CHAPTER V

Biorthogonal Polynomials

Suggested by the

Hermite Polynomials
CHAPTER V

1. Introduction

The question of constructing a pair of biorthogonal polynomials with respect to the Hermite's weight function namely \( \exp(-x^2) \) remained open for several years. In the literature one finds attempts of constructing a pair of biorthogonal polynomials with respect to the Jacobi's weight function namely, \( (1-x)^\alpha (1+x)^\beta \). However their seems to be no previous attempt at finding a pair of biorthogonal polynomials suggested by the Hermite polynomials.

In this chapter we begin from a scratch and construct a pair of polynomials that is biorthogonal with respect to the weight function \( \exp(-x^2) \) over the interval \( (-\infty, \infty) \). Thus we complete the question of finding pairs of biorthogonal polynomials suggested by the classical orthogonal polynomials.

There is a highly well-known relationship connecting the standard Laguerre polynomials with the standard Hermite polynomials that runs as follows (see Szegö [8, p. 106]):

\[
(V.1) \quad H_{2m}(x) = (-1)^m 2^{2m} m! L_m^{(-1/2)}(x^2),
\]

\[
(V.2) \quad H_{2m+1}(x) = (-1)^m 2^{2m+1} m! x L_m^{(1/2)}(x^2).
\]
This fact coupled with the well settled concept of the pair of Kohnert biorthogonal polynomials, namely \( Z_n(x;k) \) and \( Y_n(x;k) \) suggests that we introduce a pair of polynomials in the following manner.

\[
(V.3) \quad S_{2n}(x;k) = (-1)^n 2^{2n} n! \frac{\Gamma'(km+1/2)}{\Gamma'(km+1/2)} Z_n(x^2;k)
\]

\[
= (-1)^n 2^{2n} \frac{\Gamma'(km+1/2)}{\Gamma'(km+1/2)} \sum_{j=0}^{n} (-1)^j \frac{x^{2kj}}{\Gamma'(kj+1/2)}
\]

\[
(V.4) \quad S_{2n+1}(x;k) = (-1)^n 2^{2n+1} n! x^{k/2} Z_n(x^2;k)
\]

\[
= (-1)^n 2^{2n+1} \frac{\Gamma'(1+kn+k/2)}{\Gamma'(1+kn+k/2)} \sum_{j=0}^{n} (-1)^j \frac{x^{(2j+1)k}}{\Gamma'(1+kj+k/2)}
\]

\[
(V.5) \quad T_{2n}(x;k) = (-1)^n 2^{2n} n! Y_n(x^2;k)
\]

\[
= (-1)^n 2^{2n} \sum_{r=0}^{n} \frac{x^{2r}}{r!} \sum_{s=0}^{r} (-1)^s \frac{s+1/2}{r} \left( -\frac{k}{k} \right)^s \varepsilon
\]

\[
(V.6) \quad T_{2n+1}(x;k) = (-1)^n 2^{2n+1} n! x^{r/2} Y_n(x^2;k)
\]

\[
= (-1)^n 2^{2n+1} \sum_{r=0}^{n} \frac{x^{2r+1}}{r!} \sum_{s=0}^{r} (-1)^s \left( -\frac{k}{k} \right)^s \left( \frac{1+s+k/2}{k} \right)
\]

It could be easily seen that by reverting the order of summation (V.3), and (V.4), and (V.5) and (V.6) can easily be combined in the following compact forms respectively:
\begin{align*}
(\text{V.7}) \quad S_n(x;k) &= 2^n \left[ \gamma \left( \frac{\ln k - k\epsilon}{2} \right) \sum_{j=0}^{\left[ \frac{n}{2} \right]} (-1)^j \frac{x^j}{\gamma \left( \frac{kn+1+\epsilon}{2} - kj \right)} \right] \\
(\text{V.8}) \quad T_n(x;k) &= (-1)^{\left[ \frac{n}{2} \right]} 2^n .
\end{align*}

