CHAPTER - 7

ON THE DIFFUSION OF A CHEMICALLY REACTIVE SPECIES IN TRANSIENT FORCED AND FREE CONVECTIVE FLOW PAST AN INFINITE VERTICAL PLATE WITH VARIABLE TEMPERATURE
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7.1 INTRODUCTION:

Many papers have been published on unsteady free convective flows past a semi-infinite or infinite plate. Some of them are by Siegel (1), Schetz and Lichhorn (2), Menold and Yang (3), Cheng and Anderson (4), Goldstein and Briggs (5), Sugawara and Michiyoshi (6), Goldstein and Eckert (7), Mizukami and Sakurai (8) have confirmed some of the theoretical predictions by experiments. It was first pointed out by Siegel that the initial behaviour of the temperature field and the velocity field for the semi-infinite flat plate is the same as for the doubly-infinite vertical flat plate. Hence the velocity distribution in this region is independent of the vertical distance and the temperature field is given as a solution of the one-dimensional heat-conduction equation. This led Siegel to conclude that the transition from conduction to convection is only due to the leading edge effect that has propagated to a certain point. The distance of this point from the leading edge is known as the maximum penetration distance \( X_{p_{\text{max}}} \). Also \( X_p \) is a function of \( t' \) (time) and \( y' \) (co-ordinate normal to the plate).
These papers are devoted to free convective flows only. Soundalgekar and Jahagirdar (9) studied recently the unsteady free and forced convective flow of a viscous, incompressible fluid past an infinite vertical isothermal plate. But in many transport processes which occur in nature, in addition to temperature difference, the density difference is caused by chemical composition difference and gradients or by material or phase constitution. In nature, we come across such a situation viz. the atmospheric flow is driven appreciably by both temperature and H₂O concentration differences. The flow caused by the density difference which in turn is caused by concentration difference is known as the mass transfer flow. Such a physical phenomenon has good application in chemical industry.

Mass transfer effects on steady free-convective flows past semi-infinite vertical plate were studied by Gebhast and Pera (10). These problems were solved by similarity method. Soundalgekar and Jahagirdar (11,12,13) studied the mass transfer effects on the unsteady free and forced convection flows past an infinite vertical plate. In Ref. (11), the case of isothermal plate was considered, in Ref. (12), the case of the plate where wall temperature varies linearly with time was considered and in Ref. (13), the effects of the first-order chemical reaction on the unsteady forced and free convective flows past an isothermal plate was considered. It is now proposed to study the effects of the first order chemical reaction on the unsteady free and forced
convective flow past an infinite vertical plate whose temperature varies linearly with time. In Section 2, the mathematical analysis is presented and in Section 3, the conclusions are set out.

7.2 **MATHEMATICAL ANALYSIS**

Consider the unsteady free and forced connective flow of a viscous incompressible fluid past an infinite vertical plate whose temperature is initially same as that of the fluid viz. $T_\infty$. The level of the species concentration in the flow is assumed to be low which then enables us to neglect the Soret and Dufour effects. The $x'$-axis is taken along the plate in the vertically upward direction and the $y'$-axis is taken normal to the plate. The level of the concentration is also assumed to be the same everywhere initially. At time $t' \geq 0$, the fluid is supposed to flow upward, the plate temperature is assumed to vary linearly with time $t'$, and the level of concentration at the plate is raised from $C'_\infty$ to $C'_w$. The physical variables are function of $y'$ and $t'$ only under present conditions. Then assuming the usual Boussinesq's approximation, the flow is governed by the following equations: (Gebhart (15))

\[
\frac{\partial u'}{\partial t'} = \kappa \beta q (T' - T'_\infty) + \kappa \beta \phi (C' - C'_\infty) + \mu \frac{\partial^2 u'}{\partial y'^2} \quad \ldots(7.1)
\]

\[
\frac{\partial q}{\partial t'} = K \frac{\partial^2 T'}{\partial y'^2} \quad \ldots(7.2)
\]

\[
\frac{\partial C'}{\partial t'} = D \frac{\partial^2 C'}{\partial y'^2} - K_1 C' \quad \ldots(7.3)
\]
The initial and the boundary conditions are

\[ u' = 0, \quad \Gamma' = T_{\infty}', \quad C' = C_{\infty}' \quad \text{for all} \quad y', t' \leq 0 \]

\[ u' = 0, \quad \Gamma' = T_{\infty}' + T_w't', \quad C' = C_w' \quad \text{at} \quad y' = 0, \quad t' \geq 0 \]

\[ u' = U_o, \quad \Gamma' = T_{\infty}', \quad C' = C_{\infty}' \quad \text{as} \quad y' \to \infty \]

