Chapter - III

3.1 Finite bending of an incompressible anisotropic circular and rectangular blocks into shells

In the preceding chapter we discussed some problems on compressible and anisotropic materials and a number of such problems have been solved completely by adopting Seth's or Saint-Venant's form of strain energy function. According to the recent investigations there have been a number of exact solutions obtained for incompressible bodies independent of the form of strain energy function. Rivlin $^{40}$, Green and Wilkes $^{18}$, Rivlin and Ericksen $^{46}$ obtained exact solutions in terms of strain energy function $W$ for a number of problems for incompressible bodies specially for the problems of bending of rectangular blocks into right circular cylindrical shells when the body is isotropic. Green and Wilkes $^{18}$, Rivlin and Ericksen $^{46}$, considered the same problem for a orthotropic and transversely isotropic material. Adkins and Green $^{1}$ discussed the case of general anisotropy by restricting the form of strain energy function. In sections 3.3 and 3.4 problems of circular and rectangular blocks bent into hyperboloidal and hyperbolic shells are considered for anisotropic material. The problem of an initially curved rectangular block bent into a right circular cylindrical shell has been solved by A.E. Green, J.E. Adkins and R.T. Shield $^{19}$. Recently B. Kesava Rao $^{24}$ has obtained a solution for the problem of finite bending of an initially curved incompressible isotropic circular block into a part of a spherical shell. The problem
of flexure of a initially curved cuboid has been solved by A.K. Deshmukh. In section 3.5, we consider the problem of a initially curved circular block bent into a part of spherical shell when the material is anisotropic and incompressible. The particular case of a incompressible anisotropic circular block bent into a spherical shell has been deduced from above by a limiting process. The problem of finite bending of an incompressible anisotropic composite rectangular block into a cylindrical shell has been solved in section 3.6. In section 3.7 finite bending of incompressible composite block is deduced as a particular case when the material is isotropic.

3.2 Notation and formulae

Following the notation of Green and Zerna and Green and Adkins the formulae required in this chapter are given below:

Let \( x_1 \) be the initial, \( y_1 \) the final and \( \theta_1 \) the curvilinear coordinates of the strained body. The covariant and contravariant base vectors \( \xi_1^i, \xi_i \), the metric tensors \( \xi_{ij}, \xi^{ij} \) of the curvilinear system \( \theta_1 \) at a point \( P_0 \) of an unstrained body are given by

\[
\begin{align*}
\xi_i &= \frac{\partial x}{\partial \theta_i} \quad i = 1, 2, 3 \quad \xi = |\xi_{ij}| \\
\xi_{ij} &= \xi_i \cdot \xi_j \quad \xi^{ij} = \xi^i \cdot \xi^j \quad \xi^{ij} \xi_{jk} = \delta_i^j 
\end{align*}
\]

(3.2.1)

where \( x(\theta_1, \theta_2, \theta_3) \) is the position vector of \( P_0 \), the comma denotes partial differentiation and \( \delta_i^j \) is the kronecker delta.
In the strained state let the point \( P_o \) move to the point \( P \) at a time \( t \).

If the displacement vector \( \overrightarrow{P_oP} = d(\theta_1, \theta_2, \theta_3) \) the covariant and contravariant base vectors, the metric tensors at \( P \) of the curvilinear system \( \theta_i \) are given by

\[
G_i = r_i^j + d_i^j ; \quad G^i = G^i_j \delta_j \quad ; \quad G = |G_{ij}| \\
G_{ij} = G_i . G_j ; \quad G^{ij} = G^i . G^j ; \quad G^{ij} \delta_{jk} = \delta^i_k
\]

(3.2.2)

When the unstrained body is homogeneous, the strain energy function \( W \), measured per unit volume of the unstrained body is a function of strain components \( \gamma_{ij} \) so that

\[
W = W(\gamma_{ij})
\]

(3.2.3)

where

\[
2 \gamma_{ij} = G_{ij} - \delta_{ij}
\]

(3.2.4)

when the coordinates \( \theta_i \) are taken to coincide with rectangular cartesian coordinates which define points in the undeformed body, we denote \( \gamma_{ij} \) by \(\epsilon_{ij} \), so that

\[
\epsilon_{rs} = \frac{\partial \theta_i}{\partial x^r} \frac{\partial \theta_j}{\partial x^s} \gamma_{ij} = \frac{1}{2} \left( \frac{\partial y^m}{\partial x^r} \frac{\partial y^n}{\partial x^s} - \delta_{rs} \right)
\]

(3.2.5)

When the material is incompressible, the equation \( I_3 = \frac{G}{\epsilon} = 1 \) holds at all points of the body and \( T_{ij} \) is then given by

\[
2T_{ij} = \left( \frac{\partial \omega}{\partial \epsilon_{rs}} + \frac{\partial \omega}{\partial \epsilon_{sr}} \right) \frac{\partial \theta_i}{\partial x^r} \frac{\partial \theta_j}{\partial x^s} + 2PG_{ij}
\]

(3.2.6)

where \( P \) is a scalar invariant function of \( \theta_i \).
If the material is isotropic and incompressible, the strain energy $W$ is given by

$$W = W(I_1, I_2, I_3)$$  \hspace{1cm} (3.2.7)

where $I_1 = g^{rs} G_{rs}$, $I_2 = g_{rs} G^{rs}$, $I_3 = \frac{G}{\varepsilon}$  \hspace{1cm} (3.2.8)

In this case, the stress tensor $T^{ij}$ is

$$T^{ij} = \phi \varepsilon^{ij} + \psi B^{ij} + \rho G^{ij}$$  \hspace{1cm} (3.2.9)

where

$$\sqrt{I_3} \phi = 2 \frac{\partial W}{\partial I_1}, \sqrt{I_3} \psi = 2 \frac{\partial W}{\partial I_2}, P = 2 \sqrt{I_3} \left( \frac{\partial W}{\partial I_3} \right)$$  \hspace{1cm} (3.2.10)

$$B^{ij} = \varepsilon^{ij} I_1 - \varepsilon^{ir} \varepsilon^{js} G_{rs} = \varepsilon^{im} \varepsilon^{jm} G_{mm}/\varepsilon$$  \hspace{1cm} (3.2.11)

and as usual $\varepsilon$ is $+1$ or $-1$ according as $i, r, m$ is an even or odd permutation of $1, 2, 3$ and equal to zero otherwise.

In particular, if the material is isotropic and incompressible we have $I_3 = 1$, $W = W(I_1, I_2)$ and $P$ is a scalar invariant function of $\theta_i$ only. The physical components of stress $\sigma_{ij}$ are given by

$$\sigma_{ij} = \left( \frac{G_{ij}}{G_{jj}} \right)^{1/2} G_{jm} T^{im}$$  \hspace{1cm} (3.2.12)

The equation of equilibrium can be expressed in the form

$$T^{ij} + \rho F_i = 0$$  \hspace{1cm} (3.2.13)

or

$$T^{ij} + \sum_r T^{rj} + \sum_r T^{ir} + \rho F_i = 0$$  \hspace{1cm} (3.2.14)

where the vertical bar denotes the covariant differentiation.
and comma, the partial differentiation with respect to the strained body and \( F^i \) are the contravariant components of the body force vector referred to the covariant base vectors \( G_1 \).

The Christoffel symbol \( \Gamma^r_{ij} \) is given by

\[
2 \Gamma^r_{ij} = G^r_s (G_{si,j} + G_{sj,i} - G_{ij,s}) \tag{3.2.15}
\]

### 3.3 Finite bending of an incompressible anisotropic circular block into a hyperboloidal shell

Suppose, that in the undeformed state of the body it is a circular block bounded by the planes \( x_3 = a_1, x_3 = a_2 \) \((a_2 > a_1)\) and the cylinder \( x_1^2 + x_2^2 = a^2 \). The block is then bent symmetrically about \( x_3 \)-axis into a part of a hyperboloidal shell, whose inner and outer boundaries are the hyperboloids of revolution obtained by revolving the confocal hyperbolas

\[
x_3 = c \cosh \xi \cos \gamma, \quad x_1 = c \sinh \xi \sin \gamma, \quad \eta = \eta_1 \tag{i = 1, 2}
\]

about the \( x_3 \)-axis respectively and the edge \( \xi = \alpha \). Let the \( y_1 \)-axes coincide with the \( x_1 \)-axes and the curvilinear coordinates \( G_1 \) in the deformed state be a system of orthogonal curvilinear coordinates \((\xi, \eta, \varphi)\) where \( \varphi \) is the angle between \( y_1, y_3 \) plane and the plane through a point in space and \( y_3 \)-axis. Then

\[
y_1 = c \sinh \xi \sin \gamma \cos \varphi, \quad y_2 = c \sinh \xi \sin \gamma \sin \varphi
\]

\[
y_3 = c \cosh \xi \cos \gamma \tag{3.3.2}
\]
Since the deformation is symmetric about $\gamma$-axis, we see that

1) the planes $x_3 = \text{constant}$ in the undeformed state become the hyperboloidal surfaces $\eta = \text{constant}$ in the deformed state

2) the curves $x_1^2 + x_2^2 = \text{constant}$ in the undeformed state become the circles $\xi = \text{constant}$ in the deformed state

3) $\tan^{-1} \frac{x_2}{x_1} = \varphi$ .

