CHAPTER-VIII

Obtaining Generating Functions
For Classical Orthogonal
Polynomials by Hermite’s Method

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CHAPTER VIII

1. INTRODUCTION

Askey [1] used Hermite’s method to obtain the usual generating function for the standard Jacobi polynomials in the form

\[ \sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x) t^n = 2^{\alpha+\beta} R^{-1} (1-t+R)^{-\alpha} (1+t+R)^{-\beta}, \]

where

\[ R = (1-2xt+t^2)^{1/2}. \]

In this chapter we show that the Hermite’s method applies equally well to the remaining classical orthogonal polynomials, namely the Laguerre polynomials and the Hermite polynomials.

In the last section of this chapter, we adequately show that one can find the usual generating functions for the classical orthogonal polynomials including Bessel polynomials by the method of Hermite as they essentially originate merely from the various choices of parameters; see (B.7) to (B.15). In a nutshell, we exhibit the fruitfulness of the Fujiwara-Thakare approach towards the unification of the classical orthogonal polynomials including the Bessel polynomials.
The Laguerre polynomials and the Hermite polynomials are respectively defined by the following Rodrigue's formulae.

\[(\text{VIII.3}) \quad L_n^{\alpha}(x) = \frac{x^{-\alpha} e^x}{n!} \frac{D^n}{Dx^n} \left\{ e^{-x} x^{n+\alpha} \right\}, \quad D = d/dx.\]

\[(\text{VIII.4}) \quad H_n(x) = (-1)^n \exp(x^2) D^n \exp(-x^2).\]

Using the method of integration by parts it is fairly easy to obtain the orthogonality conditions, namely

\[(\text{VIII.5}) \quad \int_0^\infty x^\alpha e^{-x} L_n^{\alpha}(x) L_m^{\alpha}(x) dx = 0, \quad m \neq n, \quad \alpha > 1.\]

\[(\text{VIII.6}) \quad \int_{-\infty}^{\infty} \exp(-x^2) H_n(x) H_m(x) dx = 0, \quad m \neq n.\]

2. Generating function for Laguerre polynomials

Consider a generating function

\[f(x, t) = \sum_{n=0}^{\infty} L_n^{\alpha}(x) t^n.\]

Consider also

\[I_k = \int_0^\infty x^k f(x, t) x^\alpha e^{-x} dx.\]
Put \( x = (1+t)y \) with \( |t|<1 \).

The integral \( I_k \) is then

\[
I_k = \int_0^\infty y^k (1+t)^k f(x,t)(1+t)^{\alpha+1} y^\alpha e^{-y} e^{ty} dy.
\]

This would clearly be a polynomial of degree \( k \) in \( t \) if

(VIII.7) \( f(x,t) = (1+t)^{-\alpha-1} e^{ty} = (1+t)^{-\alpha-1} \exp \left[ xt/(1+t) \right] \).

If

(VIII.8) \( f(x,t) = \sum_{n=0}^\infty Q_n(x) t^n \), then \( Q_n(x) \) is a polynomial

of degree \( n \) in \( x \) and as \( I_k \) is going to be a polynomial

of degree \( k \) in \( t \), we must have

\[
\int_0^\infty x^k Q_n(x) x^\alpha e^{-x} dx = 0, \quad n=k+1, k+2, \ldots.
\]

This is equivalent to the orthogonality of \( Q_n(x) \) over \((0,\infty)\)

with respect to the weight function \( x^\alpha e^{-x} \). Thus

\( Q_n(x) = \lambda_n 1_n(x) \), for some constant \( \lambda_n \), since there

is only one set of polynomials that are orthogonal with

respect to a given weight function after they have been

normalized. From (VIII.7) and (VIII.8), we obtain
\[ Q_n(o) = (-1)^n \frac{(1+\alpha)_n}{n!} \cdot \]

Also from (VIII.3)

\[ L_n^\alpha(o) = \frac{(1+\alpha)_n}{n!} \quad \text{and hence} \]

\[ Q_n(x) = (-1)^n \frac{\alpha}{L_n(x)}. \]

Thus, we get

\[(VIII.9) \sum_{n=0}^{\infty} L_n^\alpha(x)t^n = (1-t)^{-\alpha-1} \exp\left(\frac{-xt}{1-t}\right), \quad |t|<1.\]

The above arguments also contain the proof of the fact that the orthogonality relation (VIII.5) for the Laguerre polynomials holds if they are defined by the generating relation (VIII.9).

