PART-A
A Study of Biorthogonal Polynomials
PART A

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1. Introduction

Let $H$ be a Hilbert space with the inner product denoted by $(,) \text{ over real or complex field}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $H$ such that

$$(x_n, y_n) = \delta_{n,m} = \begin{cases} 0 & \text{if } n \neq m; \\ 1 & \text{if } n = m, \end{cases}$$

then these two sequences form a pair of biorthogonal system in $H$. One can note that biorthogonal systems in a Hilbert space always exist; for example two replicas of an orthogonal system in $H$ would form a biorthogonal system. Such a theoretical formulation in abstract context is known for several years (see for example Schmeidler [19], Higgins [4]; et al).
This notion of biorthogonality was used in 1951 by two physicists named Spencer and Fano [20] in carrying out calculations involving the penetration of gamma rays through matter.

It is a classical result that the system of Hermite polynomials $H_n(x)$ forms a complete orthogonal system in the Hilbert space $L_2(-\infty, \infty)$. Similarly the Laguerre polynomials and the Jacobi polynomials form complete orthogonal systems in the Hilbert spaces $L_2(0, \infty)$ and $L_2(-1, 1)$ respectively. In fact, there are several concrete examples of such complete orthogonal systems in Hilbert space $L_2(a,b)$ where $a, b$ are real parameters with $a < b$. However, for several years such a concretization of the notion of biorthogonality in Hilbert space $L_2(a,b)$ was not known. Perhaps Konhauser [9, 10] was the first to build up a systematic study of general properties of biorthogonal polynomial sets in $L_2(a,b)$. He also succeeded in constructing a pair of biorthogonal polynomials in the Hilbert space $L_2(0, \infty)$. The problems of constructing biorthogonal systems of polynomials in Hilbert spaces $L_2(-1, 1)$ and $L_2(-\infty, \infty)$ remained open for several years. In the first part of the Thesis we amicably settle both these questions.
2. Preliminary Definitions

We shall now briefly review the notion of biorthogonal sets in the Hilbert space $L_2(a,b)$.

Let $r(x)$ and $s(x)$ be real polynomials in $x$ of degree $h > 0$ and $k > 0$, respectively. Let $R_m(x)$ and $S_n(x)$ denote polynomials of degree $m$ and $n$ in $r(x)$ and $s(x)$, respectively; thus $R_m(x)$ and $S_n(x)$ are polynomials of degree $mh$ and $nk$ in $x$. The polynomials $r(x)$ and $s(x)$ are called basic polynomials.

Definition:

The real-valued function $p(x)$ of real variable $x$ is an admissible weight function on the finite or infinite interval $(a,b)$ if all the moments

$$M_{i,j} = \int_a^b p(x) \left\{ r(x) \right\}^i \left\{ s(x) \right\}^j \, dx$$

$i, j = 0, 1, 2, ...$,

exist, with

$$M_{0,0} = \int_a^b p(x) dx \geq 0.$$
Definition:

The polynomial sets \( \{ R_m(x) \} \) and \( \{ S_n(x) \} \) are biorthogonal over the interval \((a, b)\) with respect to the admissible weight function \( p(x) \) and the basic polynomials \( r(x) \) and \( s(x) \) provided the conditions

\[
(A.1) \quad I_{m,n}^b = \int_a^b p(x) R_m(x) S_n(x) \, dx
\]

\[= 0, \text{ if } m, n = 0, 1, 2, \ldots, m \neq n;\]

\[\neq 0, \text{ if } m = n,\]

are satisfied.

Konhauser \([9]\) established several general properties of biorthogonal polynomial sets in the Hilbert space \( L_2(a, b) \) such as necessary and sufficient condition for the existence of biorthogonal polynomials, zeros of biorthogonal polynomials, pure recurrence relations etc. He also explicitly demonstrated the equivalence of \((A.1)\) with the following two conditions:

\[
(A.2) \quad \int_a^b p(x) \left[ x(x) \right]^j S_n(x) \, dx
\]

\[= 0, \text{ if } j = 0, 1, 2, \ldots, n-1;\]

\[\neq 0, \text{ if } j = n,\]

and
\[(A.3) \int_a^b p(x) \left[ s(x) \right]^j R_m(x) \, dx = 0, \text{ if } j = 0, 1, 2, \ldots, m-1; \]
\[ \neq 0, \text{ if } j = m. \]

We shall have an occasion to use both these equivalent formulations.

In 1967 Konhauser [10] constructed a pair of biorthogonal polynomials \( Z_n^\alpha(x;k) \) and \( Y_n^\alpha(x;k) \) of degree \( n \) in \( x^k \) and \( x \), respectively, where \( x \) is real, \( k \) is a positive integer and \( \alpha > -1 \). This pair of polynomials is biorthogonal with respect to the weight function \( x^\alpha e^{-x} \) over the interval \((0, \infty)\).

We shall now state explicitly these two polynomial sets:

\[(A.4) Z_n^\alpha(x;k) = \frac{\int (kn+\alpha+1)}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{x^j}{\int (kj+\alpha+1)}, \quad [10, \text{p.304}], \]

\[(A.5) Y_n^\alpha(x;k) = \frac{1}{n!} \sum_{r=0}^{n} \frac{x^r}{r!} \sum_{s=0}^{r} (-1)^s \binom{r}{s} \left( \frac{s+\alpha+1}{k} \right)_n; \quad [1, \text{p.427}]. \]

Hence afterwards, \( Z_n^\alpha(x;k) \) and \( Y_n^\alpha(x;k) \) would be called Konhauser biorthogonal set of first and second kind respectively.

For \( k=1 \) both these sets are reduced to the Laguerre polynomials \( L_n^\alpha(x) \); for \( k=2 \) these are the polynomials considered by Preiser [17, 18]. Several others enriched the theory of
these biorthogonal sets, for example Carlitz [1], Prabhakar [13, 14], Srivastava [21, 22], Karande and Thakare [5, 6], Patil and Thakare [11, 12], Karande and Patil [7, 8], Srivastava and Singh [23].

It will be very close to the truth that the above was the only concrete realization of the abstract notion of biorthogonality. In fact, the afore-mentioned list of references is almost complete.