\begin{align*}
&\sum_{r=0}^{\left[ \frac{n}{2} \right]} \frac{x^{n-2r}}{([n/2]-r)!} \sum_{s=0}^{\left[ \frac{n}{2} \right]-r} (-1)^s \frac{s^{n/2-x}}{s} \frac{s+(k+1)e/2+1/2}{k} \\
&\text{The value of } \epsilon \text{ is } 0 \text{ or } 1 \text{ according to } n \text{ is even or odd integer. Throughout chapter-V and chapter-VI, } \epsilon \text{ will always have this meaning. Also throughout chapter-V and chapter-VI, } \lceil p \rceil \text{ denotes the greatest integer less than or equal to } p.
\end{align*}

Let us break for a while our line of thought to recall the following existence theorem originally due to Konhauser [5, p. 255]:

**Theorem:** Given an interval \((a, b)\), an admissible weight or function \(p(x)\) which is either nonnegative/nonpositive on \((a, b)\) and basic polynomials \(r(x) = x^h, s(x) = x^k\), where \(h\) and \(k\) are positive integers, then unique biorthogonal polynomial sets exist if \(h+k\) is even. Also, if \(h+k\) is odd and if zero is not interior to the interval \((a, b)\) then unique biorthogonal polynomials sets will exist.
This theorem entails for the uniqueness of the existence of a biorthogonal pair suggested by the Hermite polynomials, that we must select \( k \) to be a positive odd integer. The precise reason for this is that the polynomials \( S_m(x;k) \) are polynomials of degree \( m \) in \( x^k \) and the polynomials \( T_n(x;k) \) are of precise degree \( n \) in \( x \) and zero is an interior point of the interval of biorthogonality \( (-\infty, \infty) \). We surmise that the said restriction on \( k \) is not at all severe. One also notes that for \( k=1 \), (V.3) and (V.4) or (V.5) and (V.6) yield (V.1) and (V.2). As such now it is needless to say that for \( k=1 \) both our polynomial sets \( \{ S_n(x;k) \} \) and \( \{ T_n(x;k) \} \) get reduced to the standard Hermite system.

In section-2 we establish explicitly that the pair \( S_n(x;k) \) and \( T_n(x;k) \) is biorthogonal with respect to the weight function \( \exp(-x^2) \) over the interval \( (-\infty, \infty) \). In section-3 we obtain several generating functions and recurrence relations for both the biorthogonal sets \( \{ S_n(x;k) \} \) and \( \{ T_n(x;k) \} \). For doing so we shall use the usual series techniques.
integral on right hand side is clearly zero and there is nothing to prove. Next, if we consider both \( n \) and \( m \) odd or even, this integral becomes after putting \( x^2 = t \)

\[
\int_0^\infty t^{-\frac{1}{2}+(nk+m)/2} e^{-t} \, dt = \Gamma\left(-\frac{kj-r}{2}+(nk+m+1)/2\right);
\]

and we have

\[
I_{n,m} = 2^{n+m} (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left(\frac{kn+k-j}{2}+\frac{\epsilon}{2}\right) \prod_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{1}{\Gamma\left(\frac{kn+1+\epsilon}{2}-\frac{kj}{2}\right)}
\]

\[
\sum_{r=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \sum_{s=0}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{k}{\left(\frac{m}{2}-r\right)!} \left(\frac{m}{2}-r\right)^s \frac{s+\left(k+1\right)\epsilon}{2} \frac{1}{\Gamma\left(\frac{m}{2}\right)}
\]

