Chapter 4

CP-ODD WEAK BASIS
INVARIANTS AND TEXTURE
ZEROS

4.1 Introduction

In the Standard Model (SM) of fundamental particles and interactions, there is no
$CP$ violation in the leptonic sector. However, in most extensions of the SM, there
can be several $CP$ violating phases. In the simplest three generation scenario, there
is a Dirac-type $CP$ violating phase in the lepton mixing matrix. However, for Ma-
ajorana neutrinos, there could be two additional phases. It is possible to work in a
parametrization in which all the three $CP$ violating phases are situated in the charged
current lepton mixing matrix. Without any loss of generality, one can work in fla-
vor basis in which charged lepton mass matrix is diagonal so that the neutrino mass
matrix carries all the information about $CP$ violation. The search for $CP$ violation
in the leptonic sector at low energies is one of the major challenges for experimental
neutrino physics. Experiments with superbeams and neutrino beams from neutrino
factories have the potential to measure either directly the Dirac phase $\delta$ through the
observation of $CP$ and $T$ asymmetries or indirectly through neutrino oscillations. An alternative method is to measure the area of the unitarity triangles defined for the leptonic sector. Thus, neutrino physics provides an invaluable tool for the investigation of leptonic $CP$ violation at low energies apart from having profound implications for the physics of the early universe.

In the flavor basis, the mass matrix for Majorana neutrinos contains nine physical parameters including three mass eigenvalues, three mixing angles and three $CP$ violating phases. Two mass squared differences ($\Delta m^2_{12}$ and $\Delta m^2_{23}$) and two mixing angles ($\theta_{12}$ and $\theta_{23}$) have been measured in solar, atmospheric and reactor experiments. The third mixing angle $\theta_{13}$ and the Dirac-type $CP$ violating phase $\delta$ are expected to be measured in the forthcoming neutrino oscillation experiments. Possible measurement of effective Majorana mass in neutrinoless double beta decay searches will provide an additional constraint on the remaining three neutrino parameters viz. neutrino mass scale and two Majorana-type $CP$ violating phases. While the neutrino mass scale will be independently determined from the direct beta decay searches and cosmological observations, the two Majorana phases will not be uniquely determined from the measurement of effective Majorana mass even if the overall neutrino mass scale is known. Thus, it is not possible to fully reconstruct the neutrino mass matrix from the observations from feasible experiments. Under the circumstances, it is natural to employ other theoretical inputs for the reconstruction of the neutrino mass matrix. The possible forms of these additional theoretical inputs are constrained by the existing neutrino data. Several proposals have been made in literature to restrict the possible forms of the neutrino mass matrix by reducing the number of free parameters which include the presence of texture zeros[1, 2, 3], the requirement of zero determinant[4] and the zero trace condition[5] amongst others.

There have been numerous attempts aimed at understanding the pattern of the fermion masses and mixings by introducing Abelian and non-Abelian flavor symmetries some of which lead to texture zeros in the fermion mass matrices. Furthermore, as discussed earlier, it is not possible to fully reconstruct the neutrino mass
matrix solely from the results of presently feasible experiments and the introduction of texture zeros is an extra ingredient aimed at reducing the number of free parameters. However, some sets of these texture zeros can be obtained by suitable weak basis transformations and have no physical meaning as such. However, a large class of sets of leptonic texture zeros considered in the literature imply the vanishing of certain \( CP \)-odd weak basis invariants and one can, thus, recognize a lepton flavor model in which the texture zeros are not explicitly present but which corresponds to a particular texture structure in a certain weak basis. The presence of texture zeros, in general, leads to a decrease in the number of independent \( CP \) violating phases. A particular texture zero structure gives rise to definite relationships between different \( CP \) violating phases\cite{2}. Such exact relations in closed form were obtained in Ref.\cite{6}. Correlations between Dirac and Majorana \( CP \) violating phases for a particular texture zero scheme were studied in detail in Ref.\cite{2}. It is, therefore, important to examine the interrelationships between the \( CP \)-odd weak basis invariants which are required to vanish as a necessary and sufficient condition for \( CP \) conservation.