These imply that

$$x_3 = f(\eta), \quad (x_1^2 + x_2^2)^{\frac{1}{2}} = f(\xi), \quad x_1 \tan \varphi = x_2$$

(3.3.3)

which give

$$x_1 = f(\xi) \cos \varphi, \quad x_2 = f(\xi) \sin \varphi, \quad x_3 = f(\eta)$$

(3.3.4)

The metric tensors for the strained and unstrained state of the body are given by

$$G_{ij} = \begin{bmatrix}
c^2(\cosh^2 \xi - \cos^2 \eta)
& 0
& 0

0
& c^2(\cosh^2 \xi - \cos^2 \eta)
& 0

0
& 0
& c^2 \sinh^2 \xi \sin^2 \eta
\end{bmatrix}$$

(3.3.5)

$$E_{ij} = \begin{bmatrix}
f' \cdot f'
& 0
& 0

0
& f' \cdot f'
& 0

0
& 0
& f' \cdot f'
\end{bmatrix}$$

(3.3.6)

where

$$f' = \frac{df}{d\gamma} \quad \text{and} \quad F' = \frac{df}{d\xi}$$
the condition \( I_3 = 1 \) gives

\[
\frac{e^3 (\cosh^2 \gamma - \cos^2 \gamma)}{f^t} = \frac{FF^t}{\sinh^2 \gamma \sin \eta} \tag{3.3.7}
\]

An approximate solution is obtained by considering \( \xi \) to be a small quantity. Then (3.3.7) becomes

\[
\frac{e^3 \sin^3 \eta}{f^t} = \frac{FF^t}{f} = A \tag{3.3.8}
\]

where \( A \) is an arbitrary constant.

Then

\[
x_3 = f(\gamma) = \frac{e^3}{A} \int \sin^3 \eta d \eta + B \tag{3.3.9}
\]

\[
= \frac{e^3}{A} \left( \frac{\cos^3 \gamma}{3} - \cos \gamma \right) + B \tag{3.3.10}
\]

and

\[
x_1^2 + x_2^2 = F^2(\xi) = A \xi^2 + D \tag{3.3.11}
\]

where \( B \) and \( D \) are constants.

As the internal and external boundaries of the hyperboloidal shell are given by \( \gamma = \gamma_i \) \( (i = 1, 2) \) respectively, which are initially the planes \( x_3 = a_1, \ x_3 = a_2 \), (3.3.10) gives

\[
a_1 = \frac{e^3}{A} \left( \frac{\cos^3 \gamma_1}{3} - \cos \gamma_1 \right) + B \tag{3.3.12}
\]

Solving these equations, we get

\[
A = \frac{e^3}{(a_2 - a_1)} \frac{\cos^3 \gamma_2 - \cos^3 \gamma_1}{3} - \left( \cos \gamma_2 - \cos \gamma_1 \right) \tag{3.3.13}
\]

\[
a_2 \left( \frac{\cos^3 \gamma_1}{3} - \cos \gamma_1 \right) - a_1 \left( \frac{\cos^3 \gamma_2}{3} - \cos \gamma_2 \right) \tag{3.3.14}
\]

\[
B = \frac{\cos^3 \gamma_1 - \cos^3 \gamma_2}{3} - \left( \cos \gamma_1 - \cos \gamma_2 \right)
\]
Since the bending is symmetric about $x_3$-axis, we must have $x_1^2 + x_2^2 = 0$, when $\xi = 0$. Then (3.3.11) becomes

$$x_1^2 + x_2^2 = F^2(\xi) = A\xi^2 \quad (3.3.15)$$

After substituting (3.3.10) and (3.3.15) in (3.3.5) and (3.3.6) we get

$$\gamma_{ij} = \begin{bmatrix}
(c \sin \gamma)^2 & 0 & 0 \\
0 & (c \sin \gamma)^2 & 0 \\
0 & 0 & \xi^2(c \sin \gamma)^2 \\
\end{bmatrix}$$

$$\epsilon_{ij} = \begin{bmatrix}
\frac{1}{(c \sin \gamma)^2} & 0 & 0 \\
0 & \frac{1}{(c \sin \gamma)^2} & 0 \\
0 & 0 & \frac{1}{\xi^2(c \sin \gamma)^2} \\
\end{bmatrix} \quad (3.3.16)$$

$$\delta_{ij} = \begin{bmatrix}
\frac{(c \sin \gamma)^6}{A^2} & 0 & 0 \\
0 & A & 0 \\
0 & 0 & A\xi^2 \\
\end{bmatrix}$$

$$\eta_{ij} = \begin{bmatrix}
\frac{A^2}{(c \sin \gamma)^6} & 0 & 0 \\
0 & 1/A & 0 \\
0 & 0 & \frac{1}{A\xi^2} \\
\end{bmatrix} \quad (3.3.17)$$
From (3.2.2), (3.2.4), (3.2.5), (3.2.8), (3.3.16) and (3.3.17) we obtain

\[ I_1 = \frac{A^2}{c^4 \sin^4 \gamma} + \frac{2c^2 \sin^2 \gamma}{A} \quad (3.3.18) \]

\[ I_2 = \frac{c^4 \sin^4 \gamma}{A^2} + \frac{2A}{c^2 \sin^2 \gamma} \quad (3.3.19) \]

and

\[ \frac{\partial \mathbf{F'}}{\partial \mathbf{e}} = \begin{bmatrix} F' \cos \varphi & 0 & -F \sin \varphi \\ F' \sin \varphi & 0 & F \cos \varphi \\ 0 & f' & 0 \end{bmatrix} \quad (3.3.20) \]

\[ \frac{\partial e^F}{\partial x^F} = \begin{bmatrix} \cos \varphi & \sin \varphi & 0 \\ \frac{1}{f'} & 0 & 0 \\ -\frac{\sin \varphi}{F} & -\frac{\cos \varphi}{F} & 0 \end{bmatrix} \quad (3.3.21) \]

From (3.2.4), (3.2.5) and (3.3.16) to (3.3.21) the strain tensors \( \gamma_{ij} \) and \( e_{ij} \) are given by

\[ 2 \gamma_{11} = \left( c^2 \sin^2 \gamma - \frac{c^6 \sin^6 \gamma}{A^2} \right) \]

\[ 2 \gamma_{22} = \left( c^2 \sin^2 \gamma - A \right) \]

\[ 2 \gamma_{33} = \frac{2}{f'} (c^2 \sin^2 \gamma - A) \]

\[ \gamma_{12} = \gamma_{13} = \gamma_{23} = 0 \quad (3.3.22) \]
\[ 2 \varepsilon_{11} = \frac{e^2 \sin^2 \gamma}{\Lambda} - 1 \]
\[ 2 \varepsilon_{22} = \frac{e^2 \sin^2 \gamma}{\Lambda} - 1 \]
\[ 2 \varepsilon_{33} = \frac{2}{4 \sin \gamma} - 1 \]
\[ \varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23} = 0 \quad (3.3.23) \]

Substituting (3.3.16) in (3.2.15) the non-zero christoffel symbols are given by
\[ \Gamma^1_{11} = - \cot \gamma \]
\[ \Gamma^1_{12} = \Gamma^1_{32} = \Gamma^2_{22} = \cot \gamma, \quad \Gamma^2_{33} = - \frac{2}{5} \cot \gamma \quad (3.3.24) \]

Also, from (3.2.6) and from (3.3.23) we obtain the components \( T^{ij} \) of the stress tensor
\[ T^{11} = \frac{1}{A} \left( \cos \phi \frac{\partial W}{\partial \varepsilon_{11}} + \sin \phi \frac{\partial W}{\partial \varepsilon_{22}} \right) + \frac{P}{e^2 \sin^2 \gamma} \]
\[ T^{22} = \frac{A^2}{c^6 \sin \gamma} \frac{\partial W}{\partial \varepsilon_{33}} + \frac{P}{c^2 \sin^2 \gamma} \]
\[ T^{33} = \frac{1}{A} \left( \sin^2 \phi \frac{\partial W}{\partial \varepsilon_{11}} + \cos^2 \phi \frac{\partial W}{\partial \varepsilon_{22}} \right) + \frac{P}{c^2 \frac{2}{5} \sin^2 \gamma} \quad (3.3.25) \]

The equation of equilibrium (3.2.13) with the help of (3.3.24) reduces to
\[ \frac{\partial T^{22}}{\partial \gamma} = - \cot \gamma \left( 4T^{22} - T^{11} - \frac{2}{5} T^{33} \right) \quad (3.3.26) \]

The other two equations being satisfied identically.
Substituting (3.3.25) in (3.3.26) we get

\[
\frac{\partial T^{22}}{\partial \eta} = \frac{\text{cot} \eta}{c^2 \sin^2 \eta} \left( \frac{c^2 \sin^2 \eta}{A} \frac{\partial W}{\partial \epsilon_{11}} + \frac{c^2 \sin^2 \eta}{A} \frac{\partial W}{\partial \epsilon_{22}} - \frac{2A^2}{c^4 \sin^4 \theta} \frac{\partial W}{\partial \epsilon_{33}} \right) \\
- \frac{2 \text{cot} \eta}{c^2 \sin^2 \eta} \left( \frac{A^2}{4} \frac{\partial W}{\partial \epsilon_{33}} \right) + \frac{p}{(3.3.27)}
\]