Consider the case of \( n, m \) even; we then have \( \left\lfloor \frac{n}{2} \right\rfloor = n/2, \left\lfloor \frac{m}{2} \right\rfloor = m/2 \). Hence we get

\[
I_{n,m} = 2^{m+n} (-1)^{m/2} \sum_{j=0}^{n/2} \left(\frac{(kn+k)}{2}\right) \prod_{j=0}^{n/2} \left(\frac{1}{\Gamma\left(\frac{(kn)/2-kj+1/2}{2}\right)}\right)
\]

\[
\sum_{r=0}^{m/2} \sum_{s=0}^{m/2-r} \frac{k}{\left(\frac{m}{2}-r\right)!} \left(\frac{m}{2}-r\right)^s \frac{s+1/2}{k} \frac{1}{\Gamma\left(m/2\right)}
\]

\[
= 2^{m+n} (-1)^{m/2} \sum_{j=0}^{n/2} \left(\frac{(kn+k)}{2}\right) \prod_{j=0}^{n/2} \left(\frac{1}{\Gamma\left(\frac{(kn)/2-kj+1/2}{2}\right)}\right)
\]

\[
\sum_{r=0}^{m/2} \sum_{s=0}^{m/2-r} (-1)^{s} \left(\frac{m/2-r}{k}\right) \left(\frac{s+1/2}{k}\right)_{m/2}
\]

\[
\sum_{r=0}^{m/2} \sum_{s=0}^{m/2-r} (-1)^{s} \left(\frac{m/2-r}{k}\right) \left(\frac{s+1/2}{k}\right)_{m/2}
\]
Using the identity (V.9) the above result reduces to

\[ I_{n,m} = 2^{m+n}(-1)^{m/2} \prod ((kn+k)/2) \sum_{j=0}^{n/2} (-1)^j \binom{n/2}{j} \binom{j-n/2}{m/2} \]

\[ = 2^{m+n}(-1)^{m+n/2} \prod ((kn+k)/2) \sum_{j=0}^{n/2} (-1)^j \binom{n/2}{j} \binom{m/2}{j-n/2} \]

\[ = 2^{m+n}(-1)^{n/2} \prod ((kn+k)/2)(m/2)! \sum_{j=0}^{n/2} (-1)^j \binom{n/2}{j} \binom{m/2}{j} \binom{j/2}{m/2}. \]

Since

\[ \sum_{j=0}^{n/2} (-1)^j \binom{n/2}{j} \binom{m/2}{j} = \sum_{j=m/2}^{n/2} (-1)^j \binom{n/2}{j} \binom{m/2}{j} \]

\[ = (-1)^{m/2} \binom{n/2}{m/2} (1-1)^{(n-m)/2}, \]

it is evident that

\[ (V.11) \quad I_{n,m} = 2^{2n} \prod ((kn+k)/2)(n/2)! \delta_{n,m}, \]

where \( \delta_{n,m} \) is the Kronecker's delta.

Similarly, if \( n, r \) both are odd, by noting that \( \lfloor n/2 \rfloor = (n-1)/2 \) and \( \lfloor m/2 \rfloor = (m-1)/2 \), it is easy to show in similar manner that

\[ (V.12) \quad I_{n,m} = 2^{2n} \prod (1+(kn+k)/2)((n-1)/2)! \delta_{n,m}. \]
Thus for any nonnegative integers \( n, m \) we have shown that

\[
(V.13) \quad I_{n,m} = 2^{2n} \sum \binom{\left\lfloor \frac{n/2}{} \right\rfloor}{n/2}! \delta_{n,m},
\]

in agreement with \((V.10)\). In particular,

\[
(V.14) \quad I_{n,n} = 2^{2n} \sum \binom{\left\lfloor \frac{n+1}{} \right\rfloor}{n/2}!.
\]

For \( k=1 \), \((V.10)\) is the orthogonality condition for the Hermite polynomials as both the sets get reduced to the Hermite polynomials for \( k=1 \).

3. Generating Functions and Recurrence Relations

We shall obtain in this section generating functions and recurrence relations for both the sets \( \{ S_n(x;k) \} \) and \( \{ T_n(x;k) \} \). The series technique alongside relations \((V.3)\) to \((V.8)\) would be employed. In fact, we would obtain generating relations for even polynomials and odd polynomials separately which would then be put in the compact form; in this connection see Brahman [1]. We begin first with the second set \( \{ T_n(x;k) \} \).