4.2 Weak Basis Invariants from the Neutrino Mass Matrix

The texture zeros are not weak basis (WB) invariants\cite{7}. This means that a given set of texture zeros which arise in a certain WB may not be present at all or may appear in different entries in another WB. A large class of sets of leptonic texture zeros considered in the literature imply the vanishing of certain \( CP \)-odd weak-basis invariants\cite{7}. Thus, one can recognize a lepton mass model in which the texture zeros are not explicitly present and which corresponds to a particular texture scheme in a certain WB. The relevance of \( CP \)-odd WB invariants in the analysis of the texture zero Ansätze is due to the fact that texture zeros lead to a decrease in the number of the independent \( CP \) violating phases. A minimum number of \( CP \)-odd WB invariants can be found which will all vanish for the \( CP \) invariant mass matrices as a necessary
and sufficient condition\[8\].

A necessary and sufficient condition for low energy CP invariance in the leptonic sector is that the following three WB invariants are identically zero\[9\]:

\[
I_1 = \Im \text{Det}[H_\nu, H_l], \quad (4.1)
\]

\[
I_2 = \Im \text{Tr}[H_l M_\nu M^*_\nu H_l^* M^*_\nu], \quad (4.2)
\]

\[
I_3 = \Im \text{Det}[M^*_\nu H_l M_\nu, H_l^*]. \quad (4.3)
\]

Here, $M_l$ and $M_\nu$ are the mass matrices for the charged leptons and the neutrinos, respectively, and $H_l = M^*_l M_l$ and $H_\nu = M^*_\nu M_\nu$. The invariant $I_1$ was first proposed by Jarlskog\[10\] as a rephasing invariant measure of Dirac-type CP violation in the quark sector. It, also, describes the CP violation in the leptonic sector and is sensitive to the Dirac-type CP violating phase. The invariants $I_2$ and $I_3$ were proposed by Branco, Lavoura and Rebelo\[11\] as the WB invariant measures of Majorana-type CP violation. The invariant $I_3$ was shown\[12\] to have the special feature of being sensitive to Majorana-type CP violating phase even in the limit of the exactly degenerate Majorana neutrinos.

The CP violation in the lepton number conserving (LNC) processes is contained in Jarlskog CP invariant $J$ which can be calculated from the WB invariant $I_1$ using the relation

\[
I_1 = -2J \left( m_2^2 - m_\mu^2 \right) \left( m_\mu^2 - m_\tau^2 \right) \left( m_\tau^2 - m_\mu^2 \right) \left( m_\tau^2 - m_\tau^2 \right) \left( m_\tau^2 - m_\tau^2 \right) \left( m_\tau^2 - m_\tau^2 \right). \quad (4.4)
\]

If the neutrino mass matrix $M_\nu$ is a complex symmetric matrix with eigenvalues $m_1$, $m_2$ and $m_3$ and the charged lepton mass matrix $M_l$ is diagonal and is given by

\[
M_l = \text{diag}\{m_e, m_\mu, m_\tau\}, \quad (4.5)
\]

then $I_1$ can be written as

\[
I_1 = 2 \left( m_\tau^2 - m_\mu^2 \right) \left( m_\mu^2 - m_\tau^2 \right) \left( m_\tau^2 - m_\tau^2 \right) \Im \left( M_{ee} A_{ee} + M_{\mu\mu} A_{\mu\mu} + M_{\tau\tau} A_{\tau\tau} \right) \quad (4.6)
\]
where the coefficients $A_{ee}$, $A_{\mu\mu}$ and $A_{\tau\tau}$ are given by

$$A_{ee} = M_{\mu\tau} M_{\mu\mu}^* M_{\mu\tau}^* \left( |M_{\mu\mu}|^2 - |M_{\mu\tau}|^2 - |M_{e\mu}|^2 + |M_{e\tau}|^2 \right) + M_{\mu\mu} M_{\mu\mu}^* \left( |M_{ee}|^2 - |M_{e\mu}|^2 \right) + M_{\mu\mu}^* M_{\mu\tau}^* M_{\mu\mu}^2,$$  