Now

\[
\frac{\partial W}{\partial \eta} = \frac{\partial W}{\partial \epsilon_{11}} \frac{\partial \epsilon_{11}}{\partial \eta} + \frac{\partial W}{\partial \epsilon_{22}} \frac{\partial \epsilon_{22}}{\partial \eta} + \frac{\partial W}{\partial \epsilon_{33}} \frac{\partial \epsilon_{33}}{\partial \eta}
\]

\[
= \text{cot} \eta \left( \frac{c^2 \sin^2 \eta}{A} \frac{\partial W}{\partial \epsilon_{11}} + \frac{c^2 \sin^2 \eta}{A} \frac{\partial W}{\partial \epsilon_{22}} \right) \\
- \frac{2A^2}{c^4 \sin^4 \theta} \frac{\partial W}{\partial \epsilon_{33}} \right) \quad (3.3.28)
\]

Substituting (3.3.28) in (3.3.27) we obtain,

\[
\frac{\partial T^{22}}{\partial \eta} = \frac{1}{c^2 \sin^2 \eta} \frac{\partial W}{\partial \eta} - \frac{2 \text{cot} \eta}{c^2 \sin^2 \eta} \left( \frac{A^2}{4} \frac{\partial W}{\partial \epsilon_{33}} \right) + \frac{p}{(3.3.29)}
\]

Again from (3.3.25) and (3.3.29) we get

\[
\frac{\partial P}{\partial \eta} = \frac{\partial W}{\partial \eta} + \frac{4A^2 \cos \eta}{c^4 \sin^5 \theta} \frac{\partial W}{\partial \epsilon_{33}} \quad (3.3.30)
\]

which gives after integration

\[
P = W + \bar{W} - \frac{A^2}{c^4 \sin^4 \theta} \frac{\partial W}{\partial \epsilon_{33}} \quad (3.3.31)
\]

where \( \bar{W} \) is a constant. From (3.3.25), (3.3.31) and (3.2.12) we get the physical components of stress
\[
\sigma_{11} = V + \dot{w}_0 + \frac{c^2 \sin^2 \gamma}{A} \left( \cos^2 \phi \frac{\partial V}{\partial \varepsilon_{11}} + \sin^2 \phi \frac{\partial \dot{w}_0}{\partial \varepsilon_{22}} \right) - \frac{A}{4} \frac{\partial^2 V}{\partial \varepsilon_{33}^2}
\]

\[
\sigma_{22} = V + \dot{w}_0
\]

\[
\sigma_{33} = V + \dot{w}_0 + \frac{c^2 \sin^2 \gamma}{A} \left( \sin^2 \phi \frac{\partial V}{\partial \varepsilon_{11}} + \cos^2 \phi \frac{\partial \dot{w}_0}{\partial \varepsilon_{22}} \right) - \frac{A}{4} \frac{\partial^2 V}{\partial \varepsilon_{33}^2}
\]

\[\text{(3.3.32)}\]

**Boundary conditions**

If the inner surface of the shell \( \gamma = \gamma_1 \) is free from traction, we must have \( \sigma_{22} = 0 \) when \( \gamma = \gamma_1 \), which on substitution in (3.3.32) gives \( \dot{w}_0 = -w(\gamma_1) \). On the outer surface of the shell \( \gamma = \gamma_2 \), we have to apply a normal traction \( R \) given by

\[R = \sigma_{22}(\gamma_2) = w(\gamma_2) - w(\gamma_1)
\]

\[\text{(3.3.33)}\]

On the edge \( \phi = \alpha \), the distribution of tractions between \( \phi \) and \( \phi + d\phi \), give rise to a force \( F_1 \) and a couple of moment \( M \) about the origin given by

\[F_1 = \alpha \int_{\gamma_1}^{\gamma_2} \sigma_{11}(c^2 \sin^2 \gamma) d\gamma
\]

\[\text{(3.3.34)}\]

\[M = \alpha \int_{\gamma_1}^{\gamma_2} \sigma_{11}(c^2 \sin^2 \gamma) \cos \gamma d\gamma
\]

\[\text{(3.3.35)}\]

Substituting (3.3.32) in these equations, we get

\[F_1 = \alpha \int_{\gamma_1}^{\gamma_2} \left\{ (V + \dot{w}_0) c^2 \sin^2 \gamma - \frac{A^2}{c^2 \sin^2 \gamma} \frac{\partial V}{\partial \varepsilon_{33}} \right. \]

\[+ \frac{c^4 \sin^4 \gamma}{A} \left( \cos^2 \phi \frac{\partial V}{\partial \varepsilon_{11}} + \sin^2 \phi \frac{\partial \dot{w}_0}{\partial \varepsilon_{22}} \right) \}

\[d\gamma
\]

\[\text{(3.3.36)}\]
which can be integrated when the strain energy function \( W \) is specified.

3.4 **Finite bending of an anisotropic incompressible rectangular block into a hyperbolic shell**

Let an initially plane rectangular block bounded by the planes \( x_1 = a_1, x_1 = a_2 \) \((a_2 > a_1)\), \( x_2 = \pm b \) and \( x_3 = \pm d \) be bent symmetrically about \( x_1 \)-axis into a part of an hyperbolic shell whose inner and outer boundaries are the confocal hyperbolas,

\[
\begin{align*}
x_1 &= c \cosh \xi \cos \eta, & x_2 &= c \sinh \xi \sin \eta, \\
\gamma &= \eta_1 \quad (i = 1, 2)
\end{align*}
\]

respectively.

Let \( y_1 \)-axes coincide with \( x_1 \)-axes and the curvilinear coordinates \( \Theta \) in the deformed state be a system of orthogonal coordinates \((\xi, \eta, z)\), so that \( y_1 = c \cosh \xi \cos \eta, \quad y_2 = c \sinh \xi \sin \eta, \quad y_3 = z). \n\]

Since the deformation is symmetric about the \( x_1 \)-axis, we see that

i) the planes \( x_1 = \text{constant} \) in the undeformed state become \( \gamma = \text{constant} \) in the deformed state

ii) the planes \( x_2 = \text{constant} \) in the undeformed state become \( \xi = \text{constant} \) in the deformed state,

iii) the planes \( x_3 = \text{constant} \) in the undeformed state become \( z = \text{constant} \) in the deformed state.
Therefore, \( x_1 = f(\gamma) \), \( x_2 = F(\bar{\gamma}) \), \( x_3 = \frac{v}{\lambda} \) \hspace{1cm} (3.4.3)

The metric tensors for the strained and unstrained states of the body are given by

\[
G_{ij} = \begin{bmatrix}
  c^2(\cosh^2 \bar{\gamma} - \cos^2 \gamma) & 0 & 0 \\
  0 & c^2(\cosh^2 \bar{\gamma} - \cos^2 \gamma) & 0 \\
  0 & 0 & 1
\end{bmatrix}
\hspace{1cm} (3.4.4)
\]

\[
G_{ij} = \begin{bmatrix}
  f'^2 & 0 & 0 \\
  0 & f'^2 & 0 \\
  0 & 0 & \frac{1}{\lambda^2}
\end{bmatrix}
\hspace{1cm} (3.4.5)
\]

where \( f' = \frac{df}{d\gamma} \) and \( f' = \frac{df}{d\bar{\gamma}} \).

The condition of incompressibility \( I_3 = \frac{G}{E} = 1 \) gives

\[
c^2(\cosh^2 \bar{\gamma} - \cos^2 \gamma) = \frac{f'}{\lambda}
\hspace{1cm} (3.4.6)
\]

As in 3.3 an approximate solution is obtained by considering \( \bar{\gamma} \) to be a small quantity.

Then (3.4.6) gives

\[
c^2 \sin^2 \gamma = \frac{f'}{\lambda} = K
\hspace{1cm} (3.4.7)
\]

where \( K \) is an arbitrary constant.

Then

\[
x_1 = f(\gamma) = \frac{c^2}{K} \int \sin^2 \gamma \, d\gamma + B = \frac{c^2}{4K} (2\gamma - \sin 2\gamma) + B
\hspace{1cm} (3.4.8)
\]

and

\[
x_2 = F(\bar{\gamma}) = K \bar{\gamma} + D
\hspace{1cm} (3.4.9)
\]
where $B$ and $D$ are constants.

As the internal and external boundaries of hyperbolic shell are given by $\eta_1 = \eta_1$ ($i = 1, 2$) respectively which were initially the planes $x_1 = a_1$ and $x_1 = a_2$, (3.4.8) gives

$$a_1 = \frac{c^2}{4k}(2\eta_1 - \sin 2\eta_1) + B, \quad i = 1, 2 \quad (3.4.10)$$

Solving these equations, we get

$$\frac{1}{k} = \frac{4(a_2 - a_1)}{c^2 \left( (\sin 2\eta_1 - \sin 2\eta_2) + (2\eta_2 - 2\eta_1) \right)} \quad (3.4.11)$$

$$B = \frac{a_1(2\eta_2 - \sin 2\eta_2) + a_2(\sin 2\eta_1 - 2\eta_1)}{(\sin 2\eta_2 - \sin 2\eta_1) + 2(\eta_1 - \eta_2)} \quad (3.4.12)$$