From the generating function (see [7])

\[223832\]
\[(V.15) \sum_{n=0}^{\infty} (m+n)^\alpha \frac{x^{2n}}{n!} t^n = (1-t)^{-(\alpha+mk+1)/k}\]

\[. \exp \left\{ x^2 \left[ 1-(1-t)^{-1/k} \right] \right\} \frac{\alpha}{m} (x^2(1-t)^{-1/k};k),\]

where \(m\) is any integer \(\geq 0\), we obtain readily, from \((V.15)\), in view of \((V.5)\) and \((V.6)\) the following generating relations:

\[(V.16) \sum_{n=0}^{\infty} T_{2m+2n}(x;t) t^n = (1+4t)^{-2-(mk+1/2)/k}\]

\[. \exp \left\{ x^2 \left[ 1-(1+4t^2)^{-1/k} \right] \right\} T_{2m}(x;k),\]

\[(V.17) \sum_{n=0}^{\infty} T_{2m+2n+1}(x;k) t^{2n+1}/n! = t \left(1+4t^2\right)^{-\left(mk+(k+1)/2\right)/k}\]

\[. \exp \left\{ x^2 \left[ 1-(1+4t^2)^{-1/k} \right] \right\} T_{2m+1}(x;k),\]

where

\[(V.18) \quad x = x(1+4t^2)^{-1/2k} \quad \text{in both (V.16) and (V.17).}\]

\((V.16)\) and \((V.17)\) can be combined fruitfully in the form

\[(V.19) \sum_{n=0}^{\infty} T_{2m+n}(x;k) t^n / [n/2]! = (1+4t^2)^{-(mk+(k+1)/2)/k}\]

\[. \exp \left\{ x^2 \left[ 1-(1+4t^2)^{-1/k} \right] \right\} \left[ (1+4t^2)^{1/2} T_{2m}(x;k) + tT_{2m+1}(x;k) \right],\]

where \(x\) is given by \((V.18)\).
Note that the generating function (V.19) can also be obtained independently. For \( k = 1 \) we have the following generating relation for the Hermite polynomials that seems to be new.

\[
(V.20) \sum_{n=0}^{\infty} \frac{H_{2m+n}(x)t^n}{[n/2]!} = (1+4t^2)^{-m+1} \exp \left\{ \frac{4x^2t^2}{(1+4t^2)} \right\} 
\cdot \left[ (1+4t^2)^{1/2} H_{2m}(x(1+4t^2)^{-1/2}) + t H_{2m+1}(x(1+4t^2)^{-1/2}) \right].
\]

From (V.19), with \( m = 0 \), we get

\[
(V.21) \sum_{n=0}^{\infty} \frac{T_n(x;k)t^n}{[n/2]!} = (1+4t^2)^{-k+1/2k} \left[ (1+4t^2)^{(k+1)/2k} + 2xt \right] \exp \left\{ x^2 \left[ 1 - (1+4t^2)^{-1/k} \right] \right\}.
\]

This may be obtained from the generating function given by Carlitz [2, p. 426, (8)] by using the same technique as used in deriving (V.19). For \( k = 1 \), (V.21) reduces to a generating function for Hermite polynomials given by Doetsch [3, p. 590, (7)]; see also Szego [8, Problem 24, p. 380].

So as to be able to obtain a generating relation for the first biorthogonal set suggested by the Hermite polynomials, we recall the following generating function obtained by Karande and Thakare [4].
(V.22) \[ \sum_{n=0}^{\infty} \frac{c_n Z_n(x;k)}{(1+\alpha)_{kn}} t^n = (1-t)^{-\alpha} \Gamma_k \left[ \begin{array}{c} c; \\ -x^k t^k / k^k (1-t) \end{array} \right] \Delta(k,1+\alpha); \]

where \( \Delta(m,\delta) \) stands for the sequence of \( m \) parameters

\[ \frac{\delta}{m}, \frac{\delta+1}{m}, \ldots, \frac{\delta+m-1}{m}, m \geq 1. \]

