PREFACE

The following few pages contain my investigations on some problems in non-linear continuum mechanics and thermoelasticity of elastic bodies carried out independently at the Defence Metallurgical Research Laboratory, Hyderabad.

During recent years, like the Latin in the humanities the elasticity theory has found considerable application in the solution of engineering problems. There are many cases in which the linear elasticity methods of strength of materials are inadequate to furnish satisfactory information regarding stress-distribution in engineering structures and recourse must be made to continuum theory of elasticity which provides a basis for a deeper understanding of general mechanical behaviour. This led to the rapid development of theory of elasticity into an important branch of continuum mechanics. The theory of continuum mechanics is a scientific discipline concerned with the global behaviour of the substance under the influence of the external disturbances. Most of our classical knowledge concerning the mechanical behaviour of the engineering materials is insufficiently physical to describe the behaviour of larger physical masses. On the other hand the continuum mechanics is physically realistic discipline to describe the gross phenomena observable in macroscopic bodies. The unified approach to continuum mechanics is based on the recognition that the same physical principles govern every process no matter what the constitution of bodies.
Whether the substance is milk or steel whether it is hot or cold, mass must be conserved, energy balance must add up. While continuum mechanics has accomplished a great deal in explaining the behaviour of rubber like materials under finite deformations it has yet to contribute to a more accurate description of response of metals.

The modern era of the non-linear elastic theory spans by now two decades, its first developments were profoundly influenced by the researches of, Seth, Reiner and Rivlin, particularly by Rivlin's discovery of important exact solution for isotropic, incompressible materials. The initial success in solving a number of simple problems by a perturbation treatment starting from the so-called universal solutions for incompressible isotropic materials led to the idea of approximate solutions for materials of small or moderate compressibility. The general formalism of this approach is developed and is then applied to anisotropic, compressible bodies.

In recent years the theory of finite deformation has been developed in considerable detail and it is understandable from the number of exact solutions that have been obtained for incompressible elastic bodies without specifying any form for the strain-energy function. The number of exact solutions obtained for compressible bodies have not been many and in general, was based on the strain energy form adopted
by B.R. Seth and the generalised stress-strain relation, Green and Zerna\textsuperscript{17} used tensor notations for the general theory of finite deformation and the results are applied to solve a number of special problems, mainly for isotropic, incompressible materials. Number of workers developed various approximation procedures depending on the nature of deformations and geometry of the body. The approximate solutions accounted for the large deformations than those admissible in the linearized elastic theory. Rivlin\textsuperscript{40-45} obtained exact solutions in terms of the strain-energy function for a number of problems pertaining to incompressible materials and gave adequate scope for exploring the potentialities of the field of finite elasticity. Murnaghan\textsuperscript{33-34} formulated a second approximation theory which involves more physical constants than the young's modulus and poisson's ratio of the linearized theory. Reiner\textsuperscript{38-39} proposed a semi-empirical theory by experimentally determining the elastic coefficients. Seth\textsuperscript{53-64} proposed a linear stress-strain relation by referring the components of strain to the strained state of the body. Signorini\textsuperscript{66} developed an exact quadratic theory by taking the stress to be quadratic function of strain. Various other developments are found in the surveys of C. Truesdell\textsuperscript{71-73}, A.E. Green and W. Zerna\textsuperscript{16-17}, A.E. Green and J.E. Adkins\textsuperscript{1}, A.C. Eringen\textsuperscript{10}, A.E. Green and R.S. Rivlin\textsuperscript{20} and Novozhilov\textsuperscript{36}. The applications of finite elasticity are many and some of which have no counterpart in the infinitesimal elasticity. Their theory has qualitatively predicted a yield point. It
showed that the large difference between the static and
dynamic values for elastic bodies like India rubber is in
a good measure due to the use of the small-strain theory.
Its application gives axial stress, neglected in classical
theory in cylinders subjected to large torsional shifts.
Seth and Shepherd employed in general graphical methods
to solve certain problems of finite bending. Apart from the
labour involved in obtaining the graphical solution it does
not give very accurate results. In Chapter II and III
various other problems on finite bending are solved intro-
ducing tensor notations. It is interesting to note that
the solutions obtained are suitable for practical applica-
tions and high speed computational work.