However, there were certain scattered efforts of constructing pairs of biorthogonal polynomials which were suggested by the Jacobi polynomials. Perhaps the only attempt in this direction needs to be credited to Chai [3] who posed the following problem in the section of Problems and Solutions in the SIAM Reviews of 1972:

"Problem 72-17: Consider the polynomial defined by the hypergeometric function

\[(6) \quad Z_n(x;k) = F(-n, 1+\alpha+\beta, 1+\alpha; x^k)\]

\[= \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{(1+\alpha+\beta+n)_{kj}}{(1+\alpha)_{kj}} x^j,\]

where \(\alpha > -1, k\) and \(\beta\) are nonnegative integers and
(a) \( j = a(a+1) \ldots (a+j-1) \), \( (a)_o = 1 \). For \( k = 1 \), equation (A.6) becomes the Jacobi polynomial satisfying the orthogonal conditions

\[
\begin{align*}
(A.7) \quad & \int_0^1 x^\alpha (1-x)^\beta Z_n(x;k)x^idx \\
& = 0, \quad i=0,1,\ldots,n-1, \\
& \neq 0, \quad i = n.
\end{align*}
\]

Show that the latter conditions are also valid for any positive integer \( k \)."

The importance of this problem prompted the editors to append the following comment:

"Editorial note: The author notes that the biorthogonality conditions are useful in the computations involving the penetration of gamma rays through matter as well as in determining the moments of a hypergeometric distribution function."

An ingenious and direct proof to the problem of Chai was supplied in 1973 by Carlitz [2].

Prabhakar and Kashyap [16] attempted to construct a pair of biorthogonal polynomials with respect to the weight function \( (1-x)^\alpha \) over the interval \((-1,1)\). Unfortunately they did not realize that this pair is just two replicas of an orthogonal system. Let us elaborate it. In particular, Prabhakar and Kashyap [16] have discussed the following pair of polynomials:
\[(A.8) \quad U_n(x;k) = \frac{1}{(1/k)_n} \sum_{j=0}^{n} \frac{(-n)^j}{j!} \left(\frac{1+\alpha+j}{k}\right)_n \left(\frac{1-x}{2}\right)^j,
\]

and

\[(A.9) \quad V_n(x;k) = \frac{1}{n!} \sum_{j=0}^{n} \frac{(-n)^j}{j!} \left(1+\alpha+kj\right)_n \left(\frac{1-x}{2}\right)^{kj}.
\]

It is a simple matter to observe that these two sets are related to each other in the manner indicated below and as such are not independent of each other:

\[(A.10) \quad \alpha V_n(x;k) = \frac{(k)_n}{n!} \frac{(\alpha+1-k)/k}{(1-2(1-x/2);1/k)} U_n\]

What we have observed above remains valid for the pair of biorthogonal polynomials constructed by Prabhakar and Tomar \[15\] that were suggested by the Legendre polynomials as they are special cases of \((A.8)\) and \((A.9)\) with \(\alpha=0\).

We shall satisfactorily and adequately solve the problem of constructing a biorthogonal pair of polynomials suggested by the Jacobi polynomials in Chapter-I, besides obtaining several properties of this pair in Chapters I-IV.

Apropos to the problem of finding a pair of biorthogonal polynomials suggested by the Hermite polynomials we could not lay our hands on any previous literature. We are happy to state that we have succeeded in solving this particular problem completely in Chapter-V. Several of their properties are obtained in Chapter-VI.

The relevant details are postponed to respective chapters.
REFERENCES


CHAPTER-I

Biorthogonal Polynomials
Suggested by the
Jacobi Polynomials

A paper based on the text of this chapter has been accepted for Publication in Pacific Journal of Mathematics, U.S.A.
CHAPTER I

1. Introduction

In the beginning chapter of the Thesis we shall introduce and study a pair of biorthogonal polynomials that are suggested by the classical Jacobi polynomials \( P_n^{(\alpha, \beta)}(x) \). Thus we put to rest the question of finding such a pair of polynomials.

There is a famous result originally due to Feldheim [5] which connects the classical Jacobi polynomials \( P_n^{(\alpha, \beta)}(x) \) with the classical Laguerre polynomials \( L_n^\alpha(x) \) in the following manner:

\[
(I.1) \quad \Gamma(1+\alpha+\beta+n) P_n^{(\alpha, \beta)}(x) = \int_0^\infty t^{\alpha+\beta+n-1} e^{-t} L_n^\alpha(\frac{1-x}{2}t) dt,
\]

\( \alpha+\beta > -1, \quad n=0,1,2,\ldots \).

The Jacobi polynomials and the Laguerre polynomials are defined as follows:

\[
P_n^{(\alpha, \beta)}(x) = \sum_{k=0}^{n} \frac{(1+\alpha)_n(1+\alpha+\beta)_n}{k!(n-k)!} \frac{n!k}{(1+\alpha)_k(1+\alpha+\beta)_n} \left( \frac{x-1}{2} \right)^k; \quad [12, p. 255],
\]

\[
L_n^\alpha(x) = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n}{k!(n-k)!} \left( \frac{x}{1+\alpha} \right)^k; \quad [12, p. 201].
\]
The result of Feldheim served as a motivation for us to define the first set \( \{ J_n(\alpha, \beta, k; x) \} \) with the help of the following Feldheim-type relation involving the Konhauser biorthogonal set of the first kind \( Z_\alpha^n(x; k) \) (see A.4):

\[
(I.2) \quad \int_0^\infty e^{-x} \sum_{n=0}^\infty n \frac{(1+\alpha+\beta+n)}{n!} (1-\frac{x}{2})^n J_n(\alpha, \beta, k; x) dt,
\]

\( \alpha+\beta > -1, \ n = 0, 1, 2, \ldots \); and \( k \) is a positive integer.

Using (A.4) we obtain by routine calculations the following expansion formula for the first set \( \{ J_n(\alpha, \beta, k; x) \} \):

\[
(I.3) \quad J_n(\alpha, \beta, k; x) = \frac{(1+\alpha) n!}{2} \sum_{j=0}^n \frac{(1+\alpha+\beta+n)_{kj}}{(1+\alpha)_{kj}} (1-x)_{kj}^{-1}.
\]

For \( k=1 \) \( J_1(\alpha, \beta, k; x) \) is \( p_1(\alpha, \beta)(x) \). Thus for \( k=1 \) (I.2) yields (I.1).