Using (V.3) we have

\[ \sum_{n=0}^{\infty} \frac{c_n S_{n2n}(x;k)}{n! (k/2)_{kn}} t^{2n} = \frac{\Gamma((k/2)}{\Gamma(1/2)} \sum_{n=0}^{\infty} \frac{c_n Z_n(x^2;k)}{(1/2)_{kn}} (-4t^2)^n. \]

Then using the result (V.22), we get

(V.23) \[ \sum_{n=0}^{\infty} \frac{c_n S_{n2n}(x;k)}{n! (k/2)_{kn}} t^{2n} = \frac{\Gamma((k/2)}{\Gamma(1/2)} (1+4t^2)^{-\alpha} \Gamma_k \left[ \begin{array}{c} c; \\ \Delta(k;1/2); \end{array} \right], \]

where

(V.24) \[ Y = 4 x^2 k^2 t^2 / k^2 (1+4t^2). \]

Next, it is easy to observe that

(V.25) \[ (1+\theta) \sum_{n=0}^{\infty} \frac{c_n S_{n2n+1}(x;k)}{n! (1+k/2)_{kn}} t^{2n} = \sum_{n=0}^{\infty} \frac{c_n S_{n2n+1}(x;k)}{n! (k/2)_{kn}} t^{2n}, \]

where \( \theta = t \, d/dt. \)
From (V.24), using (V.4) and then (V.22), we get

\[
\sum_{n=0}^{\infty} \frac{(c)_n s_{2n+1}(x;k)}{n! (k/2)_k n} t^{2n+1} = 2x^k t(1+\theta) \sum_{n=0}^{\infty} \frac{(c)_{n^2} (x^2;k)}{(1+k/2)_k n} (-4t^2)^n
\]

\[
= 2x^k t (1+\theta) (1+4t^2)^{-c} \left[ 1 F_k \left[ \begin{array}{c} c; \\ \Delta(k,1+k/2) \end{array} \right] \right],
\]

where \( Y \) is given by (V.24).

Operating \( \theta \) on the right hand side we readily get the generating function for the odd polynomials namely,

\[
(V.26) \sum_{n=0}^{\infty} \frac{(c)_n s_{2n+1}(x;k)}{n! (k/2)_k n} t^{2n+1} = \frac{2x^k t (1+4t^2-8ct^2)}{(1+4t^2)^{c+1}}.
\]

\[
\left[ 1 F_k \left[ \begin{array}{c} c; \\ \Delta(k,1+c/2) \end{array} \right] \right] + \frac{16c x^k t^3}{(1+k/2)_k (1+4t^2)^{c+2}} \left[ 1 F_k \left[ \begin{array}{c} c+1; \\ \Delta(k,1+3k/2) \end{array} \right] \right]
\]

where \( Y \) is again given by (V.24).

It can be seen that

\[
\sum_{n=0}^{\infty} \frac{(c)_n s_{2n}(x;k)}{n! (k/2)_k n} t^{2n} + \sum_{n=0}^{\infty} \frac{(c)_n s_{2n+1}(x;k)}{n! (k/2)_k n} t^{2n+1}
\]

\[
= \sum_{n=0}^{\infty} \frac{(c)[n/2]}{[n/2]! (k/2)_k [n/2]} S_n(x;k) t^n.
\]
In view of this identity and (V.23) and (V.26) we have the following generating relation for the first biorthogonal set \( \{ S_n(x; k) \} \):

\[
(V.27) \sum_{n=0}^{\infty} \binom{c}{[n/2]} \frac{S_n(x; k)}{[n/2]! (k/2)_k [n/2]} t^n
\]

\[
= \frac{P'(k/2)}{P'(1/2)} (1+4t^2)^{-c} \left[ \begin{array}{cc}
0; \\
\Delta(k,1/2);
\end{array} \right]_k Y
\]

