CHAPTER 6

ERROR-CORRECTING CODES WITH SUB-BLOCK STRUCTURE

The contents of this chapter include matter from my following research paper:

1. Linear Codes having Sub-block Structure Correcting Random and Low-Density Burst Errors, communicated.
CHAPTER 6

ERROR-CORRECTING CODES WITH SUB-BLOCK STRUCTURE

Sub-block coding techniques have been in use in communication systems for many years. In most of these situations, sub-block coding is used in conjunction with retransmission. In this mode a message is coded with a sub-block check; the coded message is transmitted; and a parity check is calculated at the receiver. If the check indicates the presence of an error in a sub-block, the sub-block containing the erroneous message is repeated over the channel.

For channels with low error rate, error detection and retransmission proves to be a practical way of improving reliability. When error rate increases, the retransmission scheme may result in low throughput as more and more sub-blocks have to be retransmitted. In these circumstances, it may be practical to consider hybrid schemes which correct a few errors but reserve the use of retransmission as a backup.

Most of the error detecting and error correcting codes consider the distribution of errors over the whole word length. In the case of random errors, it is natural to expect that the number of errors increase as the code length increases, and these additional
random errors are in fact to be seen in the additional digits of the word length. In the case of burst errors too, a frequent occurrence of disturbances may result in many distinct bursts that are separated by some error free zone. In such cases, it becomes necessary to consider the code length to be divided into various sub-blocks and an error pattern as a combination of patterns lying within individual sub-blocks.

In special cases, codes having sub-block structure have been studied by Slopien (1960) as sum codes, by Wolf and Elspas (1963) as error-locating codes, and by Wolf (1965a) as codes derivable from the tensor product of parity check matrices. The work due to Gupta (1976), Gupta and Malhotra (1976) and Sharma and Gupta (1977) is worth mentioning in this direction.

In this chapter, we study error correcting linear codes having a sub-block structure similar to that described in the previous paragraph. Firstly, we study codes detecting random errors and correcting low-density burst errors within a single sub-block, and then study codes correcting random and low-density bursts simultaneously within a single sub-block. This study not only generalize the studies made with respect to the usual codes having no sub-block structure
but significantly generalizes the studies made on error-localizing codes. Lower and upper bounds on the number of parity check digits required for the existence of such codes have been obtained. Throughout this chapter, a burst shall mean an open-loop burst. The block of \( n \) digits, consisting of \( k \) information digits, and \( r = n - k \) check digits, shall be considered to be sub-divided into \( s \) mutually exclusive sub-blocks, each of length \( t = \frac{n}{s} \) digits.

6.1. **An Upper Bound for Codes Detecting Random and Correcting Low-Density Burst Errors within a Single Sub-block.**

A linear code capable of detecting random errors of weight \( e \) or less and correcting low-density bursts of length \( b \) or less with weight \( w \) or less within a single sub-block must satisfy the following conditions:

(a) The syndrome resulting from the occurrence of an error of weight \( e \) or less within a single sub-block must be distinct from the all-zeros syndrome,

(b) The syndrome resulting from the occurrence of a low-density burst error of length \( b \) or less with weight \( w \) or less within a single sub-block must
be distinct from the syndrome resulting from any other low-density burst of length \( b \) or less with weight \( w \) or less within the same sub-block,

(c) The syndrome resulting from the occurrence of a low-density burst error of length \( b \) or less with weight \( w \) or less within a single sub-block must be distinct from the syndrome resulting from any other low-density burst of length \( b \) or less with weight \( w \) or less within some other sub-block.

In the following, we give an upper bound on the number of parity check digits required for the existence of a linear code over \( \text{GF}(q) \) that detects random errors of weight \( e \) or less and corrects low-density burst errors of length \( b \) or less with weight \( w \) or less.