It is also possible to put \( J_n(\alpha, \beta, k; x) \) in the following hypergeometric form:

\[
(I.4) \quad J_n(\alpha, \beta, k; x) = \frac{(1+\alpha) n!}{2} \left\lfloor \begin{array}{c} m, \\ \Delta(k, 1+\alpha+\beta+n); \\ \Delta(k, 1+\alpha); \\ (1-x)^{-1} \end{array} \right\rfloor,
\]

where \( \Delta(m, \delta) \) stands for the sequence of \( m \) parameters

\[
\delta, \delta+1, \ldots, \delta+m-1, m \geq 1.
\]
It is now needless to say that our polynomials $J_n(\alpha, \beta, k; x)$ are essentially the same as the polynomials $Z_n(x; k)$ introduced by Chai [4] and this fact was mentioned in the general introduction of part A of the Thesis.

Carlitz [3] supplied the proof for the biorthogonality of the polynomials $Z_n(x; k)$ to $x^i$ with respect to $x^\alpha(1-x)^\beta$ over the interval $(0,1)$. This condition is of the type (A.2). Thus one observes that Chai's proposal was on $(0,1)$ as against our proposal on $(-1,1)$ that will be explicitly established later.

This incidentally reminds one of the transition of the classical Jacobi polynomials first denoted by $P_n(\alpha, \beta; x)$ and orthogonal with respect to the weight function $x^\alpha(1-x)^\beta$ on $(0,1)$ to that of Szegö's standardized Jacobi polynomials $P_n(\alpha, \beta)(x)$ which are orthogonal with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ over the interval $(-1,1)$.

We now introduce the second set $\{K_n(\alpha, \beta, k; x)\}$ in the form of the following explicit series representation

\[(I.5) \quad K_n(\alpha, \beta, k; x) = \sum_{r=0}^{n} \sum_{s=0}^{r} (-1)^s (r)_s (1+\beta)_n \frac{n-r}{r} \frac{r+s+1}{k} \frac{x-1}{n} \frac{x+1}{2} \frac{n-r}{n-r} \]

\[= \sum_{r=0}^{n} \sum_{s=0}^{r} (-1)^s (r)_s (1+\beta)_n \frac{n-r}{r} \frac{r+s+1}{k} \frac{x-1}{n} \frac{x+1}{2} \frac{n-r}{n-r} \]
It is to be noted that for $k = 1$, $K_n(\alpha, \beta, k; x)$ also become the standard Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$. In fact with $k = 1$, (1.5) can be transcribed in the form

\[
K_n(\alpha, \beta, 1; x) = \frac{1}{n!} \sum_{r=0}^{\infty} \frac{r}{s!} \frac{(1+\alpha)_{n+s}}{(1+s)!} \frac{(-\beta-n)^r}{r!} \left( \frac{x-1}{2} \right)^r \left( \frac{x+1}{2} \right)^{n-r} \]

\[
= \frac{(1+\alpha)^n}{n!} \left( \frac{x+1}{2} \right)^n \sum_{r=0}^{\infty} \frac{r}{s!} \frac{(-n)^r}{(1+\alpha)_r} \frac{(-\beta-n)^r}{r!} \left( \frac{x-1}{2} \right)^r \left( \frac{x+1}{2} \right)^{n-r} \]

Now, in view of the known result

\[
\genfrac{[}{]}{0pt}{}{-n, b;}{c; 1} = \frac{(c-b)_n}{(c)_n}; \text{ see } [12, \text{problem 4, p. 69}],
\]

above expression becomes

\[
= \frac{(1+\alpha)^n}{n!} \left( \frac{x+1}{2} \right)^n \sum_{r=0}^{\infty} \frac{r}{s!} \frac{(-n)^r}{(1+\alpha)_r} \frac{(-\beta-n)^r}{r!} \left( \frac{x-1}{2} \right)^r \left( \frac{x+1}{2} \right)^{n-r} \]

\[
= \frac{(1+\alpha)^n}{n!} \left( \frac{x+1}{2} \right)^n \left[ \begin{array}{c} -n, -\beta-n; \\ 1+\alpha; \end{array} \right].
\]
Finally,

\[ K_n(\alpha, \beta, 1; x) = P_n^{(\alpha, \beta)}(x). \]

The selections of the polynomials are also supported by the following nice relationships between our first and second sets on one hand and the Konhauser biorthogonal sets of first and second kind on the other hand:

(I.6) \[ \lim_{\beta \to \infty} J_n(\alpha, \beta, k; 1 - \frac{2x}{\beta}) = Z_n^{\alpha}(x; k), \]

and

(I.7) \[ \lim_{\beta \to \infty} K_n(\alpha, \beta, k; 1 - \frac{2x}{\beta}) = Y_n^{\alpha}(x; k). \]

For \( k = 1 \) both the results (I.6) and (I.7) yield the following well-known connection relation between the standard Jacobi polynomials and the standard Laguerre polynomials (see Szegő [14], p. 103)

(I.8) \[ \lim_{\beta \to \infty} P_n^{(\alpha, \beta)}(1 - \frac{2x}{\beta}) = L_n^{\alpha}(x). \]
Let us digress a little bit. One readily observes that $J_n(\alpha, \beta, k; x)$ are polynomials of degree $n$ in $x^k$ and the polynomials $K_n(\alpha, \beta, k; x)$ are polynomials of degree $n$ in $x$, where $x$ is a real variable in both the cases. Thus if these two sets are to form a pair of biorthogonal polynomials with respect to the weight function $(1-x)^{\alpha} (1+x)^{\beta}$ over the interval $(-1, 1)$ we must show

\[(I.9) \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} J_n(\alpha, \beta, k; x) K_m(\alpha, \beta, k; x) dx = 0 \quad \text{for} \quad m, n = 0, 1, 2, \ldots, m \neq n;\]

\[\neq 0 \quad \text{for} \quad m=n.\]

Or equivalently in view of equivalence between $(A.1)$ and $(A.2)$, $(A.3)$ these two polynomial sets must satisfy the following two conditions

\[(I.10) \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} J_n(\alpha, \beta, k; x) x^i dx = 0 \quad \text{for} \quad i=0, 1, 2, \ldots, n-1;\]

\[\neq 0 \quad \text{for} \quad i=n;\]

and
\[
(I_{11}) \quad \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta K_n(\alpha, \beta, k; x) (1-x)^{ki} \, dx
\]

= 0 for \( i=0, 1, 2, \ldots, n-1; \)

\( \neq 0 \) for \( i=n \).