Since the bending is symmetric about $x_1$-axis, we have $x_2 = 0$ when $\xi = 0$. Then (3.4.9) becomes

$$x_2 = f(\xi) = k\lambda \xi \quad (3.4.13)$$

From (3.4.8), (3.4.13), (3.4.4) and (3.4.5) and noting that $\xi$ is small, we get

$$c_{ij} = \begin{bmatrix} c^2\sin^2\eta & 0 & 0 \\ 0 & c^2\sin^2\eta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.4.14)$$

$$s_{ij} = \begin{bmatrix} \lambda k^2 & 0 & 0 \\ 0 & \frac{c^4\sin^4}{k^2} & 0 \\ 0 & 0 & \frac{1}{\lambda^2} \end{bmatrix} \quad (3.4.15)$$
\[
\frac{\partial x^r}{\partial s} = \begin{bmatrix}
0 & \frac{c^2 \sin^2 \eta}{K} & 0 \\
\lambda K & 0 & 0 \\
0 & 0 & \frac{1}{\lambda}
\end{bmatrix}
\] (3.4.16)

\[
\frac{\partial e^r}{\partial s} = \begin{bmatrix}
0 & \frac{1}{\lambda K} & 0 \\
\frac{-K}{c^2 \sin^2 \eta} & 0 & 0 \\
0 & 0 & \lambda
\end{bmatrix}
\] (3.4.17)

From (3.4.14), (3.4.15), (3.4.16) and (3.4.17), the strain tensors $\gamma_{ij}$ and the components of strain defined by (3.2.4), and (3.2.5) are given by

\[2\gamma_{11} = c^2 \sin^2 \eta \quad - \quad \frac{2K^2}{\lambda K^2}
\]
\[2\gamma_{22} = c^2 \sin^2 \eta \quad - \quad \frac{c^4 \sin^4 \eta}{K^2}
\]
\[2\gamma_{33} = (1 - \frac{1}{\lambda^2})
\]
\[\gamma_{12} = \gamma_{13} = \gamma_{23} = 0
\] (3.4.18)

\[2e_{11} = \frac{K^2}{c^2 \sin^2 \eta} - 1
\]
\[2e_{22} = \frac{c^2 \sin^2 \eta}{\lambda^2 K^2} - 1
\]
\[2e_{33} = \frac{2}{\lambda} - 1
\]
\[e_{12} = e_{23} = e_{31} = 0
\] (3.4.19)

Also, from (3.2.6) and from (3.4.14), (3.4.15) and (3.4.17)
we obtain for the components $T^{ij}$ of the stress tensor

$$
T^{11} = \frac{1}{\lambda^2 k^2} \frac{\partial W}{\partial \varepsilon_{22}} + \frac{P}{c^2 \sin^2 \eta} \\
T^{22} = \frac{K^2}{c^4 \sin^4 \eta} \frac{\partial W}{\partial \varepsilon_{11}} + \frac{P}{c^2 \sin^2 \eta} \\
T^{33} = \lambda \frac{2}{\partial W} \frac{\partial W}{\partial \varepsilon_{33}} + P
$$

(3.4.20)

where $P$ is a scalar invariant.

Substituting (3.4.14) in (3.2.15) the non-zero christoffel symbols are given by

$$
\begin{align*}
\Gamma^{1}_{12} &= \Gamma^{1}_{32} = \Gamma^{3}_{22} = \cot \eta, \\
\Gamma^{2}_{11} &= -\cot \eta
\end{align*}
$$

(3.4.21)

The equation of equilibrium with the help of (3.4.21) reduces to

$$
\frac{\partial T^{22}}{\partial \eta} + \cot \eta (3T^{22} - T^{11}) = 0
$$

(3.4.22)

and the other two equations being satisfied identically.

Substituting (3.4.20) in (3.4.22) we get

$$
\frac{\partial T^{22}}{\partial \eta} = \cot \eta \left( \frac{c^2 \sin^2 \eta}{\lambda^2 k^2} \frac{\partial W}{\partial \varepsilon_{22}} - \frac{K^2}{c^2 \sin^2 \eta} \frac{\partial W}{\partial \varepsilon_{11}} \right) \\
- \frac{2\cot \eta}{c^2 \sin^2 \eta} \left( \frac{K^2}{c^2 \sin^2 \eta} \frac{\partial W}{\partial \varepsilon_{33}} + P \right)
$$

(3.4.23)

Now

$$
\frac{\partial W}{\partial \varepsilon_{11}} = \frac{\partial W}{\partial \varepsilon_{11}} \frac{\partial \varepsilon_{11}}{\partial \eta} + \frac{\partial W}{\partial \varepsilon_{22}} \frac{\partial \varepsilon_{22}}{\partial \eta} + \frac{\partial W}{\partial \varepsilon_{33}} \frac{\partial \varepsilon_{33}}{\partial \eta}
$$

$$
= \cot \eta \left( \frac{c^2 \sin^2 \eta}{\lambda^2 k^2} \frac{\partial W}{\partial \varepsilon_{22}} - \frac{K^2}{c^2 \sin^2 \eta} \frac{\partial W}{\partial \varepsilon_{11}} \right)
$$

(3.4.24)
Substituting (3.4.24) in (3.4.23) we get
\[
\frac{\partial T}{\partial \eta} = \frac{1}{c^2 \sin^2 \eta} \frac{\partial V}{\partial \eta} - \frac{2 \cot \eta}{c^2 \sin^2 \eta} \frac{K^2}{c^2 \sin^2 \eta} \frac{\partial V}{\partial \epsilon_{11}} + P
\]  
(3.4.25)

From (3.4.20) and (3.4.25) we obtain
\[
\frac{\partial P}{\partial \eta} = \frac{\partial V}{\partial \eta} + \frac{2K^2 \cos \eta}{c^2 \sin^3 \eta} \frac{\partial V}{\partial \epsilon_{11}}
\]  
(3.4.26)

which after integration becomes
\[
P = \int V + W_0 - \frac{K^2}{c^2 \sin^2 \eta} \frac{\partial V}{\partial \epsilon_{11}}
\]  
(3.4.27)

where \(W_0\) is a constant. From (3.4.20), (3.4.27) and (3.2.12) we get the physical components of stress
\[
\sigma_{11} = \frac{c^2 \sin^2 \eta}{K^2} \frac{\partial V}{\partial \epsilon_{22}} - \frac{K^2}{c^2 \sin^2 \eta} \frac{\partial V}{\partial \epsilon_{11}} + V + W_0
\]
\[
\sigma_{22} = V + W_0
\]
\[
\sigma_{33} = \frac{2}{c^2 \sin^2 \eta} \frac{\partial V}{\partial \epsilon_{33}} - \frac{K^2}{c^2 \sin^2 \eta} \frac{\partial V}{\partial \epsilon_{11}} + V + W_0
\]
\[
\sigma_{12} = \sigma_{23} = \sigma_{31} = 0
\]  
(3.4.28)

**Boundary conditions**

If the inner boundary of the shell \(\eta = \eta_1\) is free from traction, we must have \(\sigma_{22} = 0\) when \(\eta = \eta_1\) which on substitution from (3.4.28) gives
\[
W_0 = -W(\eta_1)
\]  
(3.4.29)

On each of the faces \(\gamma = \pm \alpha\) there acts a resultant
normal force

\[ F_1 = 2 \lambda d_1 \int \sigma_{11} c \sin \eta \, d \eta \]

\[ = 2 \lambda d_1 \int (V + V_0 + \frac{c^2 \sin^2 \eta}{\lambda^2 k^2} \frac{\partial V}{\partial e_{22}} - \frac{K^2}{c^2 \sin^2 \eta} \frac{\partial V}{\partial e_{11}}) c \sin \eta \, d \eta \]

(3.4.30)

and a resultant couple

\[ M_1 = 2 \lambda d_1 \int \sigma_{11} c^2 \sin \eta \cos \eta \, d \eta \]

\[ = 2 \lambda d_1 \int (V + V_0 + \frac{c^2 \sin^2 \eta}{\lambda^2 k^2} \frac{\partial V}{\partial e_{22}} - \frac{K^2}{c^2 \sin^2 \eta} \frac{\partial V}{\partial e_{11}}) c^2 \sin \eta \cos \eta \, d \eta \]

(3.4.31)

Also the force \( F_2 \) required to keep the length of the block constant in the direction of the \( \nu_3 \)-axis and a couple of moment \( M_2 \) about the origin in the axial plane applied per unit area between \( \xi \) and \( \xi + d \xi \) are given by

\[ F_2 = \int \sigma_{33} c^2 \sin^2 \eta \, d \eta \]

\[ = \int (V + V_0 + \frac{\lambda}{\partial e_{33}} \frac{\partial V}{\partial e_{33}} - \frac{K^2}{c^2 \sin^2 \eta} \frac{\partial V}{\partial e_{11}}) c^2 \sin^2 \eta \, d \eta \]

(3.4.32)

\[ - M_2 = \int \sigma_{33} c^2 \sin^2 \eta (c \cos \eta) \, d \eta \]

\[ = \int (V + V_0 + \frac{\lambda}{\partial e_{33}} \frac{\partial V}{\partial e_{33}} - \frac{K^2}{c^2 \sin^2 \eta} \frac{\partial V}{\partial e_{11}}) c^3 \sin^2 \eta \cos \eta \, d \eta \]

(3.4.33)

which can be integrated when the strain energy \( W \) is specified.
3.5 Finite bending of an initially curved anisotropic incompressible circular block into a spherical shell

Suppose the body in the deformed state, referred to a system of spherical polar coordinates \((r, \theta, \varphi)\), be a part of a spherical shell bounded by spherical surfaces \(r = r_1\) and \(r = r_2\) and the cone \(\theta = \alpha\). Let the body in the undeformed state be also a part of the spherical shell, the surfaces corresponding to \(r = r_1^*\), \(r = r_2^*\) being concentric spherical surfaces of radii \(K_1\) and \(K_2\) respectively; the surface corresponding to \(\theta = \alpha^*\) being a cone with vertex at the centre of the concentric spherical surfaces.