**Theorem 6.1.** Given positive integers \( e, w \) and \( b \) such that \( e < 2w < 2b \), a sufficient condition that there exists an \( (n, k) \) linear code over \( \text{GF}(q) \), subdivided into \( s \) sub-blocks of length \( t \) each, \( n = st, t > 2b \), that detects random errors of weight \( e \) or less and corrects low-density bursts of length \( b \) or less with weight \( w \) or less, occurring within a single sub-block, can always be constructed using \( r \) check digits where \( r \) is the smallest integer satisfying the inequality:
\[ q^r \geq [1 + (q-1)^{t-1}]^e + \sum_{p_1, p_2} 2^{w-1} \left( \binom{b-1}{p_1} (q-1)^{p_1} \right) \sum_{p_1 + p_2 \leq e} \]

\[ \times I(q, t-b; b, p_2) + \sum_{i=d} 2^{w-1} \left( \binom{b-1}{i} (q-1)^i \right) \]

\[ + \sum_{k=1}^{b-1} \sum_{r_1, r_2, r_3} \left( \binom{b-k-1}{r_1} \binom{k}{r_2} \binom{b-k-1}{r_3} (q-1)^{r_1 + r_2 + r_3 + 1} \right) \]

\[ + [1 + (q-1)]^{(b-1, w-1)} (s-1) \left[ q^{w-1} [ (q-1)(t-w+1) + 1 ] \right] \]

\[ + (q-1)^2 \sum_{j=w+1}^{b} \left( t-j+1 \right) [1 + (q-1)]^{(j-2, w-2)} - 1 \]

\[ \cdots (6.1) \]

where

\[ 0 \leq p_1 \leq w-1, \]
\[ 0 \leq p_2 \leq w, \]
\[ 0 \leq r_1 \leq w-2, \]
\[ 1 \leq r_2 \leq 2w-2, \]
\[ 0 \leq r_3 \leq w-1, \]
\[ r_2 + r_3 \geq w, \]
\[ e-1 \leq r_1 + r_2 + r_3 \leq 2w-2, \]
\[ d = \text{max.} (e, w), \]

\[ [1+x]^{(m, r)} = \begin{cases} 
1 + \left( \frac{m}{1} \right)x + \ldots + \left( \frac{m}{r} \right)x^r, & 0 \leq r \leq m, \\
0, & r < 0,
\end{cases} \]
and
\[
I(q,n;b,p) = \begin{cases} 
\binom{n}{p}(q-1)^p, & p = 0,1 \\
(q-1)^p \sum_{i=p}^{b} \binom{i-2}{p-2}(n-i+1), & p \geq 2.
\end{cases}
\]

**Proof.** The existence of such a code will be shown by constructing an appropriate \((n-k) \times n\) parity-check matrix for the desired code.

Select a non-zero \((n-k)\)-tuple as the first column of the parity-check matrix \(H\). To add subsequent columns to \(H\) appropriately, let us suppose that we have chosen the first \((s-1)\) columns corresponding to the first \((s-1)\) sub-blocks and the first \((j-1)\) columns \(h_1, h_2, \ldots, h_{j-1}\) of the \(s\)-th sub-block. While adding the \(j\)-th column \(h_j\) to the \(s\)-th sub-block, we must ensure conditions (a) - (c) stated at the beginning of this section.

According to the condition (a), for the detection of random errors of weight \(e\) or less in a sub-block, the syndrome resulting from the occurrence of \(e\) or fewer positions within any sub-block must be different from the all zero \((n-k)\)-tuple. So, the \(j\)-th column \(h_j\) can be added to the \(s\)-th sub-block, provided that

\(h_j\) is not a linear combination of any \((j-1)\) or fewer columns out of the \((j-1)\) columns \(h_1, h_2, \ldots, h_{j-1}\).
In other words,

$$h_j \neq (a_{11} h_{11} + a_{12} h_{12} + \ldots + a_{1q-1} h_{1q-1})$$  \hspace{1cm} (6.2)$$

where the \(h_i\) are any \(q-1\) columns among \(h_1, h_2, \ldots, h_{j-1}\).

As there are \((q-1)\) nonzero coefficients, therefore, the number of ways in which the coefficients \(a_i\), including the pattern of all zeros, can be chosen is

$$[1 + (q-1)]^{(j-1, q-1)}$$  \hspace{1cm} (6.3)$$

According to the condition (b), for the correction of low-density burst errors of length \(b\) or less with weight \(w\) or less within a sub-block, the syndrome of any low-density burst of length \(b\) or less with weight \(w\) or less within a single sub-block must be distinct from the syndrome resulting from any other low-density burst of length \(b\) or less with weight \(w\) or less within the same sub-block.