(4.7)

$$A_{\mu\mu} = M_{e\tau} M_{\mu\mu}^* M_{\mu\tau}^* \left( |M_{\tau\tau}|^2 - |M_{e\tau}|^2 - |M_{\mu\tau}|^2 + |M_{\mu\mu}|^2 \right) + M_{\tau\tau} M_{\mu\mu}^* \left( |M_{\mu\mu}|^2 - |M_{\mu\tau}|^2 \right) + M_{\tau\tau}^* M_{\tau\tau}^2 M_{\mu\mu}^2,$$  

(4.8)

and

$$A_{\tau\tau} = M_{e\mu} M_{\tau\mu}^* M_{\mu\mu}^* \left( |M_{ee}|^2 - |M_{\mu\mu}|^2 - |M_{e\tau}|^2 + |M_{\mu\tau}|^2 \right) + M_{ee} M_{\tau\tau}^* \left( |M_{\mu\mu}|^2 - |M_{\mu\tau}|^2 \right) + M_{ee}^* M_{\tau\tau}^2 M_{\mu\mu}^2,$$  

(4.9)

Therefore, the Jarlskog CP invariant measure $J$ is given by

$$J = \Im \left( \frac{M_{ee} A_{ee} + M_{\mu\mu} A_{\mu\mu} + M_{\tau\tau} A_{\tau\tau}}{(m_1^2 - m_2^2)(m_2^2 - m_3^2)(m_3^2 - m_1^2)} \right).$$  

(4.10)

This relation can be used to calculate $J$ from the mass matrices directly for any lepton mass model rotated to the WB in which $M_1$ is diagonal.

The $CP$ violation in lepton number violating (LNV) processes can be calculated from the WB invariants $I_2$ and $I_3$ which have been given below:

$$I_2 = \Im \left[ (M_{ee} M_{\mu\mu} M_{\mu\mu} \left( m_\mu^2 - m_\tau^2 \right))^2 + M_{\mu\mu} M_{\mu\tau}^2 M_{\tau\tau} (m_\mu^2 - m_\tau^2)^2 + M_{\tau\tau} M_{\tau\tau}^2 M_{ee} (m_\tau^2 - m_\mu^2)^2 + 2M_{ee} M_{\mu\mu} M_{\mu\tau} M_{\tau\tau} (m_\mu^2 - m_\tau^2) (m_\tau^2 - m_\mu^2) + 2M_{\mu\mu} M_{\mu\tau}^* M_{\tau\tau} M_{ee} (m_\mu^2 - m_\tau^2) (m_\tau^2 - m_\mu^2) + 2M_{\tau\tau} M_{\tau\tau}^* M_{ee} (m_\tau^2 - m_\mu^2) (m_\tau^2 - m_\mu^2) \right].$$  

(4.11)
\begin{equation}
I_3 = 2 \left( m^2_e - m^2_\mu \right) \left( m^2_\mu - m^2_\tau \right) \left( m^2_\tau - m^2_e \right) \otimes \left( m^2_e M_{ee} B_{ee} + m^2_\mu M_{\mu\mu} B_{\mu\mu} + m^2_\tau M_{\tau\tau} B_{\tau\tau} \right)
\end{equation}

(4.12)

where the coefficients $B_{ee}$, $B_{\mu\mu}$ and $B_{\tau\tau}$ are given by

\begin{align}
B_{ee} &= M_{\mu\tau} M^{\ast}_{\mu\tau} \left( m^4_\mu |M_{\mu\mu}|^2 - m^4_\tau |M_{\tau\tau}|^2 - m^2_\mu m^2_\tau |M_{\mu\tau}|^2 + m^2_\mu m^2_\tau |M_{\tau\mu}|^2 \right) \\
&\quad + m^2_\mu M_{\mu\mu} M^{\ast}_{\mu\mu} \left( m^2_\mu |M_{\mu\tau}|^2 - m^2_\tau |M_{\tau\mu}|^2 \right) \\
&\quad + m^2_\mu m^2_\tau M^{\ast}_{\mu\mu} M^{\ast\ast}{\mu\mu} M_{\mu\tau}^2 (M_{\tau\mu}^2),
\end{align}