...(7.4)

Here \( u' \) is the velocity of the fluid in the \( x' \)-direction, \( y' \) the density, \( g \) the acceleration due to gravity, \( \beta \) the coefficient of \textit{volume} expansion, \( \beta^* \) the coefficient of expansion with concentration, \( \Gamma' \) the temperature of the fluid, \( T_{\infty}' \) the temperature of the fluid in the \textit{main}-stream, \( T_w' \) the plate temperature, \( C' \) the species concentration, \( C_{\infty}' \) the species concentration in the free stream, \( \mu \) the coefficient of viscosity, \( C_p \) the specific heat at constant pressure, \( K \) the thermal conductivity, \( D \) the chemical molecular diffusivity, \( C_w' \) the species concentration near the plate and \( K_1 \) is the reaction rate constant. We assume that their exists a first-order chemical reaction.

On introducing the following non-dimensional quantities

\[ y = y' U_o / \nu, \quad t = t' U_o^2 / \nu, \quad u = u' / U_o, \quad \Theta = \frac{\nu g \beta (T - T_{\infty})}{U_o^3} \]

\[ G_r = \frac{\nu g \beta T_w}{U_o^3}, \quad G_C = \frac{\nu g \beta^*(C' - C_{\infty})}{U_o^2}, \quad \kappa = \frac{k_1 \nu}{U_o^2}, \quad S_C = \frac{\nu}{D} \]

\[ P = \frac{\mu C_p}{k} \]

...(7.5)
in equations (7.1) – (7.4), we get

$$\frac{\partial u}{\partial t} = \Theta + G_c C + \frac{\partial^2 u}{\partial y^2}$$  \ldots (7.6)

$$P \frac{\partial \Theta}{\partial t} = \frac{\partial^2 \Theta}{\partial y^2}$$  \ldots (7.7)

$$S_c \frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial y^2} = K S_c C$$  \ldots (7.8)

and the boundary and initial conditions are

\[ u = 0, \Theta = 0, C = 0 \quad \text{for all } y, t \leq 0 \]

\[ u = 0, \Theta = G_r t, C = 1 \quad \text{at } y = \text{at } t \geq 0 \]

\[ u = 1, \Theta = 0, C = 0 \quad \text{as } y \to \infty \]

The solutions of equations (7.6) – (7.9) are derived by the usual Laplace-transform technique and these are as follows:

$$\Theta = G_r t \left[ (1 + 2 P \eta^2) \text{erfc}(\eta \sqrt{P}) - 2 \sqrt{\frac{P}{\pi}} \eta e^{-P \eta^2} \right]$$  \ldots (7.10)

$$C = \frac{1}{2} \left\{ e^{-2 \sqrt{K S_c} t} \frac{\eta}{\text{erfc}(\sqrt{S_c} \eta - \sqrt{Kt})} + e^{2 \sqrt{K S_c} t} \frac{\eta}{\text{erfc}(\sqrt{S_c} \eta + \sqrt{Kt})} \right\}$$  \ldots (7.11)

$$u = (1 - \text{erfc} \eta) + \frac{G_r}{6(P-1)} t^2 \left[ (4 \eta^4 + 12 \eta^2 + 3) \text{erfc} \eta - 2 \eta e^{-\eta^2} (2 \eta^2 + 5) \right]$$

$$+ \frac{G_c}{K S_c} \text{erfc} \eta - \frac{G_c}{2 K S_c} \sqrt{K S_c (1 - S_c)} \int_{-2 \sqrt{K S_c \eta}}^{\sqrt{K S_c \eta}} \frac{\eta}{\text{erfc}(\eta - \sqrt{K S_c \eta})}$$

$$+ \frac{G_c}{K S_c} \text{erfc} \eta - \frac{G_c}{2 K S_c} \sqrt{K S_c (1 - S_c)} \int_{-2 \sqrt{K S_c \eta}}^{\sqrt{K S_c \eta}} \frac{\eta}{\text{erfc}(\eta - \sqrt{K S_c \eta})} + \ldots$$
\begin{align*}
&+ e^{2\sqrt{\frac{Ksct}{1-Sc}} \eta} \frac{Gr}{\delta(P-1)} t^2 \left[ (4p^2\eta^2 + 12p^2\eta^2 + 3) \right] \\
&+ \frac{2\sqrt{\frac{Ksct}{1-Sc}} \eta}{\sqrt{\pi}} e^{-\eta^2} \left[ (2\eta^2 + 5) \right] - \frac{Gc}{2Ksct} \left[ e^{-2\sqrt{\frac{Ksct}{1-Sc}} \eta} \right] \right] \\
&+ e^{2\sqrt{\frac{Ksct}{1-Sc}} \eta} \left[ \frac{Gc}{2Ksct} \left[ e^{-\frac{Ksct}{1-Sc}} \right] + \left[ \frac{Ksct}{1-Sc} + e^{\frac{Ksct}{1-Sc}} \right] \right] \\
\end{align*}

