Chapter 2

Cyclic Edge Extensions
Self-Centered graphs

2.1 Introduction

Some graphs have the property that each vertex in the graph $G$ is a central vertex. The center of a graph is defined to be a certain set of vertices, all of which are “close” to the remaining vertices of the graph. A graph $G$ is self-centered if every vertex is in the center. Any disconnected graph is self-centered. Buckley [5] determined the extremal sizes of a connected self-centered graph having $m$ vertices and radius $r$. Various results and generalizations on self-centered graphs proved by Buckley [5], Schoone et al. [18], Janakiraman [10], [11], Li Hao and Lai Zaikang [7] can be seen. This chapter mainly deals with the concept of reduction of eccentricity by some
constant for all the vertices of the graph considered. To reduce the eccentricity for all the vertices we must consider such a graph which is self-centered, and a minimal one too. Thus in general we consider the minimal self-centered graphs viz., cycles. We have reduced the eccentricity of the entire class of cycles by one, two and three. To reduce eccentricity of all vertices greater than 3, would be a tough job since the process of edge addition could not be generalized for higher values. Thus we have developed a program which in general gives the reduced eccentricity for all the vertices and for all lengths of the cycle considered.

The idea behind this chapter works on the paper developed by Janakiraman et al.[10] where the authors say that given any graph, it can be transformed to a self centered graph by adding edges using the algorithm proposed by them. They have developed an algorithm for constructing self-centered graphs from trees and self centered graphs from a given connected graph. The authors have adopted the method of Breadth Search First (BFS) to develop the algorithm.

Harary [9], introduced the concept of changing and unchanging of a graphical invariant $i$, with interest in determining those for which $i(G - v) = i(G)$ and $i(G - v) \neq i(G)$ for all vertices $v$ of
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$G$, $i(G - e) = i(G)$ and $i(G - e) \neq i(G')$ for all edges $e$ of $G$ and $i(G + e) = i(G)$ and $i(G + e) \neq i(G)$ for all edges $e$ of $\bar{G}$, the complement of $G$. The problem posed by O. Vacek [19] still remains open till date, that seeks for a characterization of addition of edges for invariance of radius of a graph.

These concepts have been studied quite well for several invariants by Dutton et al. [6], Brigham et al. [3], [4], Harary [8], [9], Lawson et al. [13], Medidi [14], Walikar et al. [20], [21], [22], [23] and Akram [1], [2]. Usually, these kind of studies reflect the variation of a parameter due to vertex or edge removal or edge addition, which find their applications in network analysis as they measure the results of link or equipment failure or network enhancement.

The concept of edge extension for graphs and digraphs was introduced by Akram [1]. This was earlier introduced in terms of concept of reducibility number for posts in lattices theory. The concept introduced by Akram was analogous to this one. Akram studied reducibility of graphs and digraphs and the characterization of reducibility number for some classes of graphs like connected graphs, regular graphs, eulerian graphs etc. He introduced the concept of vertex extension in [1]. Later he introduced the concept of con-
tractability number of graphs in [2].

Akram [1] studied the addition of edges by introducing the concepts of edge extension set of graphs, edge extensible class of graphs and the edge extensibility number of a graph. He defined a non-empty set $S$ of edges as edge-extension set, such that every edge in $S$ joins two non-adjacent vertices in $G$. And $G + S$, the graph after adding $S$ to it is called the edge extension graph.

**Definition 2.1.1.** [1] Let $\tau$ be a class of graphs satisfying certain property. Then $\tau$ is called edge extensible class, if for every graph $G \in \tau$, $G$ is complete, or there exists an extension edge $e$ such that $G + e \in \tau$.

**Definition 2.1.2.** [1] Let $G$ be a non-trivial simple graph (not complete). The simple graph obtained from $G$ by adding a non-empty set of edges $S$ such that every edge in $S$ joins two non-adjacent vertices in $G$ is called edge extension graph, and is denoted by $G + S$, $S$ is called the edge extension set. In particular, if $S$ consists of a single element $e$, then $e$ is called the extension edge, and the graph is denoted by $G + e$.