The understanding of the non-elastic behaviour of solids
at low stress levels has recently advanced to such a stage
that this behaviour can now be used to study the laws govern-
ing the motion of the individual atoms. This part develops
the theoretical and partly experimental interpretation of the
elasto-plastic effects in solids. It has commonly been
assumed that Hooke's law describes the relation between stress
and strain in solids with high degree of accuracy at low
stress levels. Recent studies have revealed that under appro-
priate conditions all solids suffer large deviations from
Hooke's law at all stress levels. Such deviations are mani-
ifested as damping, elastic after effects, stress relaxation,
creep, frequency variation of elastic moduli, etc. By damping
we understand the ability of a vibrating solid member to convert its mechanical energy of vibration into heat, even if the body is completely isolated from its surroundings. A perfectly elastic material will not produce any damping, since under oscillatory conditions, stress and strain are always in phase, and consequently no mechanical hysteresis or loss of vibrational energy can take place. Damping is therefore, a result of the non-elastic behaviour of solids. This is also observed far below the stress levels at which plastic flow occurs. It has therefore, seemed appropriate to study such behaviour as the elasto-plastic effect and other effects which are contrary to the theory of elasticity but are shown by real solids by way of damping and variation in modulus with the temperature. All such are different manifestations of the lack of uniqueness of the relation between stress and strain. The present investigation in Chapter IV is mainly concerned with the development of a method for the correction of internal damping in the domain of plastic deformation. Generally the measurements of internal damping is achieved with the inverse torsion pendulum by observing the global damping of the vibrations of the specimen. Recently many authors have proposed the methods for the calculation of local internal damping from the curves of the measured global damping vs. deformation. Here a correction is proposed for the calculation of damping in the plastic domain where the Hookean elasticity fails. This method
developed in the case of a cylindrical specimen only requires the calculation of the variation of the period with the strain amplitude of a torsion pendulum. Although this aspect has been studied previously by Lazan [76, 80] and Morrow [78], there have been few systematic studies. It is desirable to extend some recent studies made by the author at Institute National et Sciences Appliquées, LYON, France, at low temperatures to correlate the effects of plastic strain and the temperature on the damping spectrum. It is the intention to clarify many of the details regarding non-linear continuum mechanics for which not much data existed.

In passing to the group of basic problems concerning the physical foundations of the theory of elasto-plasticity, an essential question arises which some time has even a certain philosophical aspect. The question being the admissibility of a fictitious model of material continuum replacing the complex nature of the structure of matter. This formalism concerning the highly idealized material continuum is by no means new, its validity has been questioned often, the answer usually being in the affirmative. The classical theory of elasticity with its modern deviations towards non-linearity both in the physical and geometrical senses, the applied mathematicians were readily disposed to accept the notion of a continuum. It has been shown recently that the equilibrium of elastic-plastic bodies with certain restriction can be examined on the basis of general principles of the theory of
non-linear continuum.

Recently Mendelson and Manson\textsuperscript{82} presented a method for calculating the strain distributions in various shapes when the material was stressed beyond the elastic limit. This method, however, had simplifications that made it inapplicable to situations where the plastic strains were small or of the same order of magnitude as the elastic strains. The present investigation in Chapter V is an attempt to overcome this difficulty and to reduce the method to one in which the plastic strains can be added in small incremental amounts. The method is applied to the problem of a uniformly stressed infinite plate containing a hole at the origin.

Non-linear continuum mechanics developed in the context of mechanical theories, have been also applied to the thermoelasticity theories in general. From the thermo-dynamic viewpoint classical elasticity is a special case of thermo-elasticity, instead of thermo-elasticity being an extension of the classical theory. In the post war years we have seen a rapid development of thermo-elasticity stimulated by various engineering sciences. A considerable progress in the field of air-craft and machine structures mainly with gas and steam turbines, and the emergence of new topics in chemical and nuclear engineering have given rise to numerous problems in which thermal stresses play an important and frequently even a primary role. Thermo-elasticity embraces a wide field of
phenomena. It contains the theory of heat conduction and the
theory of strains and stresses due to the flow of heat, when
coupling of temperature and strain field occurs. Thermo-
elasticity makes it possible to determine the stresses pro-
duced by the temperature field and, moreover, to calculate
the distribution of temperature due to the action of internal
forces which vary with time.

The classical approach to the solutions of the thermo-
elasticity problems of the plane theory of elasticity in the
absence of the body force, is due to Airy\textsuperscript{35} who introduced a
stress function satisfying the so called biharmonic equation,
in terms of which the stress distribution can be determined
for any region under consideration. Many general solutions
of these problems have been obtained in various coordinate
systems namely:

i) in Cartesian Coordinates by M. Levy\textsuperscript{88} and L.N.G. Filon\textsuperscript{35}.

ii) in Polar Coordinates by Timoshenko and Goodier\textsuperscript{70}.

iii) in bi-polar coordinates by G.B. Jeffery\textsuperscript{36}.