3. Generating function for Hermite polynomials

As in section-2, let

\[ F(x,t) = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}, \quad \text{and} \]

\[ I_k = \int_{-\infty}^{\infty} x^k F(x,t) \exp(-x^2)dx. \]
Set \( x = y + t \), then

\[
I_k = \int_{-\infty}^{\infty} (y+t)^k P(x,t) \exp(-2yt-t^2) \exp(-y^2) \, dy.
\]

Obviously, this is a polynomial of degree \( k \) in \( t \) if

(VIII.10) \( P(x,t) = \exp(2yt+t^2) = \exp(2xt-t^2) \).

If we take

(VIII.11) \( \sum_{n=0}^{\infty} q_n(x)t^n = P(x,t) \), then \( q_n(x) \) is a polynomial of degree precisely \( n \) in \( x \) which must satisfy

\[
\int_{-\infty}^{\infty} x^k q_n(x) \exp(-x^2) \, dx = 0, \quad n=k+1, k+2, \ldots,
\]

in order to have \( I_k \) a polynomial of degree \( k \) in \( t \).

The last condition is equivalent to the orthogonality of \( q_n(x) \) with respect to the weight function \( \exp(-x^2) \) over \((-\infty, \infty)\).

Hence \( q_n(x) = h_n H_n(x) \) for some constant \( h_n \), as was done in section-2.

We can select \( h_n = 1/n! \); because from (VIII.10), (VIII.11) and (VIII.4), we have
\[ a_{2n}(0) = \frac{(-1)^n}{n!}, \quad H_{2n}(0) = \frac{(-1)^n}{n!} (2n)!; \quad \text{and} \]
\[ a_{2n+1}(0) = 2 \frac{(-1)^n}{n!}, \quad H_{2n+1}(0) = 2 \frac{(-1)^n}{n!} (2n+1)! ] .

Finally, we get

\[ (VIII.12) \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} \frac{t^n}{n!} = \exp(2xt-t^2). \]

As was mentioned in section-2 the above arguments also contain the proof of the orthogonality condition (VIII.6) for the Hermite polynomials if they are defined by the generating relation (VIII.12).

4. Fujiwara-Thakare Approach

As was explicitly indicated in the introduction to Part-B of the Thesis, we now bring out very clearly why Hermite's method becomes applicable in obtaining the generating functions for the classical orthogonal polynomials.

Consider the generating function

\[ G(x,t) = \sum_{n=0}^{\infty} P_n(\alpha,\beta;x) t^n \]

and look at the integral
\[ I_k = \int_a^b G(x, t) \, (x-a)^\alpha \, (b-x)^\beta \, dx. \]

Put

\[ (\text{VIII}, 13) \quad R = \left[ 1 + 2t \, x'(x) + \lambda^2 t^2 \right]^{1/2} = 1 + \lambda t \left[ 2(y-a)/(a-b)^2 + 1 \right], \]

where \( x(x) \) is given by (B.3). Then the integral

\[ I_k = \int_a^b \left[ y + \lambda t(y-a)/(a-b) \right]^k G(x, t) \left[ 1 + \lambda t(y-b)/(a-b) \right]^\alpha \]

\[ \cdot \left[ 1 + \lambda t(y-a)/(a-b) \right] \left[ 1 + \lambda t(1+2(y-a)/(a-b)) \right]^\beta (y-a)^\alpha (b-y)^\beta dy. \]

