CHAPTER IV

Bilateral Generating Functions For Biorthogonal Polynomials Suggested by the Jacobi Polynomials
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1. Introduction

Burchnall [1] observed that operational technique facilitated obtaining a few interesting generating functions for the standard Hermite polynomials. We saw in the previous chapter the validity of the statement of Burchnall in the context of our second biorthogonal polynomial set. The main message of this chapter is to bring out again the power of operational techniques for obtaining bilateral generating functions for our pair of biorthogonal polynomials.

We shall put to fruitful use the differential operator \( \theta \) defined by (III.1). At the outset we give in this chapter one more proof of the generating function (II.11) involving the second set \( \{ K_n(\alpha, \beta, k; x) \} \). In chapter-II we obtained this generating function by using series technique implicitly. In fact it became possible for us to obtain Rodrigues' type formula (II.16) from (II.11). However, here we use (II.16) to obtain the generating function (II.11). It again needs to be emphasized here that Rodrigues' type formula (II.16) could be obtained independently (see chapter-III, formula (III.6)) by exploiting the properties of the differential operator \( \theta \). We are readily led to one more proof of the generating function (II.9) by using earlier obtained Szegö-type
relation (II.17) for the second biorthogonal set. This considerations permit us to obtain a family of bilateral generating functions for the second set and the functions of several variables that will be defined later. As applications we give several bilateral generating functions involving both the biorthogonal polynomial sets
\[ \{ K_n(\alpha, \beta, k; x) \} \quad \text{and} \quad \{ J_n(\alpha, \beta, k; x) \}. \]

2. Generating Functions

In this section we give an additional proof of the generating function (II.11) which we state in the form of the following Lemma.

Lemma: The second biorthogonal set \( \{ K_n(\alpha, \beta, k; x) \} \) suggested by the Jacobi polynomials satisfy the following generating relation.

\[
(IV.1) \sum_{n=0}^{\infty} \binom{m+n}{n} K_{m+n}(\alpha, \beta-m-n, k; x) t^n = \left( \frac{x+1}{2} \right)^{m-\beta} \left( 1 - \frac{x+1}{2} t \right)^{-(1+\alpha+\beta+mk-m)/k} \left[ \frac{x-1}{2} + \left( 1 - \frac{x+1}{2} t \right) \right]^{1/k} K_m(\alpha, \beta-m, k; 1+(x-1)(1+ \frac{x+1}{2} - 1/k).\]

Proof. Using (III.6), we find that

\[
\sum_{n=0}^{\infty} \binom{m+n}{n} R_{m+n}^{(m,n)}(\alpha, \beta-m-n, k; 1-x) t^n
\]

\[
= (2-x)^{m-\beta} x^{-1-\alpha-mk} \frac{1}{m! (2k)^m} \sum_{n=0}^{\infty} \binom{2-x}{n} x^{-kn} t^n \theta^n \theta^m \left\{ (2-x)^{\beta} x^{1+\alpha} \right\}
\]

\[
= (2-x)^{m-\beta} x^{-1-\alpha-mk} \exp \left\{ (2-x)x^{k_0} t \theta / 2k \right\} (2-x)^{\beta-m} x^{1+\alpha+mk} K_m(\alpha, \beta-m, k; 1-x).
\]

Now applying the result (III.5) the above expression becomes

\[
(2-x)^{m-\beta} (2-x(1 - \frac{2-x}{2} x^{-1/k})^{(1-\frac{2-x}{2} x^{-1/k})} (1 - \frac{2-x}{2} x^{-1/k})^{-(1+\alpha+km)/k}.
\]

\[
K_m(\alpha, \beta-m, k; 1-x (1 - \frac{2-x}{2} x^{-1/k})).
\]

Thus we obtain

\[
\sum_{n=0}^{\infty} \binom{m+n}{n} R_{m+n}^{(m,n)}(\alpha, \beta-m-n, k; 1-x) t^n
\]

\[
= \frac{(2-x)^{m-\beta}}{2} \left[ (1 - \frac{2-x}{2} x^{-1/k})^{1/k} - \frac{x}{2} \right]^{\beta-m} x^{1+\alpha+km} x^{-1/k}.
\]

\[
K_m(\alpha, \beta-m, k; 1-x(1 - \frac{2-x}{2} x^{-1/k})).
\]

This can be put in the desired form (IV.1).
In view of Szegö-type relation (II.17) an alternative formulation (III.9) of the generating function (IV.1) can be stated as

\[(IV.2) \sum_{n=0}^{\infty} \binom{m+n}{n} K_{m+n}(\alpha-km-kn, \beta-mn, k; x)t^n\]

\[= (x+1)^{m-\beta} \left[ 2+(x-1)(1+\frac{x+1}{2}t)^{1/k} \right]^{\beta-m} (1+\frac{x+1}{2}t)^{-1-m+(1+\alpha)/k}\]

\[K_m(\alpha-km, \beta-m, k; 1+(x-1)(1+\frac{x+1}{2}t)^{1/k})\].