(4.13)

\begin{align}
B_{\mu\mu} &= M_{\tau\mu} M^{\ast}_{\tau\mu} \left( m^4_\mu |M_{\tau\mu}|^2 - m^4_\tau |M_{\mu\tau}|^2 - m^2_\tau m^2_\mu |M_{\mu\mu}|^2 + m^2_\tau m^2_\mu |M_{\tau\mu}|^2 \right) \\
&\quad + m^2_\tau M_{\tau\mu} M^{\ast}_{\tau\mu} \left( m^2_\mu |M_{\mu\mu}|^2 - m^2_\tau |M_{\tau\mu}|^2 \right) \\
&\quad + m^2_\mu m^2_\tau M^{\ast}_{\mu\mu} M^{\ast\ast}{\mu\mu} M_{\tau\mu}^2 (M_{\mu\tau}^2),
\end{align}

(4.14)

and

\begin{align}
B_{\tau\tau} &= M_{\mu\tau} M^{\ast}_{\mu\tau} \left( m^4_\mu |M_{\mu\mu}|^2 - m^4_\tau |M_{\tau\tau}|^2 - m^2_\mu m^2_\tau |M_{\mu\tau}|^2 + m^2_\mu m^2_\tau |M_{\tau\mu}|^2 \right) \\
&\quad + m^2_\mu M_{\mu\mu} M^{\ast}_{\mu\mu} \left( m^2_\mu |M_{\mu\tau}|^2 - m^2_\tau |M_{\tau\mu}|^2 \right) \\
&\quad + m^2_\mu m^2_\tau M^{\ast}_{\mu\mu} M^{\ast\ast}{\mu\mu} M_{\mu\tau}^2 (M_{\tau\mu}^2).
\end{align}

(4.15)

From the above expressions for the WB invariants $I_1$, $I_2$ and $I_3$, one can immediately conclude that

1. A sufficient condition for CP conservation in LNC and LNV processes is that all the diagonal entries in the neutrino mass matrix vanish. Therefore, the symmetric neutrino mass matrices having Zee-type structure[5] in the diagonal charged lepton basis are CP conserving.

2. A sufficient condition for CP conservation in LNC processes is that any three independent entries in the neutrino mass matrix vanish. Therefore, the symmetric neutrino mass matrices having three texture zeros in the flavor basis cannot give CP violation in LNC processes. Such a mass matrix will have three independent phases and all of them can be rephased away by field redefinitions.
However, some of the symmetric neutrino mass matrices having three texture zeros in the flavor basis can give $CP$ violation in LNV processes since $I_2$ can be non-zero. The neutrino mass matrices having three zeros in a row or a column will have non-zero $I_2$ and, hence, will exhibit Majorana-type $CP$ violation with one physical phase.

3. The neutrino mass matrix with $\mu - \tau$ symmetry satisfying the relations

$$M_{e\mu} = \pm M_{e\tau} \text{ and } M_{\mu\mu} = M_{\tau\tau}$$

(4.16)

cannot give $CP$ violation in LNC processes because $\text{Det} [H_\nu, H_i]$ vanishes if $\mu - \tau$ symmetry is present. However, unlike the LNC processes, the neutrino mass matrix with $\mu - \tau$ symmetry does not conserve $CP$ in LNV processes since $I_2$ and $I_3$ do not vanish, in general, for the neutrino mass matrix with $\mu - \tau$ symmetry.

### 4.3 Implications for Texture Zeros

The seven allowed textures of the neutrino mass matrices with Frampton, Glashow and Marfatia (FGM) texture zero structure have been summarized in Table (4.1).

<table>
<thead>
<tr>
<th>Type</th>
<th>Constraining Eqns.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$M_{ee} = 0, M_{e\mu} = 0$</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$M_{ee} = 0, M_{e\tau} = 0$</td>
</tr>
<tr>
<td>$B_1$</td>
<td>$M_{e\tau} = 0, M_{\mu\mu} = 0$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>$M_{e\mu} = 0, M_{\tau\tau} = 0$</td>
</tr>
<tr>
<td>$B_3$</td>
<td>$M_{e\mu} = 0, M_{\mu\mu} = 0$</td>
</tr>
<tr>
<td>$B_4$</td>
<td>$M_{e\tau} = 0, M_{\tau\tau} = 0$</td>
</tr>
<tr>
<td>$C$</td>
<td>$M_{\mu\mu} = 0, M_{\tau\tau} = 0$</td>
</tr>
</tbody>
</table>