It needs to be observed that the conditions \((I.9)\) to \((I.11)\) will be valid if we select \( \alpha>-1, \, \beta>-1 \). Also for \( k=1 \) each of the above three are equivalent to the orthogonality condition of the standard Jacobi polynomials.

In the very next section we shall establish \((I.9)\) and \((I.10), (I.11)\) will be established in second chapter.

2. Biorthogonality

Employing the explicit formulas \((I.3)\) and \((I.5)\) we shall show that the pair of polynomials \( K_n(\alpha, \beta, k; x) \) and \( J_n(\alpha, \beta, k; x) \) satisfies the biorthogonality condition \((I.9)\).

In fact, we have
\[ I_{n,m} = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} J_{\alpha}(\alpha, \beta, k; x) K_{\beta}(\alpha, \beta, k; x) dx \]

\[ = \frac{\Gamma(1+\alpha+k\lambda) \Gamma(1+\beta+m)}{2^m n! m! \Gamma(1+\alpha+\beta+n)} \sum_{j=0}^{\infty} \frac{(-1)^j (\frac{n}{j})}{2^k j} \frac{\Gamma(1+\alpha+\beta+n+kj)}{\Gamma(1+\alpha+kj)} \]

\[ \cdot \sum_{r=0}^{m} \left( \frac{r}{s} \right) \sum_{s=0}^{r} (-1)^s \left( \frac{s+1}{k} \right)_m \frac{1}{r! \Gamma(1+\beta+m-r)} \]

\[ \cdot \int_{-1}^{1} (1-x)^{\alpha+kj+r}(1+x)^{\beta+m-r} dx \]

\[ = 2 \frac{\Gamma(1+\alpha+k\lambda) \Gamma(1+\beta+m)}{n! m! \Gamma(1+\alpha+\beta+m)} \sum_{j=0}^{\infty} \frac{(-1)^j (\frac{n}{j})}{\Gamma(1+\alpha+\beta+m)} \frac{\Gamma(1+\alpha+\beta+n+kj)}{\Gamma(2+\alpha+\beta+m+kj)} \]

\[ \cdot \sum_{r=0}^{m} \frac{\alpha+kj+r}{r} \sum_{s=0}^{r} (-1)^s \left( \frac{s+1}{k} \right)_m \]

Recall the following result of Carlitz \[2, p. 429\].

\[ (I.12) \left( \frac{x+\alpha+1}{k} \right)_n = \sum_{r=0}^{n} (-x+r-1)^{r} \sum_{s=0}^{r} (-1)^s \left( \frac{s+1}{k} \right)_m \]

Using this, we have

\[ I_{n,m} = 2 \frac{\Gamma(1+\alpha+k\lambda) \Gamma(1+\beta+m)}{n! m! \Gamma(1+\alpha+\beta+n)} \sum_{j=0}^{\infty} \frac{(-1)^j (\frac{n}{j})}{\Gamma(1+\alpha+\beta)} (-j)^m \left( \frac{1+\alpha+\beta}{m+kj+1} \right) \]
\[= 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+km) \Gamma(1+\beta+m)}{n! \Gamma(1+\alpha+\beta+n)} (-1)^m \binom{n}{m} \]

\[\cdot \sum_{j=m}^{n} (-1)^j \binom{n-m}{j-m} \frac{(1+\alpha+\beta)^{n+kj}}{(1+\alpha+\beta)^{m+kj+1}}\]

\[= 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+km) \Gamma(1+\beta+m)}{n! \Gamma(1+\alpha+\beta+n)} \binom{n}{m} \]

\[\cdot \sum_{j=0}^{n-m} (-1)^j \binom{n-m}{j} \frac{(1+\alpha+\beta)^{n+kjm+kj}}{(1+\alpha+\beta)^{m+kjm+kj+1}}\]

\[= 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+km) \Gamma(1+\beta+m)}{n! \Gamma(1+\alpha+\beta+n)} \binom{n}{m} \]

\[\cdot \sum_{j=0}^{n-m-1} (-1)^j \binom{n-m-1}{j} x^{n-m-1} \frac{\alpha+\beta+n+km+kj}{x^{\alpha+\beta+n+km+kj+1}} \bigg|_{x=1}\]

\[= 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+km) \Gamma(1+\beta+m)}{n! \Gamma(1+\alpha+\beta+n)} \binom{n}{m} \]

\[\cdot \sum_{j=0}^{n-m-1} x^{\alpha+\beta+n+km+kj} (1-x)^{n-m-1} \bigg|_{x=1}\]

which is zero for \(n \neq m\) and nonzero for \(n = m\).

In particular,

\[(I.13) \quad I_{n,n} = 2^{1+\alpha+\beta} \frac{\Gamma(1+\alpha+km) \Gamma(1+\beta+n)}{n! \Gamma(1+\alpha+\beta+n)(1+\alpha+\beta+n+km)} .\]
3. Generating Functions and Recurrence Relations

In this section we shall obtain generating functions for both the polynomial sets \( \{ J_n(\alpha, \beta, k; x) \} \) and \( \{ K_n(\alpha, \beta, k; x) \} \), and a few recurrence relations for \( J_n(\alpha, \beta, k; x) \). In the first part of this section we shall consider \( J_n(\alpha, \beta, k; x) \) and in the second part \( K_n(\alpha, \beta, k; x) \).