\[
+ \frac{2x^k t (1+4t^2-8ct^2)}{(1+4t^2)^{c+1}} \left[ \begin{array}{cc}
0; \\
\Delta(k,1+k/2);
\end{array} \right]_k Y
\]

\[
+ \frac{16 \alpha^3 x^{3k} t^3}{(1+k/2)_k (1+4t^2)^{c+2}} \left[ \begin{array}{cc}
\alpha+1; \\
\Delta(k,1+3k/2);
\end{array} \right]_k Y
\]

where \( Y \) is given by (V.24).

For \( k=1 \), one obtains the generating function for the Hermite polynomials first obtained by Brafman [1, p.949, (33)].
From Konheuser [6, p. 305, (6)] we have

\[ \alpha x^2 D Z_n(x^2; k) = 2n k z_n(x^2; k) - 2k (kn-k+\alpha+1) z_{n-1}(x^2; k). \]

In view of (V.3) and (V.4), we then have

\[ \alpha x D S_{2n}(x; k) = 2n k S_{2n}(x; k) + 3k(n-k/2) S_{2n-2}(x; k); \text{ and} \]

\[ \alpha x D S_{2n+1}(x; k) = (2n+1) S_{2n+1}(x; k) + 3k(n-k+1/2) S_{2n-1}(x; k). \]

These two results can be combined in the form

(V.28) \[ \alpha x D S_n(x; k) = nk S_n(x; k) + 3k(\frac{n}{2})^k S_{n-2}(x; k), \]

where \( \epsilon \) has the meaning that we assigned in the beginning.

Next, in view of the following result due to Konheuser [6, p. 306, (3)]

\[ D^k x^{\alpha+1} D z_n(x; k) = -\frac{k}{2} x^\alpha (kn-k+\alpha+1) z_{n-1}(x; k) \]

and the fact

\[ (x^{\alpha+1} D^n f(x)) = x^{n+1} D^n \left \{ x^{n-1} f(x) \right \}, \quad n \geq 0, \]
we get
\[
(x^2D)^k \left\{ x^{-k+a+2} DZ_n^\alpha(x;k) \right\} = -k x^{a+k+1} (kn-k+a+1)_k \frac{\alpha}{\alpha} Z_{n-1}^\alpha(x;k).
\]

From which we have
\[
(V.29) \quad (x^2D)^k \left\{ x^{-2k+2a+3} DZ_n^\alpha(x^2;k) \right\} = -k x \frac{k+1}{2} x^{2k+2a+2} (kn-k+a+1)_k Z_{n-1}^\alpha(x^2;k).
\]

This readily gives in view of (V.3) a recurrence relation in the form
\[
(V.30) \quad (x^2D)^k \left\{ x^{-2k+2} DZ_{2n}^\alpha(x;k) \right\} = 8kn 2^k x^{2k+1} (kn-k/2)_k S_{2n-2}(x;k).
\]

Also by using \( x^{k+1} D = (xD-k) x^k y \), we obtain from (V.4) and (V.29)
\[
(V.31) \quad (x^2D)^k \left\{ x^{-2k+2} (xD-k) S_{2n+1}^\alpha(x;k) \right\} = 8kn x^{2k+2(kn+1-k/2)_k} S_{2n-1}(x;k).
\]

By combining (V.30) and (V.31), we get one more recurrence relation for the first set, namely
\[
(V.32) \quad (x^2D)^k \left\{ x^{-2k+2+\epsilon} \left( D - \frac{k\epsilon}{2} \right) S_{n}^\alpha(x;k) \right\} = 8k \left\lfloor \frac{n}{2} \right\rfloor 2^k x^{2k+1+\epsilon} \left( \frac{(n-1-\epsilon)k}{2} + \epsilon \right)_k S_{n-2}(x;k).
\]
Again using the result
\[ (x^3)^{\frac{1}{2}} (x^3 f(x)) = x^{\frac{3}{2}} [ x^3 D + 2 ] f(x) \] (See (III.2)),
the above recurrence relation reduces to
\[ (V.33) \quad [ x^2 (xD + \epsilon) ] \quad \frac{1}{\epsilon} \quad x^{-2k+2} (D - \frac{k\epsilon}{x}) S_n(x;k) \]
\[ = 8k \lfloor \frac{n}{2} \rfloor 2^k x^{2k+1} \left( \frac{2^{n-2k}}{2^n + \epsilon} \right) S_{n-2}(x;k). \]