Table 4.1: *Allowed two texture zero mass matrices.*
### 4.3.1 Class A

For neutrino mass matrices of type $A_1$

\[
I_1 = 2|M_{\tau\tau}|^2 \left( m_\tau^2 - m_\mu^2 \right) \left( m_\mu^2 - m_\nu^2 \right) \Im \left( M_{\mu\mu}M_{\tau\tau}M_{\mu\tau}^* \right),
\]

and

\[
I_2 = \left( m_\mu^2 - m_\tau^2 \right)^2 \Re \left( M_{\mu\mu}M_{\tau\tau}M_{\mu\tau}^* \right)
\]

The WB invariants for class $A_2$ can be obtained by interchanging the $\mu$ and $\tau$ indices in the above relations.

An important conclusion follows from the above equations. The $CP$ violation is brought about by the mismatch in the phases of $M_{\mu\mu}$ and $M_{\mu\nu}$, whereas the phase of the element $M_{\nu\nu}$ or $M_{\tau\tau}$ plays no role in the $CP$ violation. In other words, the phase of the element $M_{\nu\nu}$ or $M_{\tau\tau}$ of the neutrino mass matrix has no physical significance in class $A$ and it can be rephased away. However, a phase in the $\mu - \tau$ sector of the neutrino mass matrix is physical and cannot be rephased away. A rephasing transformation will only rotate the phase in the $\mu - \tau$ sector.

The neutrino mass matrices of class $A$ have four non-zero independent elements and $J$ is proportional to all of those four non-zero elements. Any of these four elements cannot vanish since this will create three texture zeros in the mass matrix making it incompatible with the existing neutrino data. Hence, a necessary and sufficient condition for $J = 0$ in a complex symmetric neutrino mass matrix is

\[
2 \arg (M_{\mu\tau}) = \arg (M_{\mu\mu}) + \arg (M_{\tau\tau}).
\]

The condition given in Eqn.(4.20) is equivalent to the condition that the neutrino mass matrix $M$ can be factorized as $PM^{(r)}P$ where $P$ is a diagonal phase matrix $\text{diag}\{e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}\}$ and $M^{(r)}$ is a real matrix whose elements are real and equal in magnitude to the corresponding elements of $M$. Therefore, the necessary and
sufficient condition for \( CP \) conservation is that the neutrino mass matrix \( M \) can be written as

\[
M = PM^{(c)}P.
\] (4.21)

This equation is trivially satisfied for a real mass matrix. However, if the neutrino mass matrix is a general complex symmetric matrix, then it is not necessary that it satisfies Eqns. (4.20) or (4.21) and, therefore, it will be \( CP \) violating, in general.

### 4.3.2 Class B

The neutrino mass matrices of class \( B \) can be divided into sub-classes on the basis of their \( CP \) behavior. The first sub-class consists of the neutrino mass matrices of type \( B_1 \) and \( B_2 \) for which \( I_2 \) is zero. In the second sub-class, we have the neutrino mass matrices of type \( B_3 \) and \( B_4 \) for which \( I_2 \) is non-zero.

**For neutrino mass matrices of type \( B_1 \)**

\[
I_1 = 2 \left( m_{e}^2 - m_{\mu}^2 \right) \left( m_{\mu}^2 - m_{\tau}^2 \right) \left( m_{\tau}^2 - m_{e}^2 \right) \Im \left( M_{ee}^* M_{ee}^* M_{ee}^* M_{ee}^* \right),
\] (4.22)

\[
I_2 = 0
\] (4.23)

and

\[
I_3 = 2m_{e}^2 m_{\mu}^2 m_{\tau}^2 \left( m_{e}^2 - m_{\mu}^2 \right) \left( m_{\mu}^2 - m_{\tau}^2 \right) \left( m_{\tau}^2 - m_{e}^2 \right) \Im \left( M_{ee}^* M_{ee}^* M_{ee}^* M_{ee}^* \right). \] (4.24)

The WB invariants for neutrino mass matrices of type \( B_2 \) can be obtained by interchanging the \( \mu \) and \( \tau \) indices in the above relations.