Let the origin of \(X_1^*\) and \(Y_1^*\)-axes be separated by a distance \(K\) with \(X_3^*\) and \(Y_3^*\)-axes coinciding, and the curvilinear coordinates \(\theta_1^*\) in the deformed state be identified with the coordinate system \((r, \theta, \varphi)\) so that

\[
y_1 = r \sin \theta \cos \varphi, \quad y_2 = r \sin \theta \sin \varphi, \quad y_3 = r \cos \theta
\]  
(3.5.1)

\[
x_1 = f(r) \sin F(\theta) \cos \varphi, \quad x_2 = f(r) \sin F(\theta) \sin \varphi, \quad x_3 = f(r) \cos F(\theta) - K
\]  
(3.5.2)

Then the metric tensors for the strained and unstrained bodies are given by

\[
G_{ij} = \begin{bmatrix}
1 & 0 & 0 \\
0 & r^2 & 0 \\
0 & 0 & r^2 \sin^2 \theta
\end{bmatrix}
\]
\[
G^{ij} = \begin{bmatrix}
1 & 0 & 0 \\
0 & \frac{1}{r^2} & 0 \\
0 & 0 & \frac{1}{r^2 \sin^2 \theta}
\end{bmatrix}
\]

\[G = r^4 \sin^2 \theta \] (3.5.3)

\[
\epsilon_{ij} = \begin{bmatrix}
r^2 & 0 & 0 \\
0 & r^2 F^2 & 0 \\
0 & 0 & r^2 \sin^2 \theta
\end{bmatrix}
\]

\[
\epsilon^{ij} = \begin{bmatrix}
\frac{1}{r^2} & 0 & 0 \\
0 & \frac{1}{r^2 F^2} & 0 \\
0 & 0 & \frac{1}{r^2 \sin^2 \theta}
\end{bmatrix}
\]

\[\epsilon = r^2 F^2 r^4 \sin^2 F \] (3.5.4)

where \( f' = \frac{df}{dr} \) and \( F' = \frac{dF}{d\theta} \)

The condition of incompressibility \( \frac{G}{\epsilon} = 1 \) gives

\[
\frac{\sin \theta}{F' \sin F} = \frac{r^2 f^1}{r^2} = A
\]

(3.5.5)

where \( A \) is an arbitrary constant.

Solving this we get

\[ f^3 = Af^3 + B \] (3.5.6)

\[ A \cos F = \cos \theta - B_1 \] (3.5.7)
where \( B \) and \( B_1 \) are constants. As \( r_2 \) and \( r_1 \) \((r_2 > r_1)\) are the radii of the external and internal boundaries of the deformed spherical shell which are initially the spherical surfaces of radii \( K_1 = f(r_1) \) and \( K_2 = f(r_2) \), (3.5.6) gives

\[
\begin{align*}
K_1^3 &= Ar_1^3 + B \\
K_2^3 &= Ar_2^3 + B
\end{align*}
\]

(3.5.8)

As the point circle \( x_1^2 + x_2^2 = 0 \), which lies in the \( X_3 \)-axis in the undeformed state goes into the point on \( Y_3 \)-axis for which \( \theta \) is zero in the deformed state, we have from \( x_1^2 + x_2^2 = f^2 \sin^2 f(\theta) \),

\[
f(0) = 0
\]

(3.5.9)

From (3.5.7) and (3.5.9) we get

\[
B_1 = 1 - A
\]

(3.5.10)

Solving (3.5.8) we obtain

\[
A = \frac{K_1^3 - K_2^3}{r_1^3 - r_2^3}
\]

\[
B = \frac{K_1^3 r_2^3 - K_2^3 r_1^3}{r_2^3 - r_1^3}
\]

(3.5.11)

From (3.5.4), (3.5.6) and (3.5.7) we get

\[
\varepsilon_{ij} = \begin{bmatrix}
\frac{A^2}{h} & 0 & 0 \\
0 & \frac{x^2 h^2 Q}{A^2} & 0 \\
0 & 0 & \frac{h^2 x^2 \sin^2 \theta}{Q}
\end{bmatrix}
\]
\begin{equation}
\varepsilon_{ij} = \begin{bmatrix}
\frac{\Delta^4}{h^4} & 0 & 0 \\
0 & \frac{\Delta^2}{r^2 h^2 Q} & 0 \\
0 & 0 & \frac{c}{h^2 r^2 \sin^2 \theta}
\end{bmatrix}
\end{equation}

(3.5.12)

where \( h = \frac{r}{x} \) and \( Q = \frac{\sin^2 \theta}{\sin^2 F} \).

As in 3.3, we take \( \theta \) to be small so that

\begin{equation}
Q = \frac{\sin^2 \theta}{\sin^2 F} = \frac{\sin^2 \theta}{1 - \cos^2 F} = \frac{\Delta^2 \sin^2 \theta}{\Delta^2 (\cos \theta - 1 + \Delta)^2} = \frac{\Delta^2 (1 + \cos \theta)}{\cos \theta - 1 + 2 \Delta}
\end{equation}

Then

\begin{equation}
\varepsilon_{ij} = \begin{bmatrix}
\frac{\Delta^4}{h^4} & 0 & 0 \\
0 & \frac{h^2 x^2}{A} & 0 \\
0 & 0 & \frac{h^2 x^2 e^2}{A}
\end{bmatrix}
\end{equation}

\begin{equation}
\varepsilon_{ij} = \begin{bmatrix}
\frac{h^4}{\Delta^4} & 0 & 0 \\
0 & \frac{\Delta^2}{h^2 x^2} & 0 \\
0 & 0 & \frac{h^2 x^2 e^2}{\Delta^2}
\end{bmatrix}
\end{equation}

(3.5.14)
\[ g^{4j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 e^2} \end{bmatrix} \]  \hspace{1cm} (3.5.15)

From (3.5.2) we get
\[
\frac{\partial \mathbf{e}^s}{\partial e^s} = \begin{bmatrix} f' \sin F \cos \varphi & f \cos F \sin F' \cos \varphi & -f \sin F \sin \varphi \\ f' \sin F \sin \varphi & f \cos F \cos F' \sin \varphi & f \sin F \cos \varphi \\ f' \cos F & -f \sin F \sin \varphi & 0 \end{bmatrix} \]  \hspace{1cm} (3.5.16)

\[
\frac{\partial X^s}{\partial e^s} = f^2 f' \sin F \sin \varphi \]  \hspace{1cm} (3.5.17)

From (3.5.16) and (3.5.17) we obtain,
\[
\frac{\partial \mathbf{e}^s}{\partial x^s} = \begin{bmatrix} \frac{\sin F \cos \varphi}{f'} & \frac{\sin F \sin \varphi}{f'} & \frac{\cos F}{f'} \\ \frac{\cos F \cos \varphi}{f'} & \frac{\cos F \sin \varphi}{f' f'} & -\frac{\sin F}{f'} \\ \frac{\sin \varphi}{f \sin F} & \frac{\cos \varphi}{f \sin F} & 0 \end{bmatrix} \]  \hspace{1cm} (3.5.18)

From (3.2.4), (3.2.5), (3.5.7) and (3.5.13) to (3.5.18) the strain tensors \( \gamma_{ij} \) and \( e_{ij} \) are given by

\[
2 \gamma_{11} = 1 - \frac{A}{h^4}, \quad 2 \gamma_{22} = r^2 - \frac{h^2}{A}, \quad 2 \gamma_{33} = e^2 (r^2 - \frac{h^2}{A})
\]

\[
\gamma_{12} = \gamma_{23} = \gamma_{31} = 0 \]  \hspace{1cm} (3.5.19)
\[ 2 \varepsilon_{11} = A/h^2 - 1, \quad 2 \varepsilon_{22} = A/h^2 - 1, \quad 2 \varepsilon_{33} = h^4/A^2 - 1 \]

\[ \varepsilon_{12} = \varepsilon_{23} = \varepsilon_{31} = 0 \quad (3.5.20) \]

Also, from (3.2.6) and (3.5.20) we obtain the components of the stress tensor as

\[ T^{11} = \frac{h^4}{A^2} \frac{\partial W}{\partial \varepsilon_{33}} + P \quad (3.5.21) \]

\[ T^{22} = \frac{A}{h^2 r^2} (\cos^2 \varphi \frac{\partial W}{\partial \varepsilon_{11}} + \sin^2 \varphi \frac{\partial W}{\partial \varepsilon_{22}}) + \frac{P}{r^2} \quad (3.5.22) \]

\[ T^{33} = \frac{1}{\varphi} \left\{ \frac{A}{h^2 r^2} (\sin^2 \varphi \frac{\partial W}{\partial \varepsilon_{11}} + \cos^2 \varphi \frac{\partial W}{\partial \varepsilon_{22}}) + \frac{P}{r^2} \right\} \quad (3.5.23) \]