3. Bilateral Generating Functions

The utility of (IV.1) will be exhibited in this section for obtaining bilateral generating relations involving the second biorthogonal set suggested by the Jacobi polynomials. For our formulation we shall need the following functions of several variables that were first discussed by Srivastava [4].

Let the functions \(S_n(\bar{y}, z)\) of several variables \(y_1, y_2, \ldots, y_N, (n \geq 1)\) and \(z\) be defined as follows:

\[(IV.3) \quad S_n(\bar{y}, z) = \sum_{r=0}^{[n/q]} \binom{m+n}{m-n} \lambda_r \delta_{r}(\bar{y}) z^r,\]

where \(\delta_n(\bar{y})\) is a nonvanishing function of \(N\) variables \(y_1, y_2, \ldots, y_N, (n \geq 1)\), \(q\) is a nonzero positive integer and \(\lambda_r\) are nonzero arbitrary constants.
A family of bilateral generating functions involving $K_n(\alpha, \beta, k; x)$ and $S_n(\overline{y}, z)$ will now be obtained in the form of the following Theorem.

**Theorem**: The second biorthogonal set suggested by the Jacobi polynomials $\{K_n(\alpha, \beta, k; x)\}$ and the functions $S_n(\overline{y}, z)$ satisfy the following bilateral generating function

\[
(IV.4) \sum_{n=0}^{\infty} K_{m+n}(\alpha, \beta-n, k; x) S_n(\overline{y}, z)t^n
\]

\[
= (\frac{x+1}{2})^{-\beta} \left[ \frac{x-1}{2} + \left(1 - \frac{x+1}{2}t\right)^{1/k}\right]^\beta \left[1 - \frac{x+1}{2}t\right]^{-(1+\alpha+\beta+mk)/k}
\]

\[
\left[ \Gamma \left[1+(x-1)(1 - \frac{x+1}{2}t)^{-1/k} ; \overline{y} ; z x^q \right] \right]
\]

where

\[(II.5) X = t(x+1)(1 - \frac{x+1}{2}t)^{-1+1/k}\left[1 - \frac{x+1}{2}t\right]^{1/k}\]

\[
\left[ \Gamma \left[x; \overline{y}; t\right] = \sum_{n=0}^{\infty} \chi_n K_{m+qn}(\alpha, \beta-n, k; x) \delta_n(\overline{y})t^n \right]
\]

and $m$ is a nonnegative integer.

**Proof**: Use of (IV.3) gives us
\[
\sum_{n=0}^{\infty} K_{m+n}(\alpha, \beta-m-n, k; x) \, S_n(\bar{y}, z) \, t^n
\]

\[= \sum_{n=0}^{\infty} K_{m+n}(\alpha, \beta-m-n, k; x) \, t^n \left[ \sum_{r=0}^{n/q} \binom{n}{nqr} \chi_r \, \delta_r(\bar{y}) \, z^r \right] \]

\[= \sum_{r=0}^{\infty} \chi_r \, \delta_r(\bar{y}) \, z^r \, t^{qr} \sum_{n=0}^{\infty} \binom{m+qr+n}{n} \, r_{m+qr+n}(\alpha, \beta-m-qr-n, k; x) \, t^n.\]

Applying (IV.1) to inner series, we can obtain (IV.4) after usual simplifications.

Use of Szegö-type relation (II.17) in the theorem permits us to have an equivalent formulation of the said bilateral generating function in the form,

\[(IV.6) \sum_{n=0}^{\infty} K_{m+n}(\alpha-nk, \beta-n, k; x) \, S_n(\bar{y}, z) \, t^n \]

\[= \left( -\frac{x+1}{2} \right)^{-\beta} \left[ 1 + \frac{x-1}{2} (1 + \frac{x+1}{2} t)^{1/k} \right]^\beta \left( 1 + \frac{x+1}{2} t \right)^{-1+(1+\alpha)/k}. \]

\[\cdot \left[ 1+(x-1)(1 + \frac{x+1}{2} t)^{1/k} ; \bar{y} ; z y^q \right], \]

where

\[(IV.7) y = t(x+1)/(1+\frac{x+1}{2} t) \left[ 2+(x-1)(1 + \frac{x+1}{2} t)^{1/k} \right] \quad \text{and} \]

\[H[x; y; t] = \sum_{n=0}^{\infty} \chi_n \, \delta_n(\bar{y}) \, t^n. \]
The above result could also be obtained independently by using (IV.2) and proceeding as in the proof of (IV.4).