We can see that the graph $G + S$ has the vertex set and the edge set as follows. $V(G + S) = V(G)$ and $E(G + S) = E(G) \cup S$. 
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Definition 2.1.3. [1] Let $\tau$ be a class of graphs with certain property and $G \in \tau$ be non-trivial. The edge extensibility number of $G$ with respect to $\tau$ is the smallest positive integer $m$, if exists, such that there exists an edge extension set $S$ of cardinality $'m'$ in such a way, the graph $G + S \in \tau$. We write $m = \text{ext}_{\tau}(G)$. If such a number does not exist for $G$, then we say that the corresponding edge extensibility number is $\infty$.

Examples for the above mentioned definitions are:
1. The class of connected graphs is edge extensible class.
2. The class of regular graphs is not edge extensible class.
3. The class of eulerian graphs is not edge extensible class.

Here in the below example, we have considered a path, which is a connected graph and then added two edges from the complement, which thus shows the resultant graph is still connected. Example of a connected graph, which is edge extensible class is shown in Figure 2.1.

![Connectedness in a path - $P_4$ extension](image)

Figure 2.1: Connectedness in a path - $P_4$ extension

It is easy to note that the class of connected graphs is edge exten-
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sible class, but not regular graphs. Here in the graph of Figure 2.2, consider a cycle of any length, adding edges from the complement, the resultant graph still remains connected.

![Figure 2.2: Example of connected graph, which belongs to edge extensible class](image)

Similarly, this also forms an example for the second class of graphs. That is, a cycle which is a regular graph, becomes an irregular graph upon addition of edges from the complement.

![Figure 2.3: Edge addition to an Eulerian graph](image)
Example for the third class of graphs specified above is by considering an eulerian graph, and thus by adding an edge from the complement results in a non-eulerian graph.

On the other hand a tree with respect to the class of trees $\tau$ has extensibility number $\infty$ as addition of an edge never remains a tree. Instead of adding a single edge if we add a set of edges then the chances are more of getting the required property satisfied. In this direction the extensibility number comes handy. Akram in [1] and [2] proved that the extension number for the class of eulerian graphs is $\frac{m}{2}$ if and only if $m$ is even and is equal to $m$ is and only if $m$ is odd. Also, we can find results related to extension number of class of graphs viz., regular graphs and digraphs.

Here in Figure 2.4, we have considered a cycle on 8 vertices and we have added one more cycle to the alternate vertices, thus again forming an Eulerian graph. In the next section self centered graphs are considered as the collection $\tau$ and obtain edge extensibility number of some self centered graphs. Since cycles are the minimum sized self centered graphs, we find out the $ext_\tau(G) = k$. It is clear that $k \neq 1$ for $\tau = C_m$. Hence, it is interesting and challenging to find the $ext_\tau(C_m)$. 
2.2 Edge Extension for Cycles

In this section edge extensions for cycles are considered. As stated above it is clear that \( k \neq 1 \), with \( k = ext_\tau(C'_m) \) where \( \tau \) is the class of self-centered graphs. That is the class of cycles. When an edge(set) is added to a cycle one can see that it does not remain a self-centered graph. Our aim is to add the edge/edge set to the graph(cycle) by retaining the property of self-centeredness.

So the first result discusses the minimum number of edges required to be added to a cycle such that the resulting graph is a self-centered graph. And denote \( k_i = ext_\tau(C'_m) \) where \( i \) denotes the amount by which the radius(diameter) of the cycle \( C'_m \) is reduced.

Lemma 2.2.1. Let \( C'_m \) be the class of cycles of length (order) \( m \). Let
\( \tau \) be the set of self-centered graphs. Then,

\[
k_1 = \text{ext}_\tau(C_m) = \begin{cases} 
3, & \text{if } m \text{ is even and } m \geq 12, \\
4, & \text{if } m \text{ is odd and } m \geq 11, \\
2, & \text{if } m = 4, 8, 9, 10, \\
5, & \text{if } m = 5, \\
3, & \text{if } m = 6, 7.
\end{cases}
\]

**Proof.** Label the vertices of the cycle as \( u_1, u_2, u_3, \ldots, u_m \).

For \( m \geq 12 \), with \( m \) even, join \( u_1u_3, u_1u_{m-1} \) and \( u_2u_4 \) so that the resultant graph is a self-centered graph of radius \((m/2)-1\).