With the aid of (I.3) we find that

\[
\sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)^n}{(1+\alpha)^{kn}} J_n(\alpha, \beta, k; x) t^n
\]

\[
= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1}{n!} (-1)^j \binom{n}{j} \frac{(1+\alpha+\beta)^{n+k+j}}{(1+\alpha)^{kJ}} \frac{1-x}{2}^{kJ}
\]

\[
= \sum_{j=0}^{\infty} \left( \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta+j)^{n}}{n!} t^n \right) \frac{(1+\alpha+\beta)^{k+j}}{j! (1+\alpha)^{kJ}} (-t) \frac{1-x}{2}^{kJ}
\]

\[
= (1-t)^{-1-\alpha-\beta} \sum_{j=0}^{\infty} \frac{(1+\alpha+\beta)^{k+j}}{j! (1+\alpha)^{kJ}} \frac{1-x}{2}^{kJ} \left[ \frac{-t}{(1-t)^{k+1}} \right]^j.
\]

Thus we obtain

\[ \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)^n}{(1+\alpha)^{kn}} J_n(\alpha, \beta, k; x) t^n = (1-t)^{-1-\alpha-\beta} \]

(I.14)
Also from (I.3), we have
\[ \sum_{n=0}^{\infty} \frac{J_n(\alpha, \beta-n, k; x)}{(\alpha)_n} t^n = \sum_{n=0}^{\infty} \frac{\Gamma(n+a)}{n!} \sum_{j=0}^{\infty} \frac{(1+\alpha+a)_j}{(\alpha)_j} \frac{(1-x)^j}{2^j} (-t)^j. \]

From which we establish
\[ (I.15) \sum_{n=0}^{\infty} \frac{J_n(\alpha, \beta-n, k; x)}{(\alpha)_n} t^n = e^t \Gamma_k \left[ \begin{array}{c} \Delta(k, 1+\alpha+\beta); \\ \Delta(k, 1+\alpha); \\ -t \left( \frac{1-x}{2} \right)^k \end{array} \right]. \]

For \( k=1, (I.15) \) reduces to a generating function for Jacobi polynomials which is due to Feldheim [5].

Next we establish the differential recurrence relation
\[ (I.16) (x-1)dJ_n(\alpha, \beta, k; x) = nk J_n(\alpha, \beta, k; x) - k (k\alpha-k+\alpha+1)_k J_{n-1}(\alpha, \beta+1, k; x). \]

Using (I.3), the term
\[ (I.17) nk J_n(\alpha, \beta, k; x) - k (k\alpha-k+\alpha+1)_k J_{n-1}(\alpha, \beta+1, k; x) \]

can be written in the form
\[
\frac{k}{(n-1)!} \sum_{m=0}^{n-1} (-1)^m \binom{n}{m} \frac{\Gamma(1+\alpha+\beta+n+km)}{\Gamma(1+\alpha+km)} \left( \frac{1-x}{2} \right)^{km} \\
- \frac{n-1}{(n-1)!} \sum_{m=0}^{n-1} (-1)^m \binom{n}{m} \frac{\Gamma(1+\alpha+\beta+n+km)}{\Gamma(1+\alpha+km)} \left( \frac{1-x}{2} \right)^{km}
\]

\[
= \frac{k}{(n-1)!} \sum_{m=0}^{n} (-1)^m \binom{n}{m} \left( \frac{\Gamma(1+\alpha+\beta+n+km)}{\Gamma(1+\alpha+km)} \left( \frac{1-x}{2} \right)^{km} \right)
\]

\[
= \frac{(x-1)}{n!} \sum_{m=0}^{n} (-1)^m \binom{n}{m} \left( \frac{\Gamma(1+\alpha+\beta+n+km)}{\Gamma(1+\alpha+km)} \left( \frac{1-x}{2} \right)^{km} \right)
\]

\[
= (x-1) \mathcal{D}_n(\alpha, \beta, k; x); \quad \text{and (I.16) is established.}
\]

Alternatively, we may write (I.17) as

\[
\frac{k(1-x)^k}{2^k (n-1)!} \left[ \frac{\Gamma(k[n-1]+[\alpha+k]+1)}{\Gamma(1+[\alpha+k]+[\beta+1]+[n-1]+k[\lambda-1])} \left( \frac{1-x}{2} \right)^{k(j-1)} \right]
\]

\[
= - \frac{k(1-x)^k}{2^k (n-1)!} \left[ \frac{\Gamma(k[n-1]+[\alpha+k]+1)}{\Gamma(1+[\alpha+k]+[\beta+1]+[n-1]+k[\lambda-1])} \left( \frac{1-x}{2} \right)^{kj} \right]
\]

\[
= \sum_{j=0}^{n-1} (-1)^j \binom{n-1}{j} \left( \frac{\Gamma(1+[\alpha+k]+[\beta+1]+[n-1]+k[\lambda-1])}{\Gamma(1+[\alpha+k]+[n-1]+[\lambda-1])} \left( \frac{1-x}{2} \right)^{kj} \right)
\]
from which we obtain

\[(I.18) \quad D^k J_n(\alpha, \beta, k; x) = 2^{-k} (1+\alpha+\beta+n)_k (1-x)^{k-1} J_{n-1}(\alpha+k, \beta+1, k; x).\]

Alternatively (I.18) can be derived by differentiating with respect to \(x\).

Again we may write the expression (I.17) as

\[
\frac{k}{(n-1)!} \frac{(kn+\alpha+1)}{\Gamma(1+\alpha+\beta+n)} \sum_{j=0}^{n} \frac{(-1)^j}{(j-1)!} \frac{\Gamma(1+\alpha+\beta+n+j)}{\Gamma(1+\alpha+k)} \frac{(1-x)^{kj}}{2^j}.\]

Multiply this by \((1-x)^{\alpha}\) and then taking the \(k\)-th derivative, we obtain

\[
k(-1) \frac{k}{(n-1)!} \frac{\Gamma(kn+\alpha+1)}{\Gamma(1+\alpha+\beta+n)} \sum_{j=0}^{n-1} \frac{(-1)^j}{(j-1)!} \frac{\Gamma(1+\alpha+\beta+n+j)\Gamma(1+\alpha+k)}{\Gamma(1+\alpha+k-j)\Gamma(1+\alpha+k)} (1-x)^{kj-k+\alpha}
\]

\[= k(-1) 2^{-k} \frac{k}{(n-1)!} \frac{(kn-k+\alpha+1)}{(1+\alpha+\beta+n)} \frac{\Gamma(kn-k+\alpha+1)}{\Gamma(1+\alpha+2+\beta+n)} \sum_{j=0}^{n-1} \frac{(-1)^j}{(j-1)!} \frac{\Gamma(1+\alpha+\beta+n+j\Gamma(1+\alpha+k)}{\Gamma(1+\alpha+k-j)} \frac{(1-x)^{kj}}{2^j}.\]