In order to get a physical insight, we have carried out the numerical values of the velocity and the concentration profiles. These are shown on Figures 1 and 2. We observe from Figure 1 that an increase in Sc or k leads to a decrease in the velocity but an increase in Gc or t leads to an increase in the velocity. From Fig. 2, we conclude that an increase in Sc, K or t leads to a decrease in the concentration.

**Leading edge effect:**

We now study the mass transfer effects on the penetration distance which is derived by integrating u with respect to t and the maximum penetration distance $X_p_{\text{max}}$ at any time can be determined by differentiating $X_p$ with respect to $y$ holding $t$ constant and then by setting the derivative to zero.
Thus the penetration distance is given by

\[ X_p = \int_0^t u(y,t) \, dt \]  \hspace{1cm} \ldots (7.13)

In terms of the Laplace-transform and its inverse, it can be expressed as, with \( t \) as variable,

\[ X_p = L^{-1} \left\{ \frac{1}{s} L(u(y,t)) \right\} = L^{-1} \left\{ \frac{1}{s} \bar{u}(y,s) \right\} \]  \hspace{1cm} \ldots (7.14)

Substituting for \( \bar{u} \) and taking the inverse, we get

\[
X_p = t - t \left[(1 + 2\eta^2) \text{erf} \eta - \frac{2\eta}{\sqrt{\pi}} e^{-\eta^2} \right] + \frac{64 \text{Gr} \ t^3}{(P-1)} \left\{ \frac{(5 + 2\eta^2)}{60 \times 96} \right\}^4 \eta + 12\eta^2 \text{erf} \eta - \frac{2\eta}{\sqrt{\pi}} (2\eta^2 + \bar{\eta}) - \frac{\eta}{360} \left[ e^{-\eta^2 (1+\eta^2)} - \frac{\eta (3+2\eta^2)}{2} \right] \]

\[ \text{erf} \eta \left\{ \right. \]  + \frac{G\text{c}(1-Sc)}{(K\text{Sc})^2} \]  \text{erf} \eta + \frac{G\text{c} \ t}{K\text{Sc}} \left[ (1 + 2\eta^2) \text{erf} \eta \right] - \frac{2\eta e^{-\eta^2}}{\sqrt{\pi}} \left. \right]

\[
- \frac{G\text{c}(1-Sc)}{2(K\text{Sc})^2} \times \frac{K\text{Sc} \ t}{e^{1-Sc}} \left\{ e^{-2\sqrt{K\text{Sc}t} \eta} \right\} \text{erfc} \left( \eta - \sqrt{K\text{Sc}t} \right) + e^{2\sqrt{K\text{Sc}t} \eta} \left[ \eta + \sqrt{K\text{Sc}t} \right] \frac{64 \text{Gr} \ t^3}{(P-1)} \right\{ \frac{(5 + 2\eta^2)}{60 \times 96} \right\}^4 \eta + 12\eta^2 \text{erf} \eta \left( \frac{1}{\sqrt{\pi}} \right) \]

\[ \frac{2\sqrt{P}}{\sqrt{\pi}} \eta \left[ e^{-\eta^2 (2P + \bar{\eta})} \right] - \frac{\sqrt{P}}{360} \eta \left[ e^{-\eta^2 (1+ \bar{\eta}^2)} \right] - \]
\[
-\frac{\sqrt{\frac{5}{2}} \gamma}{2} (\beta_2 \eta^2 \text{erfc}(\sqrt{\eta})) \bigg]\frac{Gc(1-SC)}{2(KSc)^2} \bigg[ -\frac{e^{-\frac{1}{2}KSc\eta}}{1-Sc} + x \text{erfc}(\sqrt{SC} \eta - \sqrt{Kt}) + e^{\frac{KSc}{1-Sc}} \text{erfc}(\sqrt{SC} \eta + \sqrt{Kt}) \bigg]
\]

\[
+ (1-Sc)Gc e^{\frac{KSc}{(1-Sc)^2}} \bigg[ -\frac{2\sqrt{KSc} \eta}{1-Sc} \text{erfc}(\sqrt{SC} \eta - \sqrt{Kt}) + e^{\frac{2\sqrt{KSc}}{1-Sc}} \text{erfc}(\sqrt{SC} \eta + \sqrt{Kt}) \bigg] - \frac{Gc}{2KSc} \bigg[
\left[(t - \frac{\sqrt{SC} \eta}{K}) e^{-\frac{1}{2}KSc \eta} \text{erfc}(\sqrt{SC} \eta - \sqrt{Kt}) +
\right.