For \( m \geq 11 \), with \( m \) odd, join \( u_1u_3, u_1u_{m-1}, u_2u_4 \) and the fourth edge from \( u_{m+1/2} \) to \( u_{m+3/2} \), to get a self-centered graph with reduced diameter, by one.

For \( C_4 \) and \( C_5 \) it is clear that \( \text{ext}_\tau(C_4) = 2 \) and \( \text{ext}_\tau(C_5) = 5 \) respec-
tively, as they result into $K_4$ and $K_5$ on addition of edges. 
For $C_6$, $ext_\tau(C_6) = 3$ and for $C_7$, $ext_\tau(C_7) = 3$.
And for $C_8, C_9, C_{10}$, the extension number is 2 as we can add the edges $u_1u_3$ and $u_1u_{m-1}$. Hence the proof. □

**Lemma 2.2.2.** Let $C_m$ denote the class of cycles of length (order) $m$, with $m \geq 11$. Then,

$$k_2 = ext_\tau(C_m) = \begin{cases} 
4, & \text{if } m \text{ is even and } m \geq 10, \\
5, & \text{if } m \text{ is odd and } m \geq 11, \\
7, & \text{when } m = 7, \\
4, & \text{when } m = 8, \\
9, & \text{when } m = 9.
\end{cases}$$

**Proof.** Label the vertices of the cycle as $u_1, u_2, u_3, \ldots, u_m$.
For $m \geq 10$, with $m$ even, join the edges $u_1u_3, u_1u_{m-1}, u_3u_6$ and $u_5u_8$, to get a graph $G$ whose diameter is two less than that of $C_m$.
For $m \geq 11$, with $m$ odd, join the edges $u_1u_3, u_1u_{m-1}, u_mu_{m-3}, u_3u_6$ and $u_5u_8$.
For $m = 7$, adding one more $C_7$ to the existing one reduces the diameter by 2. For $m = 8$, adding 4 edges to each of their eccentric nodes reduces the diameter by 2. For $m = 9$, adding 9 more edges(one more $C_9$) reduces the diameter by 2. Hence the proof. □
Lemma 2.2.3. Let $C_m$ denote the class of cycles with $m \geq 14$. Then,

$$k_3 = \text{ext}_\tau(C_m) = \begin{cases} 
6 & \text{when } m \text{ is even, } (m \geq 14), \\
8 & \text{when } m \text{ is odd, } (m \geq 15).
\end{cases}$$

Proof. Label the vertices of the cycle $C_m$ as $u_1, u_2, u_3, \ldots, u_m$.

If $m$ is even, with $m \geq 14$, then join the edges $u_1 u_3, u_3 u_6, u_5 u_8$,
$u_1 u_{m-1}, u_{m-1} u_{m-4}$ and $u_{m-3} u_{m-6}$, so that the resultant graph has its diameter reduced by $3$.

If $m$ is odd, with $m \geq 15$, then join $u_1 u_3, u_2 u_4, u_3 u_6, u_5 u_8, u_1 u_{m-1}, u_m u_{m-3}, u_{m-2} u_{m-5}$ and $u_{m-4} u_{m-7}$ to obtain the required result. Hence the proof.

To obtain a complete graph one can add edges to the cycle to reduce the radius/diameter. The above results give reduction of eccentricity of each vertex to be reduced by one, two or three. As generalization seems difficult, to find extension number for cycles, to be in class of self-centered graphs, the above results help to measure the number of edges to be added.

In the next results I do not add edge set, instead add paths, but the resultant graph is a self-centered graph. As known any disconnected graph is a self centered graph. For a connected graph to be self-centered, the radius must be equal to the diameter. This way of approach is motivated by Buckley [5] in which he considered graphs under edge operation. These graphs were constructed by Buckley using the following definitions.

**Definition 2.2.1.** [1] For integers $a$ and $b$ a cycle is said to be an $(a$
mod b)-cycle if its length is a mod b.

Akira Saito et al. [15], [16], and [17] had considered properties of cycles of particular length and thus proved that every graph of minimum degree at least three contains a (0 mod 3) cycle. We combine both these approaches in the coming results to prove that is resultant graph still remains self-centered.