Based on the knowledge that any harmonic function can be
represented as the sum of the two analytic functions of the
complex variable $Z$ and $\overline{Z}$ and the use of conformal trans-
formations, the method proposed by G.V. Kolossov\textsuperscript{37} and
developed by Muskhelishvilia\textsuperscript{35} has not only provided the
means of obtaining the solutions of the thermal stress pro-
blems for regions bounded by a single contour, but also for
the regions by any arbitrary number of contours. The generalization of these methods to the anisotropic bodies has been done by Lakhmitskii\textsuperscript{29}. Using these complex variable techniques in one form or other, a variety of problems have been solved by a number of workers\textsuperscript{12, 13, 32, 52, 53-64} in the last few decades.

This thesis deals with the discussions of the non-linear elastic deformation of the elastic bodies, elasto-plastic effects in solids, and the thermo-elastic deformation under the influence of a variety of external disturbances.

The contents of this thesis are divided into three parts:

\textbf{P a r t - I}

In this part Chapter-I provides a summary of the essential basic equations of the theory of deformation and deals with the study of finite deformation of elastic bodies under the influence of external forces. Although the finite elasticity theory has been applied to number of problems very few attempts have been made to obtain the solutions for bending of circular and rectangular blocks into shells. Rivlin\textsuperscript{41} solved the problem of bending of an isotropic incompressible rectangular block into a right circular cylindrical shell. Green and Wilkes\textsuperscript{18}, Rivlin and Ericksen\textsuperscript{46}, considered the same problem when the material is orthotropic and transversely isotropic. Green and Adkins\textsuperscript{1}, discussed the case of the general anisotropy with certain restrictions on the strain-energy functions.
G. Lakshminarayana considered the problem of bending of an isotropic circular block into a spherical shell.

In this part we discussed various exact solutions for the problems of finite bending of anisotropic, compressible, incompressible composite and initially curved circular and rectangular blocks into hyperboloidal and hyperbolic, spherical and cylindrical shells which forms the subject matter of Chapter-II and III. In addition various results are obtained as particular cases. By way of practical application of this approach some numerical results have been obtained for steel and some metals. In section 2.6 and 3.6 some of the existing results are extended to composite bodies as very little work has been done in this direction. In attempting this we have derived solutions for the problems of bending of incompressible/compressible anisotropic composite circular blocks into cylindrical and spherical shells respectively. In section 3.5 we examined the problem of finite bending of a initially curved anisotropic incompressible circular block into a spherical shell.

**Part - II**

This part deals with the elasto-plastic effects in solids and the equilibrium of elastic plastic bodies.

In this part Chapter-IV provides studies on the non-elastic behaviour of solids. The elasto-plastic effects and other effects which are contrary to the theory of elasticity are
shown by real solids by way of damping and variation in the modulus with temperatures, etc. All such effects are different manifestations of the lack of uniqueness of the relation between stress and strain. The present investigation is mainly concerned with the development of a method for the correction of the internal damping in the domain of plastic deformation where an attempt is made to calculate internal damping from the global damping of the specimen under vibration. The study is mainly theoretical in approach and partly experimental.

In Chapter-V we discuss the solutions of equilibrium of the elastic-plastic bodies. Mendelson and Manson presented a method for calculating the strain distribution in various shapes when the material was stressed beyond elastic limit. The method, however, had simplification that made it in applicable to situations where the plastic strains were small or of the same order of magnitude as the elastic strains. In the present investigation an attempt is made to overcome this difficulty and to reduce the method to one in which the plastic strains can be added in small incremental amounts. The method is applied to the problem of a uniformly stressed infinite plate, containing a hole at the origin which forms the subject matter of Chapter-V.

Part III

In this part we have slightly touched upon the thermoelastic problems which occur in isotropic and homogeneous bodies, under the assumption of small displacement and linear
stress-strain laws, wherein the uniform heat flow in the body is disturbed by a cavity or an inclusion. We assume, more over, that the material constant are independent of temperature. These assumptions are peculiar to the linearised theory of elasticity but all the same they make it possible to investigate a large class of engineering problems. The linear thermo-elastic problem is solved by complex variable technique for the insulated cross-sections of various shapes. We discuss the solutions of the problems of thermal stresses due to disturbance of uniform heat flow by an insulated hole of various cross-sections viz., aerofoil and epitrochoidal have been discussed in Chapter-VI. The problem is developed within the frame-work of the classical theory. Complex integral representations of the thermal stresses have been obtained and the results compared with those of the earlier ones. Thermal stresses have also been calculated in an ovaloidal and elliptical cross-sections which are in agreement with $^{23}$ The results for cardioid and a circle have also been deduced as particular cases.