\left( \frac{\Delta^2}{\alpha^2} \right)^{\frac{1}{2}}} \left( \frac{\Delta^2}{\alpha^2} \right)^{\frac{1}{2}} \exp \left( -\Phi_1 \frac{v_1 t}{\sqrt{\alpha}} \right) \frac{v_1^2}{v_2^2}
\]

\[
[z^+ - z^-] = \frac{n}{\left( \frac{\Delta^2}{\alpha^2} \right)^{\frac{1}{2}}} \left( \frac{\Delta^2}{\alpha^2} \right)^{\frac{1}{2}} \exp \left( -\Phi_2 \frac{v_2 t}{\sqrt{\alpha}} \right) \frac{v_2^2}{v_1^2}
\]

\[
[s^+_r - s^-_r] \approx \frac{n}{\left( \frac{\Delta^2}{\alpha^2} \right)^{\frac{1}{2}}} \left( \frac{\Delta^2}{\alpha^2} \right)^{\frac{1}{2}} \exp \left( -\Phi_1 \frac{v_1 t}{\sqrt{\alpha}} \right) \frac{v_1^3}{v_2^3}
\]

\[
[s^+_r - s^-_r] \approx \frac{n}{\left( \frac{\Delta^2}{\alpha^2} \right)^{\frac{1}{2}}} \left( \frac{\Delta^2}{\alpha^2} \right)^{\frac{1}{2}} \exp \left( -\Phi_2 \frac{v_2 t}{\sqrt{\alpha}} \right) \frac{v_2^3}{v_1^3}
\]

These jumps decay exponentially with respect to time.

6.6 PARTICULAR CASES

(a) In case of conventional coupled theory of thermoelasticity, i.e. \( \xi = \xi_0 = 0 \) and \( \xi_0 = 0 \), in cases of Green - Lindsay and Lord - Shulman theories respectively. Then, we have

\[
\lambda_1 = 1 + \xi, \quad \lambda_2 = 1, \quad \nu_1 = 1, \quad v_2 \to \infty, \quad \Phi_1 = \frac{\xi}{2}, \quad \Phi_2 \to \infty.
\]

The displacement and stress in case of Green - Lindsay
theory, temperature in both the theories are found to be continuous at both the wave-fronts, whereas the stress in Lord-Shulman theory is found to be continuous at the thermal wavefront but experiences an infinite jump at the elastic wavefront. The results agree with those of [17].

(b) When the strain and thermal fields are uncoupled to each other, i.e. $\varepsilon = 0$. then, for Green-Lindsay theory, we have

$$\lambda_1 = 1, \quad \lambda_2 = 1 + \kappa_0, \quad \nu_1 = 1, \quad \nu_2 = 1 / \sqrt{\kappa_0}, \quad \phi_1 = \varepsilon / 2, \quad \phi_2 = \nu_2 / 2,$$

and for Lord-Shulman theory, we get

$$\lambda_1 = 1, \quad \lambda_2 = 1 + \kappa_0, \quad \nu_1 = 1, \quad \nu_2 = 1 / \sqrt{\kappa_0}, \quad \phi_1 = \varepsilon / 2, \quad \phi_2 = \nu_2 / 2.$$

In case of Green-Lindsay theory, the temperature is continuous at both the wavefronts, and displacement as well as stress are continuous at the thermal wavefront but discontinuous at the elastic wavefront. The discontinuities are given by

$$[u^+ - u^-] \approx \left( \frac{\eta}{R + \eta} \right)^{1/2} \kappa(1 - \kappa_0)^2 / \kappa_0^2$$

and

$$[S^+ - S^-] \approx \left( \frac{\eta}{R + \eta} \right)^{1/2} \left\{ \frac{3/2 \kappa_0 + \kappa_0 + \kappa_0}{\eta \kappa_0^{3/2} (1 + \sqrt{\kappa_0})} + \frac{4 (1 + \kappa_0) \sqrt{\kappa_0} (1 - b)}{\kappa_0^{3/2} (\kappa_0 - 1) (R + \eta)} \right\}$$

$$- b \eta / \sqrt{\kappa_0} - 4 \left[ 2 \kappa_0 - \kappa(1 - \kappa_0) - 2 \kappa \right] / \kappa(1 - \kappa_0) / 8 \kappa_0^2.$$
In case of Lord – Shulman theory the temperature is continuous at the elastic wavefront but discontinuous at the thermal wavefront. The stress is found to be discontinuous at both the wavefronts. The discontinuities are given by

\[ [v^+ - v^-] \approx \left( \frac{\eta}{R + \eta} \right)^{1/2} \left( 1 + c_o^2 \right) c_o \exp \left( -\frac{R}{2} \sqrt{c_o} \right), \quad R = v_2 t \]

\[ [S^+_{rr} - S^-_{rr}] \approx \left( \frac{\eta}{R + \eta} \right)^{1/2} \left( c_o - 1 \right) c_o^{-5/2} \quad \text{if } t \to 0 \]

\[ [S^+_{rr} - S^-_{rr}] \approx \left( \frac{\eta}{R + \eta} \right)^{1/2} \left( 1 - c_o \right) c_o^{-3/2} \exp \left( -\frac{R}{2} \sqrt{c_o} \right) \quad \text{if } t \to \infty \]

In this case these jumps also vanish as the radial distance increases.

(c) If \( t = 0 \), \( s = s_o = 0 \), i.e. the coupling and relaxation effects are ignored. The, we have

\[ \lambda_1 = 1, \lambda_2 = 1, v_1 = 1, v_2 \to \infty, \phi_1 = 0, \phi_2 \to \infty. \]

These quantities also take the same values in case of Lord – Shulman theory if we take \( t = 0 \) and \( c_o = 0 \). The displacement and temperature are found to be continuous both the wavefronts in Green-Lindsay and Lord-Shulman theories respectively. The stress, in both the theories experiences an infinite jump at the elastic wavefront but found to be continuous in case of Lord-Shulman theory at the thermal wavefront.
The short time solutions obtained in the previous sections show that these consist of two types of waves namely, elastic wave and thermal wave travelling with velocities $v_1$ and $v_2$ respectively. The temperature and stress are found to be discontinuous in both the theories. The displacement is found to be continuous in case of Lord-Shulman theory but found to be discontinuous in Green-Lindsay theory. The stress experiences a strong discontinuity in case of Green-Lindsay theory as compared to that in Lord-Shulman theory. The jumps also decay exponentially with respect to time.

In case of conventional coupled thermoelasticity the stress in Lord-Shulman theory experiences an infinite jump at the elastic wavefront, whereas the other quantities are found to be continuous in both the theories. When the strain and thermal fields are uncoupled then, the displacement and stress at the elastic wavefront in Green-Lindsay theory experience finite jumps, whereas the temperature at the thermal wavefront and stress at both the wavefronts in Lord-Shulman theory also experience finite jumps, and other quantities are found to be continuous. In this case the jumps discussed at the different wavefronts also vanish as the radial distance increases. If the coupling and the relaxation effects are ignored, then the stress at the elastic wavefront in both the theories experiences an infinite jump, whereas all other quantities are continuous in both the theories.