One observes that with the substitutions $m=0$, $k=1$ and $\delta_n(y) = 1$, for all $n$ in (IV.4) and (IV.6), one obtains known results for the Jacobi polynomials; see Srivastava and Lavoie [3, Cor. 2, Cor. 3].

4. Applications

Our considerations of section 3 would be shown to have several interesting applications. First we begin with showing that (III.4) and (III.6) entail as limiting cases bilateral generating functions for the second Konhauser set

\[ \{ c_n(x; \gamma) \} \]

By resorting to (I.5) one can easily show after some computation,

\[ \lim_{\beta \to \infty} K_n(\alpha, \beta, \gamma; 1 - \frac{2x}{\beta} (1 - t + \frac{x}{\beta} t)^{-1/\beta} ) = Y_n(x(1-t)^{-1/\beta}) \]

Also the expression

\[ \left( \frac{x+1}{2} \right)^{-\beta} \left[ \frac{x-1}{2} + (1 - \frac{x+1}{2} t)^{1/\beta} \right]^{\beta} (1 - \frac{x+1}{2} t)^{-(1+\alpha+\beta+\beta)/\beta} \]

becomes after replacing $x$ by $(1 - \frac{2x}{\beta})$.
\[(1 - \frac{x}{\beta}) (1 - t + \frac{x}{\beta} t)^{-(1+\alpha+mk)/k} \left[ 1 - \frac{x}{\beta} (1 - t + \frac{x}{\beta} t)^{-1/k} \right]^\beta.\]

In the limit $\beta \to \infty$ the above expression reduces to

\[
\exp(x)(1-t)^{-(1+\alpha+mk)/k} \lim_{\beta \to \infty} \left[ 1 - \frac{x}{\beta} (1 - t + \frac{x}{\beta} t)^{-1/k} \right]^\beta
\]

\[
= \exp(x)(1-t)^{-(1+\alpha+mk)/k} \lim_{\beta \to \infty} \sum_{n=0}^{\infty} \frac{(-\beta)^n}{n!} \left( \frac{x}{\beta} \right)^n (1-t)^{-n/k}
\]

\[
= \exp(x)(1-t)^{-(1+\alpha+mk)/k} \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} (1-t)^{-n/k}
\]

\[
= (1-t)^{-(1+\alpha+mk)/k} \exp(x [1 - (1-t)^{-1/k}]).
\]

Then (IV,4) yields in conjunction with (I.7) the following bilateral generating function for the Konhauser set

\[
\{ Y_n(x;k) \}.
\]

(IV.8) \[
\sum_{n=0}^{\infty} \frac{\alpha}{m+n} (x;x) S_n(\bar{y};z) t^n = (1-t)^{-(1+\alpha+mk)/k}
\]

\[
\cdot \exp(x [1 - (1-t)^{-1/k}] u[x(1-t)^{-1/k}; \bar{y}; z t/(1-t)^q],
\]

where

\[
u[x; \bar{y}; t] = \sum_{n=0}^{\infty} \gamma_n \frac{\alpha}{m+n} (x;k) \delta_n (\bar{y}) t^n.
\]

Similarly from (IV.2), we can obtain
\[(IV.9) \sum_{n=0}^{\infty} \frac{\alpha-kn}{\gamma_m+n} (x;k) s_n(y;z) t^n = (1+t)^{-1+(1+\alpha)/k} \exp(x[(1-(1+t)^{1/k}]) v[x(1+t)^{1/\tau}; y; zt^q/(1+t)^q], \]

where

\[v[x; y; t] = \sum_{n=0}^{\infty} \frac{\gamma_n}{\gamma_m+n} (x;k) \delta_n(y)t^n.\]

Results (IV.8) and (IV.9) are essentially the main results very recently obtained by Srivastava [4].

Taking \(m=0\), \(q=1\) = \(\delta_n(y)\) and \(\gamma_n = (1+\gamma+\delta+n)_{\gamma_n}/(1+\gamma)_{\gamma_n}\), in (IV.4) and (IV.6) and using (I.3), we get respectively the following bilateral generating relations for the biorthogonal polynomial sets suggested by the Jacobi polynomials

\[(IV.10) \sum_{n=0}^{\infty} \frac{n!}{(1-y)^{\gamma_n}_{\gamma_n}} x_n(\alpha, \beta-n,k;x) j_n(y, \delta, p;y) t^n \]

\[= \left(\frac{x+1}{2}\right)^{-\beta} \left[ \frac{x-1}{2} + (1 - \frac{x+1}{2} t)^{1/k} \right]^{\beta} \left(1 - \frac{x+1}{2} t\right)^{-(1+\alpha+\beta)/k} \cdot G[1+(x-1)(1 - \frac{x+1}{2} t)^{-1/\kappa}, -(\frac{1-y}{2})^p X];\]