**For neutrino mass matrices of type \( B_3 \)**

\[
I_1 = 2 |M_{\mu\tau}|^2 \left( m_{e}^2 - m_{\mu}^2 \right) \left( m_{\mu}^2 - m_{\tau}^2 \right) \left( m_{\tau}^2 - m_{e}^2 \right) \Im \left( M_{ee} M_{ee}^* M_{ee}^* M_{ee}^* \right),
\] (4.25)

\[
I_2 = \left( m_{e}^2 - m_{\tau}^2 \right) \Im \left( M_{ee} M_{ee}^* M_{ee}^* \right) \] (4.26)

and

\[
I_3 = 2m_{e}^2 m_{\mu}^2 |M_{\mu\tau}|^2 \left( m_{e}^2 - m_{\mu}^2 \right) \left( m_{\mu}^2 - m_{\tau}^2 \right) \left( m_{\tau}^2 - m_{e}^2 \right) \Im \left( M_{ee} M_{ee}^* M_{ee}^* \right). \] (4.27)
The WB invariants for class $B_4$ can be obtained from the interchange of the $\mu$ and $\tau$ indices in the above relations.

The conditions for the $CP$ invariance of neutrino mass matrices of class $B$ have been summarized in Table (4.2). These conditions on the phases of the neutrino mass matrix can be viewed as the fine tunings required to have $CP$ invariance.

### 4.3.3 Class $C$

For neutrino mass matrices of type $C$

$$I_1 = 2 \left( |M_{\mu\mu}|^2 - |M_{\mu\tau}|^2 \right) \left( m_{\mu}^2 - m_{\mu}^2 \right) \left( m_{\tau}^2 - m_{\tau}^2 \right) \Im \left( M_{e\mu}^* M_{e\tau}^* M_{e\mu} M_{e\tau} \right), \quad (4.28)$$

$$I_2 = 2 \left( m_{e}^2 - m_{\mu}^2 \right) \left( m_{\tau}^2 - m_{\tau}^2 \right) \Im \left( M_{e\mu}^* M_{e\tau}^* M_{e\mu} M_{e\tau} \right) \quad (4.29)$$

and

$$I_3 = 2m_{e}^4 \left( |M_{e\mu}|^2 - |M_{e\tau}|^2 \right) \left( m_{\mu}^2 - m_{\mu}^2 \right) \left( m_{\tau}^2 - m_{\tau}^2 \right) \left( m_{\tau}^2 - m_{e}^2 \right) \Im \left( M_{e\mu}^* M_{e\tau}^* M_{e\mu} M_{e\tau} \right) \quad (4.30)$$

The neutrino mass matrix of class $C$ will be $CP$ invariant if the phases of the mass matrix are fine tuned to satisfy the condition

$$\arg (M_{ee}) + \arg (M_{e\mu}) = \arg (M_{e\mu}) + \arg (M_{e\tau}). \quad (4.31)$$

The $CP$ structure of class $C$ is much different from the $CP$ violating structure of classes $A$ and $B$. In class $C$, the quantity $I_1$ depends upon the difference between absolute squares of two matrix elements $M_{e\mu}$ and $M_{e\tau}$ unlike other classes. So, $I_1$ will vanish if $M_{e\mu} = M_{e\tau}$. Since, $M_{\mu\mu} = M_{\tau\tau}$ in class $C$, this special case is $\mu - \tau$ symmetric. Hence, the neutrino mass matrices of class $C$ having $\mu - \tau$ symmetry can not exhibit $CP$ violation in LNC processes. However, they can exhibit $CP$ violation in the LNV processes since $I_2$ and $I_3$ can be non-zero. In this special case, the neutrino
Table 4.2: Conditions for CP invariance for the neutrino mass matrices of class A, B and C.