This yields differential recurrence relation

\[(I.19) \quad D^k \left[ (1-x)^{1+\alpha} D J_n(\alpha, \beta, k; x) \right] = k(-1) 2^{-k} (1+\alpha+\beta+n)_k \]

\[\cdot (1-x)^{\alpha} (kn-k+\alpha+1)_k J_{n-1}(\alpha, \beta+k+1, k; x).\]
By combining (I.16) and (I.19) we have,

\[(I.20)\] \[D^k \left[ (1-x)^{1+\alpha} D J_n(\alpha, \beta, k; x) \right] = (-1)^k 2^{-k} (1-x)^{\alpha} (1+\alpha+\beta+n)_k \]

\[\cdot \left[ (1-x) D J_n(\alpha, \beta+k, k; x) + nk J_n(\alpha, \beta+k, k; x) \right].\]

Recall the following results involving \(Z_n^\alpha(x;k)\), the Konhauser biorthogonal set of the first kind:

\[(I.21)\] \[x D Z_n^\alpha(x;k) = nk Z_n^\alpha(x;k) - k(\alpha-\alpha+1)_k Z_{n-1}^\alpha(x;k);\]

\[\left[ 7,\text{ eq.}(6),\text{ p.}\ 305 \right].\]

\[(I.22)\] \[D^k \left[ x^{\alpha+1} D Z_n^\alpha(x;k) \right] \]

\[= -kx^\alpha (\alpha-\alpha+1)_k Z_{n-1}^\alpha(x;k);\]

\[\left[ 7,\text{ eq.}(9),\text{ p.}\ 306 \right].\]

\[(I.23)\] \[-kx^k Z_{n-1}^\alpha(x;k) = (\alpha+\alpha) Z_{n}^{\alpha-1}(x;k) - \alpha Z_n^\alpha(x;k);\]

\[\left[ 6,\text{ p.}\ 638 \right].\]

\[(I.24)\] \[x D Z_n^\alpha(x;k) = (\alpha+\alpha) Z_n^{\alpha-1}(x;k) + Z_n^\alpha(x;k);\]

\[\left[ 6,\text{ p.}\ 638 \right].\]

\[(I.25)\] \[x^k Z_n^{\alpha+k}(x;k) = (\alpha+\alpha+1)_k Z_n^\alpha(x;k) - (\alpha+1) Z_{n+1}^\alpha(x;k);\]

\[\left[ 9,\text{ eq.}\ (2.6),\text{ p.}\ 215 \right].\]
From (I.21) and (I.22) we can get very easily with the help of the defining relation (I.2) the already obtained recurrence relations (I.16) and (I.19) respectively. Similarly from (I.23), (I.24) and (I.25) it is fairly easy to get the following recurrence relations for $J_n(\alpha,\beta,k;x)$ respectively.

(I.26) $\frac{-k}{2^k} (1-x) (1+\alpha+\beta+n)_k J_{n-1}(\alpha+k, \beta+1,k;x)$

$$= \alpha J_n(\alpha,\beta,k;x) - (kn+\alpha) J_{n-1}(\alpha-1, \beta+1,k;x).$$

(I.27) $(x-1)dJ_n(\alpha,\beta,k;x) = (kn+\alpha) J_{n-1}(\alpha-1, \beta+1,k;x)$

$$- \alpha J_n(\alpha,\beta,k;x).$$

(I.28) $\frac{-k}{2^k} (1-x) (1+\alpha+\beta+n)_k J_n(\alpha+k,\beta,k;x)$

$$= ((n+\alpha+1)_k J_n(\alpha,\beta,k;x) - (n+1) J_{n+1}(\alpha,\beta-1,k;x).$$

If we denote the right hand side of (I.14) by $F(x,t)$ and $1+\alpha+\beta = c$ then we have the partial differential equation

(I.29) $(1-x)(1+kt) \frac{\partial F}{\partial x} + kt(1-t) \frac{\partial F}{\partial t} = kctF.$

Above equation can be put in the following various forms

(I.30) $(1-x) \frac{\partial F}{\partial x} + kt \frac{\partial F}{\partial t} = kctF + kt^2 \frac{\partial F}{\partial t} - (1-x)kt \frac{\partial F}{\partial x}.$
\[(I.31) \quad (1-x) \frac{\partial F}{\partial x} + kt \frac{\partial F}{\partial t} = \frac{kct}{1-t} F - \frac{(1-x)(k-1)}{1-t} t \frac{\partial F}{\partial x} .\]

\[(I.32) \quad (1-x) \frac{\partial F}{\partial x} + kt \frac{\partial F}{\partial t} = \frac{kct}{1+kt} F + \frac{k(k+1)}{1+kt} t \frac{\partial F}{\partial t} .\]

As \( F(x,t) = \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)^n}{(1+\alpha)_kn} J_n(\alpha,\beta,k;x) t^n ,\)

\[(I.30) \] gives,

\[\sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)^n}{(1+\alpha)_kn} [ (1-x)DJ_n(\alpha,\beta,k;x)+nJ_n(\alpha,\beta,k;x) ] t^n .\]

\[= kc \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)^n}{(1+\alpha)_kn} J_n(\alpha,\beta,k;x)t \frac{n+1}{n+1} F + k \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)^n}{(1+\alpha)_kn} nJ_n(\alpha,\beta,k;x)t \]

\[-k(1-x) \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)^n}{(1+\alpha)_kn} DJ_n(\alpha,\beta,k;x)t .\]

Equating the coefficients of \( t^n \) and simplifying we have,

\[(I.33) \quad (x-1) \left[ (\alpha+\beta+n)DJ_n(\alpha,\beta,k;x)+k(1+\alpha+kn-k)kDJ_{n-1}(\alpha,\beta,k;x) \right] \]

\[= (\alpha+\beta+n) \left[ nJ_n(\alpha,\beta,k;x)-k(1+\alpha+kn-k)kJ_{n-1}(\alpha,\beta,k;x) \right] .\]