Definition 2.2.2. [5] If \( a \geq 4 \), then \( C_a * sP_b \) consists of the graph formed from \( C_a \) by joining two vertices \( u \) and \( v \) of \( C_a \) at distance \( b \) from one another by \( s \) additional paths of length \( b \) (\( b > 1 \)).

Definition 2.2.3. [5] If \( a \geq 4 \) and \( 1 < d < b \), then \( C_a * sP_b * P_d \) is the graph formed from \( C_a * sP_b \) by joining the vertex \( u \) to a vertex \( w \) in \( C_a \) at distance \( d \) from \( u \) by an additional path of length \( d \).

![Figure 2.8: Concatenation of Paths to a cycle](image)
Lemma 2.2.4. Let $C_m$ be a cycle of odd length, where $m \geq 7$. Then a path of length $m$ concatenated with two eccentric vertices in a cycle results in a self-centered graph.

Proof. Case(i) - Consider a cycle $C_{7+4k}$, which is of length - 7 modulo 4 where $k = 0, 1, 2, 3, \ldots$. Let $P_{2m}$ be a path, where $m \in \mathbb{Z}$ and $m \geq 2$.

On concatenating one end vertex of the path to any vertex say $u$

![Figure 2.9: Length 7 modulo 4k](image)

of $C_m$ and the other end vertex (of the path) to the eccentric vertex of $u$, say $v$, results in a self-centered graph. The length of the path varies depending on the radius of the cycle. Hence the path length $2m$ is one less or two less than the radius of the cycle.

Similarly, we can prove for the below two cases, by concatenating the specified path with a vertex and its eccentric vertex.
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Case(ii) - Consider a cycle $C_{9+4k}$, which is of length 9 modulo 4 where $k = 0, 1, 2, 3, \ldots$. Let $P_{2m+1}$ be a path, where $m \in \mathbb{Z}$ and $m \geq 2$.

Case(iii) - Consider a cycle $C_{4+2k}$, which is of length 4 modulo 2 where $k = 0, 1, 2, 3, \ldots$. Let $P_m$ be a path, where $m \in \mathbb{Z}$ and $m = \text{rad}(C_m - 1)$ or $m = \text{rad}(C_m - 2)$ or $m = \text{rad}(C_m)$.

Hence the proof.
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Remark 2.2.1. A path of length 2 can be added to $C_7$ and a path of
length 3 can be added to $C_{11}$ to obtain a self centered graph. A path
of length 2 and 3 can be added to $C_9$ and a path of length 4 can be
added to $C_{13}$ to obtain a self centered graph.

In the next part the number of iterations is found, required for a
cycle and a path to become a complete graph, using the concept of
powers and the following three algorithms give the same.

Algorithm 2.2.1. In this algorithm number of iterations required for
cycle to be a complete graph, can be obtained.
Let $C_m$ be a cycle of length $m$.

STEP 1: Input the cycle length $m$.

STEP 2: Find the eccentricity of the given cycle by using $e =
(m/2)$ if $m$ is even or $e = \lfloor m/2 \rfloor$.

STEP 3: If $e > 1$, then increase the iteration. Next we start
adding edges for the next iteration such that the distance between
any two vertices is less than or equal to the iterated power. This is
done for all the vertices of the cycle.

STEP 4: Again checking for the eccentricity of all the vertices.
If $e_{\text{new}} = 1$, then goto STEP 6 else goto STEP 3.

STEP 5: Print the iteration number.

STEP 6: Print the number of iterations to get a complete graph.

STEP 7: STOP.
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Illustration:
Consider the cycle on 4 vertices as shown in Figure 2.12. The eccentricity(e) of this graph is 2. Since \( e > 1 \), start adding edges to vertices at distance two for the first iteration. This results in, \( u_1 \) being adjacent to \( u_3 \) and \( u_2 \) adjacent to \( u_4 \). This results in a complete graph as shown in Figure 2.12.

The algorithm 2.2.1 is illustrated in the following graph.