From (3.2.12) and (3.5.21) to (3.5.23) the physical components of stress are given by

\[ \sigma_{11} = \frac{h^4}{A^2} \frac{\partial W}{\partial \varepsilon_{33}} + P \quad (3.5.24) \]

\[ \sigma_{22} = \frac{A}{h^2} (\cos^2 \varphi \frac{\partial W}{\partial \varepsilon_{11}} + \sin^2 \varphi \frac{\partial W}{\partial \varepsilon_{22}}) + P \quad (3.5.25) \]

\[ \sigma_{33} = \frac{A}{h^2} (\sin^2 \varphi \frac{\partial W}{\partial \varepsilon_{11}} + \cos^2 \varphi \frac{\partial W}{\partial \varepsilon_{22}}) + P \quad (3.5.26) \]

\[ \sigma_{12} = \sigma_{23} = \sigma_{31} = 0 \quad (3.5.27) \]

Since the strain components \( \varepsilon_{rs} \) are purely functions of \( r \), the strain energy function \( W \) is also a function of \( r \) only and is independent of \( \theta \) and \( \varphi \). Hence the only equation of equilibrium to be satisfied reduces to

\[ \frac{\partial \varepsilon_{11}}{\partial r} + \frac{(2 \sigma_{11} - \sigma_{22} - \sigma_{33})}{r} = 0 \quad (3.5.28) \]
Substituting (3.5.24) to (3.5.26) in (3.5.28) we get

$$\frac{\partial \sigma_{11}}{\partial r} = \frac{1}{r} \left\{ \frac{A}{h^2} \left( \frac{\partial W}{\partial \sigma_{11}} + \frac{\partial W}{\partial \sigma_{22}} \right) - \frac{2h^4}{A^2} \frac{\partial W}{\partial \sigma_{33}} \right\}$$  \hspace{1cm} (3.5.29)

From (3.5.20) we have

$$\frac{\partial \sigma_{11}}{\partial r} = \frac{2A(h^3 - A)}{rh^5}$$  \hspace{1cm} (3.5.30)

$$\frac{\partial \sigma_{22}}{\partial r} = \frac{2A(h^3 - A)}{rh^5}$$  \hspace{1cm} (3.5.31)

$$\frac{\partial \sigma_{33}}{\partial r} = \frac{4h(A - h^2)}{ra^2}$$  \hspace{1cm} (3.5.32)

From (3.5.29) to (3.5.32) we get

$$\frac{\partial \sigma_{11}}{\partial r} = \frac{\partial W}{\partial r} \frac{h^3}{h^3 - A}$$  \hspace{1cm} (3.5.33)

which gives

$$\sigma_{11} = \int_{h^3 - A}^{h^3} \frac{\partial W}{\partial r} \, dr + C$$  \hspace{1cm} (3.5.34)

where C is an arbitrary constant.

From (3.5.24) and (3.5.34) we get

$$P = \int_{h^3 - A}^{h^3} \frac{\partial W}{\partial r} \, dr + C - \frac{h^4}{A^2} \frac{\partial W}{\partial \sigma_{33}}$$  \hspace{1cm} (3.5.35)

Substituting the value of P in (3.5.25) and (3.5.26) we get

$$\sigma_{22} = \sigma_{33} = \sigma_{11} + \frac{rh^3}{2(h^3 - A)} \frac{\partial W}{\partial r}$$  \hspace{1cm} (3.5.36)

Substituting the value of h in (3.5.34) and (3.5.36) and integrating, we get
\[ \sigma_{11} = V(r) + C + \frac{A}{B} \int_{r_1}^{r} \frac{W(r') r'^3}{2} - 3 \int_{r_1}^{r} W(r') r'^2 dr \right) \quad (3.5.37) \]

\[ \sigma_{22} = \sigma_{11} + \frac{A}{B} \int_{r_1}^{r} \frac{W(r') r'^3}{2} \frac{dV}{dr} \right) \quad (3.5.38) \]

\[ \sigma_{33} = \sigma_{11} + \frac{A}{B} \int_{r_1}^{r} \frac{W(r') r'^3}{2} \frac{dV}{dr} \right) \quad (3.5.39) \]

**Boundary conditions**

If the inner surface of the shell \( r = r_1 \) is free from tractions, we must have \( \sigma_{11} = 0 \) when \( r = r_1 \) which on substitution from (3.5.37) gives

\[ \sigma_{11} = V(r) - W(r_1) + \frac{A}{B} \int_{r_1}^{r_2} \left[ W(r)r^3 - W(r_1)r_1^3 - 3 \int_{r_1}^{r_2} W(r) r^2 dr \right] \quad (3.5.40) \]

On the outer surface of the shell \( r = r_2 \), we have to apply a radial force \( R_1 \) given by

\[ R_1 = \sigma_{11}(r_2) = W(r_2) - W(r_1) + \frac{A}{B} \int_{r_1}^{r_2} \left[ W(r_2)r_2^3 - W(r_1)r_1^3 - 3 \int_{r_1}^{r_2} W(r) r^2 dr \right] \quad (3.5.41) \]

The resultant force \( F_1 \) and the couple \( M_1 \) acting on the edge \( \theta = \alpha \) per unit arc between \( \varphi \) and \( \varphi + d\varphi \) are given by

\[ F_1 = \sin \alpha \int_{r_1}^{r_2} \sigma_{22} r dr, \quad M_1 = \sin \alpha \int_{r_1}^{r_2} r^2 \sigma_{22} dr \quad (3.5.42) \]

which gives

\[ 2F_1 = \sin \alpha \left[ \left( W(r_2) - W(r_1) \right) r_2^2 + \frac{A}{B} \left( W(r_2)r_2^3 - W(r_1)r_1^3 \right) \right] \]

\[ - 3 \int_{r_1}^{r_2} W(r) r^2 dr - 6 \int_{r_1}^{r_2} W(r) r^2 dr \quad (3.5.43) \]
\[ 2M_1 = \sin \left[ \left\{ v(r_2) r_2^3 - v(r_1)(r_1^3 + 2r_2^3) - \int_{r_1}^{r_2} vr^2 \, dr \right\} \right. \]
\[ \left. + \frac{A}{B} \left\{ v(r_2) r_2^6 - \frac{v(r_1)}{3} r_1^6 (r_1^3 + 2r_2^3) \right. \right. \]
\[ \left. \left. - 4 \int_{r_1}^{r_2} vr^5 \, dr - 6 \int_{r_1}^{r_2} r^2 \, dr (\int vr^2 \, dr) \right\} \right] \]
\[ (3.5.44) \]

For obtaining a complete spherical shell from part of a spherical shell, obtained by finite bending of incompressible anisotropic circular block, we adopt the principle of limiting process. Supposing

\[ K_2 = K + a_2, \quad K_1 = K + a_1 \quad (3.5.45) \]

and letting \( K \to \infty \) whilst \( a_1 \) and \( a_2 \) remain fixed, we see from (3.5.11)

\[ A \to \frac{3(a_1 - a_2)}{r_3^3 - r_2^3} (K + \frac{a_1 + a_2}{2})^2 \quad (3.5.46) \]

\[ B \to K^2 + \frac{3(a_2 r_1^3 - a_1 r_2^3)}{r_1^3 - r_2^3} K^2 + \frac{3(a_2^2 r_1^3 - a_1^2 r_2^3)}{r_1^3 - r_2^3} \quad (3.5.47) \]

\[ \frac{A}{B} \to 0 \quad (3.5.48) \]

From (3.5.46) to (3.5.48), (3.5.6), (3.5.7), (3.5.2), (3.5.38) to (3.5.44) we get

\[ f(r) \to K + \frac{a_2 r_1^3 - a_1 r_2^3}{r_1^3 - r_2^3} + \frac{r_3^3(a_1 - a_2)}{r_1^3 - r_2^3} \quad (3.5.49) \]

\[ F(\theta) \to \frac{2(1 - \cos \theta)}{A} \frac{1}{K} \quad (3.5.50) \]
\[ x_1 \rightarrow \frac{2(1 - \cos \theta)}{\lambda} \cos \phi \]

\[ x_2 \rightarrow \frac{2(1 - \cos \theta)}{\lambda} \sin \phi \]

\[ x_3 \rightarrow \frac{a_2 x_1^3 - a_1 x_2^3}{r_1^3 - r_2^3} + \frac{(a_1 - a_2)x^3}{r_1^3 - r_2^3} \]

\[ \sigma_{11} \rightarrow \left\{ \frac{\bar{W}(r) - \bar{W}(r_1)}{r} \right\} \]

\[ \sigma_{22} = \sigma_{33} \rightarrow \frac{\bar{W}(r) - \bar{W}(r_1)}{r} + \frac{E}{2} \frac{\partial \bar{W}}{\partial r} \]

\[ R_1 \rightarrow \left\{ \bar{W}(r_2) - \bar{W}(r_1) \right\} \]

\[ 2F_1 \rightarrow \sin \alpha \left\{ \bar{W}(r_2) - \bar{W}(r_1) \right\} r_2^2 \]

\[ 2M_1 \rightarrow \sin \alpha \left\{ \bar{W}(r_2) r_2^3 - \frac{\bar{W}(r_1)}{3} (r_1^3 + 2r_2^3) - \int \bar{W}_r^2 dr \right\} \]

which tallies with the results of A. K. Deshmukh.

3.6 Finite bending of an incompressible anisotropic composite rectangular block into a cylindrical shell

Let the curvilinear system of coordinates \( \phi_1 \) in the strained body be identified with a system of cylindrical coordinates \( (r, \theta, z) \), so that

\[ \theta_1 = r, \quad \theta_2 = \theta, \quad \theta_3 = z \]

\[ y_1 = r \cos \theta, \quad y_2 = r \sin \theta, \quad y_3 = z \]

Let \( X_1 \)-axes coincide with the \( Y_1 \)-axes.