So, the \(j\)-th column \(h_j\) to be added to the \(s\)-th sub-block should be such that

\(h_j\) is not a linear combination of \((w-1)\) or fewer columns out of the immediately preceding \((b-1)\) columns \(h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1}\) of the \(s\)-th
sub-block together with a linear combination of any \( w \) or fewer columns out of any \( b \) or fewer consecutive columns from among the \((j-1)\) columns \( h_1, h_2, \ldots, h_{j-1} \).

In other words,

\[
h_j \not\in \left( b_{j_1} h_{j_1} + b_{j_2} h_{j_2} + \cdots + b_{j_{w-1}} h_{j_{w-1}} \right) \\
+ \left( c_{k_1} h_{k_1} + c_{k_2} h_{k_2} + \cdots + c_{k_w} h_{k_w} \right)
\]  \hspace{1cm} (6.4)

where the \( h_j \) are any \((w-1)\) columns among the immediately preceding \((b-1)\) columns \( h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1} \), and the \( h_k \) are any \( w \) columns within a set of \( b \) or fewer consecutive columns among \( h_1, h_2, \ldots, h_{j-1} \).

Since all possible linear combinations of \( e-1 \) or fewer columns are included in (6.3), therefore the coefficients \( b_j \) and \( c_k \) should be so chosen that at least \( e \) or these taken together are non-zero. To obtain the number of all possible distinct linear combinations, we analyze three different cases.

**Case 1.** When the \( h_k \) are taken from the first \( j-b \) columns of the \( s \)-th sub-block.

Choose a number \( p_1 \) of the \( b_j \) and \( p_2 \) of the \( c_k \) such that \( p_1 + p_2 \geq e \). The largest values which \( p_1 \)
and \( p_2 \) can attain are \( w-1 \) and \( w \) respectively. Now \( p_1 \) number of the \( b_j \) can be chosen in

\[
\binom{b-1}{p_1} (q-1)^{p_1}
\]

(6.5)

ways. To choose \( p_2 \) of the \( c_k \) is equivalent to evaluating the number of bursts of length \( b \) or less with weight \( p_2 \) in a vector of length \( j-b \). This can be done in (refer Sharma and Dass (1974))

\[
I(q, j-b; b, p_2)
\]

(6.6)

ways, where \( I(q, j-b; b, p_2) \) stands for the expression stated in the theorem.

Therefore, the total number of choices of coefficients in this case is

\[
2w-1 \sum_{p_1, p_2} \binom{b-1}{p_1} (q-1)^{p_1} I(q, j-b; b, p_2)
\]

(6.7)

\( p_1 + p_2 \geq c \)

**Case 2.** When the \( h_k \) are taken from the immediately preceding \( b-1 \) columns \( h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1} \) of the \( s \)-th sub-block

In this case, we have to select the coefficients \( b_j \) and \( c_k \), which are \( (2w-1) \) or less in number, from
among \((b-1)\) components. In view of the possibilities considered earlier, the additional number of ways in which the coefficients \(b_j\) and \(c_k\) can be selected are

\[
\sum_{i=d}^{2w-1} \binom{b-1}{i}(q-1)^i,
\]

(6.8)

where \(d = \max\ (e, w)\).

Case 2. When the \(h_{jk}\) are selected from the columns

\[h_{j-2b+2}, h_{j-2b+3}, \ldots, h_{j-1}\]

of the \(s\)-th sub-block such that all are neither taken from \(h_{j-2b+2} \ldots h_{j-2b+3} \ldots h_{j-b} \) nor from \(h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1}\).

In this case, let us suppose that the burst starts from the \((j-2b+k+1)\)-th component, which may obviously continue up to the \((j-b+k)\)-th component. Our object is to select \(w-1\) or fewer nonzero components among \((j-2b+k+2, j-2b+k+3, \ldots, j-b+k)\)-th positions together with \(w-1\) or less nonzero components among \((j-b+1, j-b+2, \ldots, j-1)\)-th positions.