Using the above technique one can obtain from \((I.31)\) and \((I.32)\) respectively the following mixed recurrence relations

\[(I.34) \quad (x-1)DJ_n(\alpha,\beta,k;x) - nJ_n(\alpha,\beta,k;x) = - \frac{(1+\alpha)_kn}{(1+\alpha+\beta)_n} \]

\[\sum_{m=0}^{n-1} \frac{(1+\alpha+\beta)^m}{(1+\alpha)_km} \left[ k(1+\alpha+\beta)J_m(\alpha,\beta,k;x)+(k+1)(x-1)DJ_m(\alpha,\beta,k;x) \right] ;\]
and

\[(I.35) \quad (x-1)D^n J_n(\alpha, \beta, k; x) - n J_n(\alpha, \beta, k; x) \]

\[= \frac{(1+\alpha)^{kn}}{(1+\alpha+\beta)^n} \sum_{m=0}^{n-1} (-k)^m \frac{(1+\alpha+\beta)^m}{(1+\alpha)^{km}} (1+\alpha+\beta+m+mk) J_m(\alpha, \beta, k; x).\]

For \( k=1 \), \((I.33), (I.34) \) and \((I.35) \) reduce to well known recurrence relations for Jacobi polynomials; see Rainville \[12, p. 262 \].

Since \( J_n(\alpha, \beta, k; x) \) are essentially \( F_{k+1} \) type generalized hypergeometric polynomials, we can obtain appropriate relations involving these polynomials from the results \((14), (15), (19) \) and \((21) \) of Rainville \[12, p. 82, 84, 85 \]. Moreover in view of \((I.4) \) we have the following differential equation of order \( k+1 \)

\[
\left\{ \frac{x-1}{k} D \prod_{j=1}^{k} \left( \frac{x-1}{k} D + \frac{j+\alpha}{k} - 1 \right) - \left( \frac{1-x}{2} \right)^k \left( \frac{x-1}{k} D - n \right) \prod_{i=1}^{k} \left( \frac{x-1}{k} D + \frac{i+\alpha+\beta+n}{k} \right) \right\} J_n(\alpha, \beta, k; x) = 0,
\]

where \( |2^k(1-x)|^k < 1, \alpha > -1 \); see Rainville \[12, p. 75-76 \].

Above differential equation may be written in a more usable form
\[
\left( (x-1)^{\alpha} (x+\alpha-k+1)^{\beta} \right)_{k} - 2^{-k} (1-x)^{\gamma} ((x-1)^{\alpha} (x+\beta+n+1))_{k} \right] J_{n}(\alpha, \beta, k; x) = 0,
\]

where \( |2^{-k} (1-x)| < 1, \alpha > -1 \).

For \( k=1 \) this differential equation reduces to the usual differential equation satisfied by the Jacobi polynomials.

Now we shall take up the second set \( \left\{ K_{n}(\alpha, \beta, k; x) \right\} \).

From (I.5), we have after routine computation

\[
\sum_{n=0}^{\infty} K_{n}(\alpha, \beta-n, k; x) \frac{x^{-n} n^{\alpha}}{(x+1)^{n}} = \sum_{r=0}^{\infty} \frac{(-\beta)^{r}}{r!} \left( \frac{x-1}{x+1} \right)^{r} \sum_{s=0}^{\infty} \frac{(-x)^{s}}{s!} \left( \frac{x-1}{x+1} \right)^{s} \sum_{n=0}^{\infty} \frac{(s+\alpha+1)^{s}}{s! n!} \frac{w^{n}}{n!} \]

\[
= \left( 1-w \right)^{-s/k} \sum_{r=0}^{\infty} \frac{(-\beta)^{r}}{r!} \left( \frac{x-1}{x+1} \right)^{r} \sum_{s=0}^{\infty} (-1)^{s} \frac{(-x)^{s}}{s!} \left( \frac{x-1}{x+1} \right)^{s} \left( 1-w \right)^{s/k} \\
= \left( 1-w \right)^{-s/k} \sum_{r=0}^{\infty} \frac{(-\beta)^{r}}{r!} \left( \frac{x-1}{x+1} \right)^{r} \left( 1-(1-w) \right)^{-r/k} \\
(I.36) = \left( \frac{x+1}{2} \right)^{\beta} \left( 1-w \right)^{-s/k} \left( 1-(1-w) \right)^{-1/k} \\
\left( \frac{x-1}{2} + \frac{1}{k} \right)^{\beta}.
\]

This can be put in the form

(I.37) \( \sum_{n=0}^{\infty} K_{n}(\alpha, \beta-n, k; x) \frac{w^{n}}{n!} = \left( \frac{x+1}{2} \right)^{\beta} \left( 1 - \frac{x+1}{2} \right)^{-1/(k)(\beta)} \left( \frac{x-1}{2} + \frac{1}{k} \right)^{\beta}. \)
For \( k=1 \), we shall get a result for the Jacobi polynomials originally due to Feldheim [5].

The generating relation (I.36) readily yields

\[
(I.38) \quad K_n(\alpha, \beta-n,k;x) = \frac{1}{n!} \left( \frac{x+1}{2} \right)^{n-\beta} \left\{ \frac{\partial^n}{\partial t^n} \left( 1-t \right)^{-n-(\alpha+\beta)/k} \left[ \frac{x-1}{2} + (1-t)^{1/k} \right]^\beta \right\} \bigg|_{t=0}.
\]

In order to obtain rather more general generating function for the second set \( \{ K_n(\alpha,\beta,k;x) \} \) we need to employ the following famous result:

Lagrange's expansion formula [8, 146] - If \( f(t) \) is a holomorphic function in a neighbourhood of \( t = 0 \), then

\[
\frac{f(t)}{1-u \phi'(t)} = \sum_{n=0}^{\infty} \left( \frac{u^n}{n!} \left\{ \frac{d^n}{dt^n} \left[ f(t)(\phi(t))^n \right] \right\} \right)_{t=0}
\]

where \( u = t/\phi(t) \), \( \phi(t) \) is a holomorphic function in a neighbourhood of zero with \( \phi(0) \neq 0 \).