\left.+(t + \frac{\sqrt{SC} \eta}{K}) e^{\frac{2\sqrt{KSc}}{1-Sc}} \text{erfc}(\sqrt{SC} \eta + \sqrt{Kt}) \right] \bigg] \ldots(7.15)
\]

The numerical values of \(X_p\) are shown on Fig. 3. Figure 3 shows that an increase in \(Sc\) or \(K\) leads to a decrease in the value of \(X_p\) but \(X_p\) increases with increasing \(Gr, fc\) or \(t\).

We now study the skin-friction. It is given in non-dimensional form as

\[
\tau = -\frac{1}{2\sqrt{\tau}} \frac{du}{d\eta} \bigg|_{\eta = 0} \ldots(7.16)
\]

Substituting for \(u\) from (7.12) in (7.16) and carrying out the operation, we get
\[
\tau = \frac{1}{\sqrt{\pi t}} + \frac{4Gr t^{3/2}}{3\sqrt{\pi} (1+t/F)} - \frac{Gc}{\sqrt{KSc}} \left\{ \frac{KSc}{\sqrt{1-Sc}} \right\} \right\} 
- \text{erfc} \left( \sqrt{Kt} \right) + (\text{erfc} \sqrt{Kt} - 1) \right] 
\]

The numerical values of \( \tau \) are entered in Table 7.1. We conclude from this table that an increase in \( Sc \) or \( t \) leads to a decrease in the value of \( \tau \) whereas \( \tau \) increases with increasing \( Gc \) or \( K \).

7.3 CONCLUSIONS:

1. The velocity, the penetration distance decrease with increasing \( Sc \) and \( K \) and increase with increasing \( Gc \) or \( t \).

2. The skin-friction increases with increasing \( Gc \) or \( Gr \) and decreases with increasing \( Sc \) or \( t \).
TABLE 7.1

VALUES OF SKIN-FRICTION

<table>
<thead>
<tr>
<th>t</th>
<th>( \kappa )</th>
<th>Gr</th>
<th>Go</th>
<th>Sc</th>
<th>t</th>
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<td>0.2</td>
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<td>0.7</td>
<td>1.3887</td>
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<td>0.2</td>
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<td>0.2</td>
<td>0.5</td>
<td>1.4659</td>
</tr>
<tr>
<td>0.2</td>
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<td>2.0</td>
<td>0.2</td>
<td>0.5</td>
<td>1.3924</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5</td>
<td>2.0</td>
<td>0.2</td>
<td>0.5</td>
<td>1.1800</td>
</tr>
</tbody>
</table>
REFERENCES:


9. Soundalgekar V.M. and Jahagirdar M.D. (To be Published).


11. Soundalgekar V.M. and Jahagirdar M.D. (To be published).

12. Soundalgekar V.M. and Jahagirdar M.D. (To be published).

13. Soundalgekar V.M. and Jahagirdar M.D. (To be published).
FIG 1. VELOCITY PROFILES

<table>
<thead>
<tr>
<th>t</th>
<th>k</th>
<th>Gr</th>
<th>Gc</th>
<th>Sc</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.5</td>
<td>2.0</td>
<td>0.2</td>
<td>0.5 — I</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5</td>
<td>2.0</td>
<td>0.2</td>
<td>0.7 — II</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5</td>
<td>2.0</td>
<td>0.6</td>
<td>0.5 — III</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5</td>
<td>4.0</td>
<td>0.2</td>
<td>0.5 — IV</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8</td>
<td>2.0</td>
<td>0.2</td>
<td>0.5 — V</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5</td>
<td>2.0</td>
<td>0.2</td>
<td>0.5 — VI</td>
</tr>
</tbody>
</table>
FIG 2 CONCENTRATION PROFILES
FIG 3 PENETRATION DISTANCE

| t  | K  | Gr | Gc | Sc |  
|----|----|----|----|----|---
| 0.2| 0.5| 2.0| 0.2| 0.5| I 
| 0.2| 0.5| 2.0| 0.2| 0.7| II
| 0.2| 0.5| 2.0| 0.6| 0.5| III
| 0.2| 0.5| 4.0| 0.2| 0.5| IV
| 0.2| 0.8| 2.0| 0.2| 0.5| V 
| 0.4| 0.5| 2.0| 0.2| 0.5| VI

FIG 3 PENETRATION DISTANCE