This integral is a polynomial of degree \( k \) in \( t \) if

\[ (\text{VIII}, 14) \quad G(x, t) = 2^{\alpha+\beta} \left( 1 + \lambda t + R \right)^{-\alpha} \left( 1 - \lambda t + R \right)^{-\beta} R^{-1}, \]

where \( R \) is given by (VIII.13). Thus, if

\[ (\text{VIII}, 15) \quad G(x, t) = \sum_{n=0}^{\infty} Q_n(x) t^n, \]

then \( Q_n(x) \) is a polynomial of degree \( n \) in \( x \) and as \( I_k \) is to be a polynomial of degree \( k \) in \( t \), we must have

\[ \int_a^b x^k Q_n(x)(x-a)^\alpha (b-x)^\beta \, dx = 0, \quad n = k+1, k+2, \ldots. \]
Thus, we have to select

\[ Q_n(x) = d_n \, F_n(\alpha, \beta; x) \]

for some constant \( d_n \) as there is only one set of polynomials orthogonal over \((a, b)\) with respect to the normalized weight function \((B, S)\). But with \( x=b \), (VIII.14) and (VIII.15) yield \( Q_n(b) = \lambda^n (1+\beta)_n / n! \); and from (B.7) to (B.11), we have \( F_n(\alpha, \beta; b) = \lambda^n (1+\beta)_n / n! \) so that we select \( d_n = 1 \) for all \( n \). And thus, we have

\[
(VIII.16) \quad \sum_{n=0}^{\infty} F_n(\alpha, \beta; x) t^n = 2^{\alpha+\beta - 1} \frac{1}{R} \left( 1 + \lambda t + R \right)^{-\alpha} \left( 1 - \lambda t + R \right)^{-\beta}
\]

where \( R \) is given by (VIII.13); see Fujiwara [3] and Thakare [4, 5] for additional proofs of (VIII.16).

In view of (B.12) one readily obtains the usual generating function (VIII.1) for the Jacobi polynomials.

Put \( a=0 \), \( \lambda = 1 \) and \( \beta=b \) in (VIII.16). Using the relationship (B.13), we obtain on account of (VIII.13) and

\[
(VIII.17) \quad \lim_{b \to \infty} (1-t/b)^b = e^{-t},
\]

the generating function (VIII.9) for the Laguerre polynomials.
Put $\beta = \alpha$, $-a = b = \sqrt{a}$, $(\alpha > 0)$ in (VIII.16). We have then, as a consequence of (B.14), (VIII.13) and (VIII.16), the generating function (VIII.12) for the Hermite polynomials.

Lastly, we consider the case of the Bessel polynomials. Put $-a = b = \lambda = 1$, $\alpha = r-\sigma-1$, $\beta = \sigma-1$ and replace simultaneously $x$ by $1+ (2x\sigma/s)$, $t$ by $sw/2\sigma$ in (VIII.16) to obtain in the limit $\sigma \to \infty$

$$G(x,w) = 2^{r-2} (1-2xw)^{-1/2} \left[ 1+\sqrt{1-2xw} \right]^{2-r}$$

$$\lim_{\sigma \to \infty} \left[ 1+(sw/2\sigma) \right]^{1+r-\sigma} \lim_{\sigma \to \infty} \left[ 1-(sw/2\sigma) \right]^{1-\sigma},$$

where

$$T = 1+ \sqrt{1 - \frac{sw}{\sigma} (1+2x\sigma/s) + \left( \frac{sw}{2\sigma} \right)^2}.$$ 

In view of (3.15) and after some simplification one obtains

$$\sum_{n=0}^{\infty} \left( \frac{s}{2} \right)^n y_n(x,r,s) \frac{w^n}{n!} = G(x,w)$$

$$= (1-2xw)^{-1/2} \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1-2xw} \right]^{2-r} \exp\left[ \frac{s}{2x} \left( 1- \sqrt{1-2xw} \right) \right].$$

The above generating function was first given by Burchnall [2].
REFERENCES


4. Thakare, N.K., Generating function in the unified form for the classical orthogonal polynomials by using operator calculus, Ganita, 28 (1977), 55-63.

5. ______________, A unified approach to the study of classical orthogonal polynomials, a monograph under private circulation.