Eliminating \( S_{n-2}(x;k) \) from (V.28) and (V.33), we get the differential equation satisfied by the first biorthogonal set suggested by the Hermite polynomials in the form:
\[ (V.34) \quad [ x^2(xD + \epsilon) ] \quad \frac{1}{\epsilon} \quad x^{-2k+2} (D - \frac{k\epsilon}{x}) S_n(x;k) \]
\[ = 2^k x^{2k+1} \left\{ x D S_n(x;k) - nk S_n(x;k) \right\}_j. \]

Direct proof of (V.34) is possible from Konheuser's result \[ 6, \text{ p. 306, (10)}. \]

For \( k = 1 \) result (V.34) yields differential equations for even and odd Hermite polynomials as follows
\[ (V.35) \quad D^2 H_{2m}(x) = 2x D H_{2m}(x) - 2(2m) H_{2m}(x), \]
\[(V.36) \ (xD+1) \left[ DH_{2m+1}(x) - \frac{1}{x} H_{2m+1}(x) \right] = 2x \left\{ xDH_{2m+1}(x) - (2m+1) H_{2m+1}(x) \right\}.\]

Differential equation \((V.36)\) can also be further simplified as follows:

\[(V.37) \ D^2 H_{2m+1}(x) = 2xD H_{2m+1}(x) - 2(m+1) H_{2m+1}(x).\]

Clearly \((V.35)\) and \((V.37)\) can be combined to have the following Hermite's differential equation:

\[D^2 H_n(x) - 2xD H_n(x) - 2nH_n(x) = 0.\]

When it comes to obtaining particular cases with \(k=1\), we shall always adopt such an approach without much explanation.

Recall the following recurrence relation involving the second Kostka set:

\[(V.38) \ k(n+1) Y_{n+1}^\alpha(x;k) = xD Y_n^\alpha(x;k) + (kn+\alpha+1-x) Y_n^\alpha(x;k).\]

This result was first proved by Kostka [6, p.308, eq.(16)].

Using our defining relations \((V.5)\) and \((V.6)\) in \((V.38)\) we are readily led to the following recurrence relations for even and odd members of the second set \(\left\{ T_n(x;k) \right\}^\alpha\):
\[(V.39) \quad k T_{2n+2}(x;k) = -2x DT_{2n}(x;k) - 2(2kn+1-2x^2) T_{2n}(x;k),\]

and

\[(V.40) \quad k T_{2n+3}(x;k) = -2x DT_{2n+1}(x;k) - 2[k(2n+1)+1-2x^2] T_{2n+1}(x;k).\]

By combining last two results, we have

\[(V.41) \quad k T_{n+2}(x;k) = -2x DT_n(x;k) - 2(kn+1-2x^2) T_n(x;k).\]

We shall also give an independent proof of the differential recurrence relation \((V.41)\) in the next chapter.

4. Concluding Remarks

1. The classical orthogonal polynomials namely, the Jacobi polynomials, the Laguerre polynomials and the Hermite polynomials have several interesting characteristic properties in common. Now the biorthogonal pairs suggested by the classical orthogonal polynomials have been satisfactorily constructed, it would be worthwhile to find common properties, if they exist, of the biorthogonal pairs suggested by the classical orthogonal polynomials.

2. One also observes that nothing much has been added to the general theory of biorthogonal polynomials since the fundamental work of Konhauser [5].
REFERENCES