Suppose a composite block bounded by the planes \( x_1 = a_1 \), \( x_1 = a_2 \), \( x_1 = a_3 \) \((a_1 < a_2 < a_3)\) and \( x_2 = \pm b \), \( x_3 = \pm c \) in the undeformed state is bent symmetrically with respect to \( X_1 \)-axis into a part of cylindrical shell of inner radius \( r_1 \).
common radius \( r_2 \) and outer radius \( r_3 \).

Then it follows that

1) the planes \( x_1 = \text{constant} \) in the undeformed state become the curved surfaces \( r = \text{constant} \) in the deformed state.

2) the planes \( x_2 = \text{constant} \) in the undeformed state become \( \theta = \text{constant} \) in the deformed state;

3) there will be uniform extension \( \lambda \) in the direction of \( x_3 \)-axis.

These imply that

\[
x_1 = f(r), \quad x_2 = f(\theta), \quad x_3 = \frac{z}{\lambda} \quad (3.6.3)
\]

The metric tensors for the strained and the unstrained bodies are given by

\[
G_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
G = r^2 \quad (3.6.4)
\]

\[
E_{ij} = \begin{bmatrix} \frac{f_r^2}{r} & 0 & 0 \\ 0 & \frac{f_\theta^2}{r} & 0 \\ 0 & 0 & \frac{1}{\lambda^2} \end{bmatrix}, \quad g^{ij} = \begin{bmatrix} \frac{1}{f_r^2} & 0 & 0 \\ 0 & \frac{1}{f_\theta^2} & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix} \\
g = \frac{f_r^2 f_\theta^2}{2} \quad (3.6.5)
\]

where \( f_r = \frac{df}{dr}, \quad f_\theta = \frac{df}{d\theta} \)

The conditions of incompressibility \( I_3 = \frac{g}{e} = 1 \) gives

\[
\frac{f_r}{r} = \frac{\lambda}{f_\theta} = \Lambda \quad (3.6.6)
\]
where $A$ is a constant to be determined.

Solving (3.6.6), we get

$$x_1 = f(x) = \frac{Ax^2}{2} + B \quad (3.6.7)$$

$$x_2 = \frac{A\theta}{A} + C \quad (3.6.8)$$

where $B$ and $C$ are constants to be determined.

Since the deformation is symmetrical with respect to the $x_1$-axis, the equation (3.6.8) gives

$$C = 0 \quad (3.6.9)$$

As $r_1$ and $r_2$ are the radii of the inner and outer surfaces of the deformed body which are initially the planes $x_1 = a_1$ and $x_1 = a_3$ respectively, the equation (3.6.7) gives

$$a_1 = \frac{Ax_1^2}{2} + B \quad (3.6.10)$$

$$a_3 = \frac{Ax_3^2}{2} + B \quad (3.6.11)$$

Solving these we get

$$A = \frac{2(a_3 - a_1)}{(x_3^2 - x_1^2)} , \quad B = \frac{(a_1^2x_3^2 - a_3^2x_1^2)}{(x_3^2 - x_1^2)} \quad (3.6.12)$$

The radius $r_2$ of the common surface of the deformed body which is initially the plane $x_1 = a_2$ is given by (3.6.7) and (3.6.12) as

$$r_2^2(a_3 - a_1) = a_2(x_3^2 - x_1^2) + (a_3^2x_1^2 - a_1^2x_3^2) \quad (3.6.13)$$

Substituting the values of $f_r$ and $F_0$ from (3.6.6)
in (3.6.5), we get
\[ \varepsilon_{ij} = \begin{bmatrix} \frac{A^2 r^2}{\lambda^2} & 0 & 0 \\ 0 & \frac{\lambda^2}{A^2} & 0 \\ 0 & 0 & \frac{1}{\lambda^2} \end{bmatrix} \]
\[ \gamma_{ij} = \begin{bmatrix} \frac{1}{\lambda^2} & 0 & 0 \\ 0 & \frac{A^2}{\lambda^2} & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix} \]
\[ \varepsilon = r^2 \quad (3.6.14) \]

Now the equations of equilibrium in the absence of body forces reduce to
\[ \frac{dT_{11}}{dr} + \frac{T_{11}}{r} - r^2 T_{22} = 0 \quad (3.6.15) \]
\[ \frac{\partial P_i}{\partial \theta} = 0, \quad \frac{\partial P_i}{\partial \zeta} = 0 \quad (3.6.16) \]

The equations (3.6.16) show that \( P_i \) are functions of \( r \) only. From (3.6.5), (3.6.7) and (3.6.8) we get
\[ \frac{\partial \varepsilon_i^r}{\partial \varepsilon_i^r} = \begin{bmatrix} A r & 0 & 0 \\ 0 & \frac{\lambda}{A} & 0 \\ 0 & 0 & \frac{1}{\lambda} \end{bmatrix} \quad (3.6.17) \]
\[ \frac{\partial \varepsilon_i^z}{\partial \varepsilon_i^z} = \begin{bmatrix} 0 & \frac{A^2}{\lambda} & 0 \\ 0 & 0 & \lambda \end{bmatrix} \quad (3.6.18) \]

From (3.2.4), (3.2.5), (3.6.4), (3.6.14), (3.6.17) and (3.6.18) the strain tensors \( \gamma_{ij} \) and \( \varepsilon_{ij} \) are given by
\[ 2 \gamma_{11} = 1 - \lambda^2 r^2, \quad 2 \gamma_{22} = r^2 - \frac{\lambda^2}{A^2}, \quad 2 \gamma_{33} = 1 - \frac{1}{\lambda^2} \]
\[ \gamma_{12} = \gamma_{13} = \gamma_{23} = 0 \quad (3.6.19) \]
\[ 2e_{11} = \frac{1}{A^2} \frac{r^2}{2} - 1 \quad , \quad 2e_{22} = \frac{\lambda^2}{\lambda^2} - 1 \quad , \quad 2e_{33} = \frac{2}{\lambda} - 1 \]
\[ e_{12} = e_{13} = e_{23} = 0 \]

(3.6.20)

Also from (3.2.7) and (3.6.20) we obtain the components of stress tensor as

\[ T_{11}^{11} = \frac{1}{\lambda^2} \frac{\partial W_i}{\partial e_{11}} + P_i \quad , \quad T_{12}^{22} = \frac{\lambda}{\lambda^2} \frac{\partial W_i}{\partial e_{22}} + \frac{P_i}{r^2} \]
\[ T_{13}^{33} = \frac{\lambda}{\lambda^2} \frac{\partial W_i}{\partial e_{33}} + P_i \quad , \quad T_{11}^{12} = \frac{1}{2\lambda} \left( \frac{\partial W_i}{\partial e_{12}} + \frac{\partial W_i}{\partial e_{21}} \right) \]
\[ T_{11}^{23} = \frac{\lambda}{2} \left( \frac{\partial W_i}{\partial e_{23}} + \frac{\partial W_i}{\partial e_{32}} \right) , \quad T_{11}^{31} = \frac{\lambda}{2\lambda} \left( \frac{\partial W_i}{\partial e_{31}} + \frac{\partial W_i}{\partial e_{13}} \right) \]

(3.6.21)

The equations of equilibrium (3.6.15) reduce with the help of (3.6.21) to

\[ \frac{dT_{11}^{11}}{dr} = \frac{1}{r^2} \left( \frac{\lambda^2}{\lambda} \frac{\partial W_i}{\partial e_{22}} - \frac{1}{A^2} \frac{\partial W_i}{\partial e_{11}} \right) \]

(3.6.22)

Now

\[ \frac{\partial W_i}{\partial x} = \frac{\partial W_i}{\partial e_{11}} \frac{\partial e_{11}}{\partial x} + \frac{\partial W_i}{\partial e_{22}} \frac{\partial e_{22}}{\partial x} + \frac{\partial W_i}{\partial e_{33}} \frac{\partial e_{33}}{\partial x} \]

\[ = \frac{1}{r^2} \left( \frac{\lambda^2}{\lambda} \frac{\partial W_i}{\partial e_{22}} - \frac{1}{A^2} \frac{\partial W_i}{\partial e_{11}} \right) \]

(3.6.23)

Substituting (3.6.23) in (3.6.22) we get

\[ \frac{dT_{11}^{11}}{dr} = \frac{dW_i}{dr} \]

(3.6.24)

Integrating this, we get

\[ T_{11}^{11} = W_i(r) + d \]

(3.6.25)

where \( d \) are constants to be determined.
Substituting the value of \( T_{11} \) from (3.6.21) in (3.6.25) we get

\[
P_1 = W_1(r) + \mathbf{d}_1 - \frac{1}{\Lambda} \frac{\partial}{\partial x_1} W_1
\]

(3.6.26)