![Diagram showing the process of adding edges to form a complete graph from a cycle]

Figure 2.12: One iteration for \( C_4 \) to become a complete graph

Algorithm 2.2.2. In this algorithm the number of iterations required for path to be a complete graph, is found.

Let \( P_m \) be a path on \( m + 1 \) vertices.

\[ \text{STEP } - 1 : \text{ Input the path length.} \]

\[ \text{STEP } - 2 : \text{ Find the radius (minimum eccentricity) and the diameter (maximum eccentricity). The radius is denoted by } e_{\text{min}} = a \text{ and the diameter is denoted by } e_{\text{max}} = b. \]

\[ \text{STEP } - 3 : \text{ If } e_{\text{min}} > 1 \text{ and } e_{\text{max}} > 1, \text{ increase the power else goto} \]
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Step-7

STEP - 4: Add edges to the consecutive vertex whose length is \( \leq \) iterated power.

STEP - 5: Find the new eccentricities, \( e_{new\text{min}} = \frac{a}{2} \) and \( e_{new\text{max}} = \frac{b}{2} \)

STEP - 6: If \( \frac{a}{2} = 1 \) and \( \frac{b}{2} = 1 \), goto STEP 7 else goto STEP 3.

STEP - 7: Print the number of iterations.

STEP - 7: Stop.

Illustration:

Consider a path \( P_m \) on 4 vertices. Find the minimum and the maximum eccentricity of the path and denote it by \( \text{min } ecc = a \) and \( \text{max } ecc = b \). Here in the graph considered \( a = 2 \) and \( b = 3 \). Since \( a > 1 \) and \( b > 1 \), iterate the first power by adding edges to vertices at distance two. Continue iteration until the eccentricities of the remaining vertices equal one. Thus, this results in a complete graph \( K_4 \) as shown in Figure 2.13.

The algorithm 2.2.2 is illustrated in the following graph.

![Figure 2.13: Path to become a complete graph in two iterations](image)

Algorithm 2.2.3. In this algorithm we find the iteration number for
cycles to be self-centered by adding edges.

Let $C_m$ be a cycle of length $m$.

**STEP - 1**: Input the cycle length $m$.

**STEP - 2**: Find the eccentricity of the given cycle by using $e = (m/2)$ if $p$ is even or $e = \lfloor m/2 \rfloor$, if $m$ is odd.

**STEP - 3**: Find the edges of the cycle. Since length = $m$, edges = $m$.

**STEP - 4**: Input the eccentricity reduction value.

**STEP - 5**: From the first vertex start adding edges one by one such that the eccentricity of the other vertices remain the same as inputed by the user.

**STEP - 6**: Perform **STEP 5** until all the vertices have the same eccentricity.

**STEP - 7**: Check the eccentricity of the $2^{nd}$, $3^{rd}$ and so on up to the $p^{th}$ vertex. Check the eccentricity of all the vertices. If they are same goto **STEP - 8** or goto **STEP 5**.

**STEP - 8**: Output list of available vertices.

**STEP - 9**: Output list of edges added and the number of edges added.

**STEP - 10**: Output list of Invalid vertices where the edge addition is not possible.

**STEP - 11**: STOP.
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Illustration:
Consider the cycle on 15 vertices as shown in Figure 2.14. Find the eccentricity of the cycle, \( e = 7 \). Input the eccentricity reduction value. Start adding edges from vertices until required eccentricity is obtained for all the vertices. Repeat this until the resultant graph is self-centered as shown in Figure 2.14.

The algorithm 2.2.3 is illustrated in the following example where \( C_{15} \) requires 8 edges to reduce eccentricity by three numbers and the same cycle requires 3 edges to reduce the eccentricity by four numbers.

![Figure 2.14: Cycle to become self-centered upon addition of edges](image-url)
2.3 Conclusions

Characterization on the number of edges to be added to a general graph seems to be difficult at this point of time. Hence, particular cases give insight about the edge additions to retain a particular property. The results discussed in this chapter deal with additions done to a cycle to retain its self-centeredness. We are happy that the findings of this chapter were presented at One day UGC sponsored International conference and workshop in Graph Theory held at Mysore University.

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