and
\[(IV.11) \quad \sum_{n=0}^{\infty} \frac{n!}{(1+y)^n} \frac{K_n(\alpha-nk, \beta-n,k;x)J_n(\gamma, \delta,p;y)}{p^n} \]

\[= \left( \frac{x+1}{2} \right)^{-\frac{\beta}{2}} \frac{1}{1+\frac{x-1}{2}(1+\frac{x+1}{2}t)} \frac{1}{\left(1+\frac{x+1}{2}t\right)^{1/k}} \left[ 1+(1+\alpha)/k \right]^{\beta} \left[ \frac{1}{1+\frac{x-1}{2}t} \right]^{1/k} \times \left[ -\left(\frac{1-y}{2}\right)^p \right] \]

where \(X\) and \(Y\) are respectively given by \(\text{(IV.5)}\) and \(\text{(VI.7)}\), and

\[G[x,t] = \sum_{n=0}^{\infty} \frac{(1+y+\delta+n)}{\left(1+y\right)^n} \frac{K_n(\alpha-n,k;x)}{p^n} t^n \quad \text{and} \]

\[M[x,t] = \sum_{n=0}^{\infty} \frac{(1+y+\delta+n)}{\left(1+y\right)^n} \frac{K_n(\alpha-nk, \beta-n,k;x)}{p^n} t^n \]

In fact \(\text{(IV.10)}\) and \(\text{(IV.11)}\) are equivalent due to the Szegő-type relation \(\text{(II.17)}\). For \(p=k=1\) we get corresponding results for the Jacobi polynomials.

Making use of \(\text{(I.6)}\), \(\text{(I.7)}\) and \(\text{(I.5)}\) in \(\text{(IV.10)}\) and \(\text{(IV.11)}\) we have respectively the generating relation for the product \(Y_n^\alpha(x;k)Z_n^\beta(x;\lambda)\) (see Srivastava \[2, \text{p.} 491, \text{eq.} (14)\]) and the generating function for the product \(Y_n^{\alpha-kn}(x;k)Z_n^\beta(y;\lambda)\) (see Srivastava \[4, \text{p.} 199, \text{eq.} (11)\]).

It is observed that on specialization of the functions \(S_n(\bar{y},z)\), \(\text{(IV.4)}\) and \(\text{(IV.6)}\) would yield a great variety of bilateral generating functions involving the polynomials.
\( K_n(\alpha, \beta-n, k; x) \) and \( K_n(\alpha-nk, \beta-n, k; x) \), respectively. In fact if \( \delta_n(y) \) can be written as product of several simple functions, then our results also include a large variety of multilateral generating functions. However, we shall not go into those details.

5. Generalization.

We shall indicate in this section a possible generalization of our formulas (IV.4) and (IV.6).

More generally, let \( \{S_n(x) \mid n=0, 1, 2, \ldots\} \) be generated by

\[
\sum_{n=0}^{\infty} A_{m,n} S_{m+n}(x) t^n = f(x,t) \left\{ g(x,t) \right\}^{-m} S_m(h(x,t)),
\]

where \( m > 0 \) is an integer, the \( A_{m,n} \) are arbitrary constants and \( f, g, h \) are arbitrary functions of \( x \) and \( t \), then

\[
\sum_{n=0}^{\infty} S_{m+n}(x) \mu(y;z) t^n = f(x,t) \left\{ g(x,t) \right\}^{-m} W[h(x,t); y; z(t/g(x,t))],
\]

where
\[ w[ x; \bar{y}; t ] = \sum_{n=0}^{\infty} \lambda_n S_{m+qn}(x) \delta_n(\bar{y}) t^n , \]

\[ \mathcal{M}(\bar{y};z) = \sum_{r=0}^{[n/q]} \lambda_r A_{m+qr, n-qr} \delta_r(\bar{y}) z^r ; \text{ and} \]

\[ q, \lambda_n \text{ and } \delta_n(\bar{y}) \text{ are the same as in (IV.4).} \]

For \( m=0 \), \( \delta_n(\bar{y}) = 1 \), we get theorem-1 of Srivastava and Lavoie [3]. Also taking \( \lambda_n \delta_n(\bar{y}) = \delta_{m,n} \neq 0 \), we obtain the result (107) of Srivastava and Lavoie [3].

**Remarks:** Subsequently we have also obtained several multilinear and multilateral generating functions involving both the biorthogonal polynomial sets suggested by the Jacobi polynomials which have not been included in this Thesis in order to keep the size of the Thesis within limits.
REFERENCES.