The physical components of stress (3.2.12) reduce to

\[
(\sigma_{11}^1) = W_1(r) - W_1(x_1)
\]

\[
(\sigma_{22}^1) = \frac{r dW_1}{dr} + W_1(r) - W_1(x_1)
\]

\[
(\sigma_{33}^1) = \frac{2}{\Lambda} \frac{\partial W_1}{\partial e_{33}} - \frac{1}{\Lambda} \frac{\partial W_1}{\partial e_{11}} + W_1(r) - W_1(x_1)
\]

\[
(\sigma_{12}^1) = \frac{1}{2\Lambda} (\frac{\partial W_1}{\partial e_{12}} + \frac{\partial W_1}{\partial e_{21}}),
(\sigma_{23}^1) = \frac{4\pi}{2} (\frac{\partial W_1}{\partial e_{23}} + \frac{\partial W_1}{\partial e_{32}})
\]

\[
(\sigma_{31}^1) = \frac{\lambda}{2\pi} (\frac{\partial W_1}{\partial e_{31}} + \frac{\partial W_1}{\partial e_{13}}),
(\sigma_{11}^2) = W_2(r) - W_2(x_3)
\]

\[
(\sigma_{22}^2) = \frac{r dW_2}{dr} + W_2(r) - W_2(x_3)
\]

\[
(\sigma_{33}^2) = \frac{2}{\Lambda} \frac{\partial W_2}{\partial e_{33}} - \frac{1}{\Lambda} \frac{\partial W_2}{\partial e_{11}} + W_2(r) - W_2(x_3)
\]

\[
(\sigma_{12}^2) = \frac{1}{2\Lambda} (\frac{\partial W_2}{\partial e_{12}} + \frac{\partial W_2}{\partial e_{21}}),
(\sigma_{23}^2) = \frac{4\pi}{2} (\frac{\partial W_2}{\partial e_{23}} + \frac{\partial W_2}{\partial e_{32}})
\]

\[
(\sigma_{31}^2) = \frac{\lambda}{2\pi} (\frac{\partial W_2}{\partial e_{31}} + \frac{\partial W_2}{\partial e_{13}})
\]

(3.6.27)

**Boundary conditions**

If the block is bent by forces applied to the edge only, the boundary conditions require that
1) the normal tractions across the curved surfaces 
\( r = r_1 \), \( r = r_3 \) must vanish. This is satisfied if 
\[
T_{11}^{11} = 0 \text{ when } r = r_1, \quad T_{22}^{11} = 0 \text{ when } r = r_3
\] (3.6.28)

ii) the normal tractions should be continuous across the common surface \( r = r_1 \). This is satisfied if 
\[
T_{11}^{11} = T_{22}^{11} \text{ when } r = r_2
\] (3.6.29)

The two equations (3.6.28) and (3.6.29) are necessary and sufficient to determine \( d_1 \), \( d_2 \) and \( \lambda \). Therefore, substituting (3.6.26) in (3.6.28) and (3.6.29) we get 
\[
d_1 = -W_1(r_1), \quad d_2 = -W_2(r_3),
\]

\[
W_1(r_2) - W_1(r_1) = W_2(r_2) - W_2(r_3)
\] (3.6.30)

Substituting the values of \( P_1 \) from (3.6.26) in (3.6.21), we get 
\[
T_{11}^{11} = \frac{d}{dr}(W_1) + d_1
\] (3.6.31)

\[
T_{11}^{33} = \lambda^2 \frac{\partial W_2}{\partial e_{33}} - \frac{1}{\lambda^2} \frac{\partial W_4}{\partial e_{11}} + W_1 + d_1
\] (3.6.32)

The resultant normal force \( Z \) applied to the surface normal to the \( z \)-axis is given by
\[
Z = \int_{r_1}^{r_2} r T_{11}^{33} dr + \int_{r_2}^{r_3} r T_{22}^{33} dr
\]
\[
= \int_{r_1}^{r_2} \left( \lambda^2 \frac{\partial W_1}{\partial e_{33}} - \frac{1}{\lambda^2} \frac{\partial W_2}{\partial e_{11}} + r W_1 + r d_1 \right) dr
\]
\[
+ \int_{r_2}^{r_3} \left( \lambda^2 \frac{\partial W_2}{\partial e_{33}} - \frac{1}{\lambda^2} \frac{\partial W_2}{\partial e_{11}} + r W_2 + r d_2 \right) dr
\] (3.6.33)
The resultant force $F$ acting on the surface which is initially at $x_2 = \pm b$ is given by

$$F = 2\alpha \int_{r_1}^{r_2} r^2 T_1^{22} \ dr + 2\alpha \int_{r_2}^{r_3} r^2 T_2^{22} \ dr$$

Using (3.6.15), (3.6.28) and (3.6.29), we have

$$F = 2\alpha \int_{r_1}^{r_2} \frac{d(r T_1^{11})}{dr} \ dr + 2\alpha \int_{r_2}^{r_3} \frac{d(r T_2^{11})}{dr} \ dr$$

$$= 2 \left( r T_1^{11} \right)_{r_1}^{r_2} + 2 \left( r T_2^{11} \right)_{r_2}^{r_3} = 0 \quad (3.6.34)$$

Therefore, it follows that the forces acting on the surfaces initially at $x_2 = \pm b$ are equivalent to a couple of moment $M$ given by

$$M = 2\alpha \int_{r_1}^{r_2} r^3 T_1^{22} \ dr + 2\alpha \int_{r_2}^{r_3} r^3 T_2^{22} \ dr$$

$$\frac{M}{2\alpha} = r_2^2 \left\{ \frac{W_1(x_2)}{2} - \frac{W_2(x_2)}{2} \right\} - \frac{W_1(r_1)}{2} \left( r_1^2 + r_2^2 + r_3^2 \right)$$

$$+ \frac{W_2(r_3)}{2} \left( r_2^2 + r_3^2 \right) - \int_{r_1}^{r_2} r \ W_1 \ dr - \int_{r_2}^{r_3} r \ W_2 \ dr \quad (3.6.35)$$

If $r_3 = r_2$ this equation with the help of (3.6.30), reduces to

$$M = 2\alpha \left\{ \frac{W_1(r_1)}{2} \left( r_2^2 - x_1^2 \right) - \int_{r_1}^{r_2} r \ W_1 \ dr \right\} \quad (3.6.36)$$

which agrees with the results obtained by Green and Adkins when the block is not composite.
3.7 Particular case

If the material is isotropic and incompressible the strain energy function $W_1$ reduces to

$$W_1 = W_1(I_1, I_2) \quad (3.7.1)$$

In this case the deformation and the metric tensors for the strained and unstrained states of the body are still given by (3.6.3), (3.6.4) and (3.6.14). From (3.2.8), (3.2.9) (3.2.11), (3.6.4) and (3.6.14), we obtain

$$I_1 = \frac{1}{A^2 x^2} + \frac{A^2 x^2}{\lambda^2} + \frac{1}{\lambda^2} \quad (3.7.2)$$

$$I_2 = A^2 x^2 + \frac{\chi^2}{A^2 x^2} + \frac{1}{\lambda^2} \quad (3.7.3)$$

$$B_{1j} = \begin{bmatrix}
\frac{\lambda^2}{A^2 x^2} + \frac{1}{\lambda^2} & 0 & 0 \\
0 & A^2 + \frac{1}{\lambda^2 x^2} & 0 \\
0 & 0 & \frac{\lambda^2}{A^2 x^2} + A^2 x^2
\end{bmatrix} \quad (3.7.4)$$

$$T_{11}^{11} = \frac{\phi_1^2}{A^2 x^2} + \left(\frac{\lambda^2}{A^2 x^2} + \frac{1}{\lambda^2}\right) \Psi_1 + p_1$$

$$T_{11}^{22} = \frac{A^2 \phi_1}{\lambda^2} + (A^2 + \frac{1}{\lambda^2 x^2}) \Psi_1 + \frac{p_1}{x^2}$$

$$T_{11}^{33} = \lambda^2 \phi_1 + \left(\frac{\lambda^2}{A^2 x^2} + A^2 x^2\right) \Psi_1 + p_1$$

$$T_{11}^{33} = T_{11}^{31} = T_{11}^{12} = 0 \quad , \quad (3.7.5)$$
Then the equations corresponding to (3.6.27), (3.6.34) and (3.6.35) are

\[
\begin{align*}
(\sigma_{11})_1 &= v_1(r) - v_1(r_1) \\
(\sigma_{22})_1 &= r \frac{d v_1}{d r} + v_1(r) - v_1(r_1) \\
(\sigma_{33}) &= (\lambda - \frac{1}{\lambda^2})(\phi_1 + \frac{A^2 x^2}{\lambda^2} \psi_1) v_1(r) - v_1(r_1) \\
(\sigma_{11})_2 &= v_2(r) - v_2(r_3) \\
(\sigma_{22})_2 &= r \frac{d v_2}{d r} + v_2(r) - v_2(r_3) \\
(\sigma_{33})_2 &= (\lambda - \frac{1}{\lambda^2})(\phi_2 + \frac{A^2 x^2}{\lambda^2} \psi_2) v_2(r) - v_2(r_3)
\end{align*}
\]

(3.7.6)

\[
F = (r^2 T_{11})_1^2 + (r^2 T_{22})_2^3 = 0
\]

(3.7.7)

\[
M = r_2^2 \left\{ v_1(r_2) - v_2(r_2) \right\} - \frac{v_1(r_1)}{2} \left( r_1^2 + r_2^2 \right)
\]
\[
+ \frac{v_2(r_3)}{2} \left( r_2^2 + r_3^2 \right) - \int_{r_1}^{r_2} r v_1 \, d r - \int_{r_2}^{r_3} r v_2 \, d r
\]

(3.7.8)

If \( r_3 = r_2 \) this equation reduces with the help of (3.6.30) to

\[
M = \frac{v_1(r_1)}{2} \left( r_2^2 - r_1^2 \right) - \int_{r_1}^{r_2} r v_1 \, d r
\]

(3.7.9)

which agrees with the result obtained by Rivlin\(^40\),\(^41\) when the block is not composite.