In (I.38) replace \( \alpha \) by \( \alpha + \gamma n \), \( \beta \) by \( \beta + \delta n \) and apply to the resulting expression the Lagrange's expansion formula with
\[ f(t) = \left( \frac{x+1}{2} \right)^{-\beta} \left( 1-t \right)^{-(1+\alpha+\beta)/k} \left[ \frac{x-1}{2} + (1-t)^{1/k} \right]^{\delta}, \]

\[ \phi(t) = \left( \frac{x+1}{2} \right)^{-\delta} \left( 1-t \right)^{-(1+\gamma+\delta)/k} \left[ \frac{x-1}{2} + (1-t)^{1/k} \right]^{1+\delta} \]

so that

\[ 1-u \phi'(t) \]

\[ = \left\{ k(1-t) \left[ \frac{x-1}{2} + (1-t)^{1/k} \right] - t(1+\delta)(\frac{x-1}{2})^{-\gamma t} \left[ \frac{x-1}{2} + (1-t)^{1/k} \right] \right\} \]

\[ \cdot \left\{ k(1-t) \left[ \frac{x-1}{2} + (1-t)^{1/k} \right] \right\}^{-1} \]

and thus we shall finally get the following generating function

\[ (I,39) \sum_{n=0}^{\infty} K_n(\alpha+\gamma, \beta+\delta, n; x) u^n = k \left( \frac{x+1}{2} \right)^{\beta} \left( \frac{x-1}{2} + (1-t)^{1/k} \right)^{-(1+\alpha+\beta)/k} \]

\[ \cdot \left[ \frac{x-1}{2} + (1-t)^{1/k} \right]^{\beta} \left\{ \frac{k(1-t)-\gamma t}{1-t} - \frac{(1+\delta)t(x-1)/2}{(1-t)\left[ \frac{x-1}{2} + (1-t)^{1/k} \right]} \right\}^{-1}, \]

where

\[ u = t \left( \frac{x+1}{2} \right)^{\delta} \left( 1-t \right)^{-(1+\gamma+\delta)/k} \left[ \frac{x-1}{2} + (1-t)^{1/k} \right]^{-1-\delta}. \]

With the substitution \( t = 1 - (1+w) \), the above generating function can be put in the following form.
\[(I.40) \sum_{n=0}^{\infty} K_n(\alpha+\delta n, \beta+\delta n, k; x) u^n = k(1+w)^{\alpha} \left[ 1 + \frac{x-1}{x-1} \right]^\beta \]
\[\frac{k}{w+1} \left[ (w+1) \right]^{-1} \left[ \frac{x+1+xw}{w} \right]^{-1-\delta} \]
\]
\[
\text{where}
\]
\[u = 2(x+1)^{\delta} \left[ (w+1)^{\delta} \right]^{-1} \left[ (w+1)^{\delta} \right]^{-1-\delta} \]
\[
\text{It is clear that for } \gamma = 0, \delta = -1, \text{ we shall get the result } (I.37) \text{ with which we began and it has a nice compact form. Similarly if we set } \gamma = -k, \delta = -1 \text{ in } (I.40) \text{ we shall get a generating function in a compact form given below}
\]
\[(I.41) \sum_{n=0}^{\infty} K_n(\alpha-k n, \beta-n, k; x) u^n = \left( \frac{x+1}{2} \right)^{-\beta} \]
\[\left[ 1 + \frac{x-1}{x-1} \right]^{1/k} \left( 1 + \frac{x+1}{2} \right)^{1/k} \]
\[
\text{For } k=1, \text{ we get a known generating function for the Jacobi polynomials originally due to Carlitz [1].}
\]
\[
\text{We shall also mention in the passing that a rather direct proof of } (I.41) \text{ will also be given in the next chapter.}
\]
4. Concluding Remarks:

(i) Spencer and Fano [13] utilized biorthogonality of polynomials in \(x\) and polynomials in \(x^2\) with respect to the
weight function $x^\alpha e^{-x}$ ($\alpha$, nonnegative integer) over the interval $(0, \infty)$, in carrying out calculations involving the penetration of gamma rays through matter. We are also optimistic that the particular cases of our pair of biorthogonal polynomials would, certainly, be of use in physical problems.

(ii) It is needless to say that these polynomials will yield for $\alpha = \beta = 0$, a pair of biorthogonal polynomials suggested by Legendre polynomials, for $\alpha = \beta = \pm 1/2$ a pair of biorthogonal polynomials related to Chebyshev polynomials of the second kind and first kind respectively and when $\alpha = \beta$ we would have a pair of biorthogonal polynomials suggested by Ultraspherical polynomials.

(iii) Prabhakar and Kashyap [11] study the pair of biorthogonal polynomials $U_n^\alpha(x;k)$ and $V_n^\alpha(x;k)$ that are explicitly given by (A.3) and (A.9) respectively.

One easily notes that

\[ \frac{(1+\alpha)n}{(1+\alpha)^n} \alpha \choose \frac{x}{(1+\alpha)n} \frac{x}{V_n(x;k)} = J_n(\alpha,0,k;x), \]

and further

\[ \frac{(k)n}{n!} \frac{(1+\alpha)n}{(1+\alpha)^n} \frac{(1+\alpha-k)/k}{n} U_n^{(1+\alpha-k)/k} (1-2(\frac{1-x}{2}) ; 1/k) = J_n(\alpha,0,k;x). \]
As was observed earlier \( U_n^{\alpha}(x;k) \) and \( V_n^{\alpha}(x;k) \) are not independent of each other. Thus, essentially these two sets are two replicas of the same system of orthogonal polynomials. This applies equally well to the biorthogonal pairs considered by Prabhakar and Tomar [10] as they happen to be particular cases of the above mentioned pair.
REFERENCES


10. T.R. Prabhakar and R.C. Tomar, Biorthogonal polynomials suggested by the Legendre polynomials, J. Indian Math. Soc. (Accepted for publication) (See also Notices Amer. Math. Soc. 79T-315, 26, No.2, (1979)).