<table>
<thead>
<tr>
<th>Type</th>
<th>CP invariance condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$2 \arg (M_{\mu\tau}) = \arg (M_{\mu\mu}) + \arg (M_{\tau\tau})$</td>
</tr>
<tr>
<td>$B_1$</td>
<td>$\arg (M_{ee}) + 2 \arg (M_{\mu\tau}) = \arg (M_{\tau\tau}) + 2 \arg (M_{e\mu})$</td>
</tr>
<tr>
<td>$B_3$</td>
<td>$2 \arg (M_{e\tau}) = \arg (M_{ee}) + \arg (M_{\tau\tau})$</td>
</tr>
<tr>
<td>$C$</td>
<td>$\arg (M_{ee}) + \arg (M_{\mu\tau}) = \arg (M_{e\mu}) + \arg (M_{e\tau})$</td>
</tr>
</tbody>
</table>

mass matrix contains only a Majorana phase and there is no Dirac phase. This special case is important in the sense that one can tell whether the physical phase in the mass matrix gives rise to CP violation in LNC or LNV processes.

The three WB invariants for neutrino mass matrices are related with one another in each class of neutrino mass matrices with FGM textures. The relations have been summarized in Table (4.3). Therefore, $I_1$, $I_2$ and $I_3$ are related with each other and only one of them is independent. The two Majorana phases contributing to CP violation in LNV processes will manifest in $I_2$ and $I_3$ for neutrino mass matrices of types $A_1$, $A_2$, $B_3$, $B_4$ and $C$. However, they will depend only on the single invariant $I_3$ for classes $B_1$ and $B_2$. The Dirac phase contributing to CP violation in LNC processes depends upon $I_1$. Hence, the three CP violating phases are not independent. There is only one independent physical phase in the mass matrix and it contributes to both LNC and LNV processes. However, such a phase cannot be labelled either Dirac or Majorana unambiguously since it contributes to the CP violation in both LNC and LNV processes. Hence, the distinction between Dirac and Majorana phases can not be maintained in the presence of two texture zeros. This is contrary to the conclusions reported in Ref.[13] where it has been claimed that the lone CP violating phase is of Dirac type and there are no Majorana phases.
<table>
<thead>
<tr>
<th>Type</th>
<th>$\frac{I_A}{I_1}$</th>
<th>$\frac{I_A}{I_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$\frac{1}{2</td>
<td>M_{\mu\nu}</td>
</tr>
<tr>
<td>$A_2$</td>
<td>$\frac{1}{2</td>
<td>M_{\mu\nu}</td>
</tr>
<tr>
<td>$B_1$</td>
<td>0</td>
<td>$m_\tau^2 m_\mu^2 m_\tau^2$</td>
</tr>
<tr>
<td>$B_2$</td>
<td>0</td>
<td>$m_\tau^2 m_\mu^2 m_\tau^2$</td>
</tr>
<tr>
<td>$B_3$</td>
<td>$\frac{1}{2</td>
<td>M_{\mu\nu}</td>
</tr>
<tr>
<td>$B_4$</td>
<td>$\frac{1}{2</td>
<td>M_{\mu\nu}</td>
</tr>
<tr>
<td>$C$</td>
<td>$\frac{1}{(m_2^2-m_1^2)(M_{\mu\nu}^2-M_{\mu\tau}^2)}$</td>
<td>$m_\tau^2 \frac{M_{\mu\nu}^2-M_{\mu\tau}^2}{M_{\mu\nu}^2-M_{\mu\tau}^2}$</td>
</tr>
</tbody>
</table>

Table 4.3: Ratios $\frac{I_A}{I_1}$ and $\frac{I_A}{I_1}$ for class A, B and C neutrino mass matrices.

### 4.4 Conclusions

The CP-odd WB invariants both for LNC and LNV processes, in a basis in which charged lepton mass matrix is diagonal, must vanish for CP invariance. A neutrino mass matrix with three or more zeros gives CP invariance in LNC processes. However, it can give CP violation through the WB invariant $I_2$ in LNV processes if the three zeros are situated along a row of the mass matrix. The neutrino mass matrices with two texture zeros can, in general, be CP violating if they are complex and if their phases are not fine tuned. The neutrino mass matrices of class C in FGM scheme having $\mu - \tau$ symmetry contain no Dirac phase and show CP violation in LNV processes only. In general, all the seven classes of the neutrino mass matrices in FGM texture zero scheme have only one physical phase which gives rise to CP violation in both LNC and LNV processes and, hence, cannot be labelled as either Dirac or Majorana. The physical distinction between a Dirac and a Majorana phase cannot be maintained in the FGM texture zero scheme [14].
Bibliography