In order to do so, let us choose \(r_1\) components from the \((j-2b+k+2, j-2b+k+3, \ldots, j-b)\)-th positions, \(r_2\) components from the \((j-b+1, j-b+2, \ldots, j-b+k)\)-th positions and \(r_3\) components from the \((j-b+k+1, j-b+k+2, \ldots, j-1)\)-th positions, where
0 \leq r_1 \leq w-2
\begin{array}{l}
1 \leq r_2 \leq 2w-2
0 \leq r_3 \leq w-1
\end{array}
(6.9)

Keeping in view the situations considered earlier, \( r_1, r_2, r_3 \) should follow the constraints:
\[
\begin{array}{l}
r_2 + r_3 \geq w
r_1 + r_2 + r_3 \leq 2w-2
\end{array}
(6.10)

Such a selection of coefficients gives us
\[
\begin{array}{l}
\sum_{r_1, r_2, r_3} \binom{b-k-1}{r_1} \binom{k}{r_2} \binom{b-k-1}{r_3} (q-1)^{r_1+r_2+r_3}
\end{array}
(6.11)

possible linear combinations where \( r_1, r_2, r_3 \) satisfy the constraints stated in (6.9) and (6.10).

Also, the \((j-2b+k+1)\)-th component can be selected in \((q-1)\) ways.

Then in this case the total number of choices of coefficients turns out to be
\[
\sum_{k=1}^{b-1} \sum_{r_1, r_2, r_3} \binom{b-k-1}{r_1} \binom{k}{r_2} \binom{b-k-1}{r_3} (q-1)^{r_1+r_2+r_3+1}
(6.12)

Thus, from (6.7), (6.8) and (6.12), the total number of choices of the coefficients \( b_j \) and \( c_k \) in (6.4) turns
out to be

\[ 2w-1 \sum_{p_1, p_2} \left[ \binom{b-1}{p_1} \binom{q-1}{p_1} I(q, j-b; b, p_2) \right] \]

\[ + \sum_{i=d}^{2w-1} \binom{b-1}{i} \binom{q-1}{i} \]

\[ + \sum_{k=1}^{b-1} \sum_{r_1, r_2, r_3} \binom{b-k-1}{r_1} \binom{k}{r_2} \binom{b-k-1}{r_3} \binom{q-1}{r_1 + r_2 + r_3 + 1} \]

\[ \cdots \]  \quad (6.13)

Finally, according to the condition (c) that the syndromes of any two error patterns lying within different sub-blocks be distinct, the syndrome of any low-density burst of length \( b \) or less with weight \( w \) or less within a single sub-block must be distinct from the syndrome resulting from any other low-density burst of length \( b \) or less with weight \( w \) or less lying within some other sub-block. So, the \( j \)-th column \( h_j \) can be added to the \( s \)-th sub-block, provided that

\( h_j \) is not a linear combination of any \((w-1)\) or fewer columns out of the immediately preceding \((b-1)\) columns \( h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1} \) of the \( s \)-th sub-block together with a linear combination of
any \( w \) or fewer columns out of any \( b \) or fewer consecutive columns from among the columns corresponding to any one of the first \( (s-1) \) sub-blocks.

In other words,

\[
h_j \neq (d_{r_1} h_{r_1} + d_{r_2} h_{r_2} + \cdots + d_{r_{w-1}} h_{r_{w-1}}) \\
+ (e_{s_1} h_{s_1} + e_{s_2} h_{s_2} + \cdots + e_{s_w} h_{s_w})
\]

where the \( h_r \) are any \((w-1)\) columns among the immediately preceding \((b-1)\) columns \( h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1} \), and the \( e_s \) are any \( w \) columns within a set of \( b \) or fewer consecutive columns among the columns corresponding to any one of the first \( (s-1) \) sub-blocks.

Now, the number of ways in which the coefficients \( d_r \) can be chosen is

\[
[1 + (q-1)]^{(b-1,w-1)},
\]

whereas to choose the coefficients \( e_s \) is equivalent to compute the number of bursts of length \( b \) or less with weight \( w \) or less in a vector of length \( t \). This can be done in (refer Sharma and Dass (1974))
\[ q^{w-1} \left[ (q-1)(t-w+1) + 1 \right] + (q-1)^2 \sum_{j=w+1}^{b} (t-j+1) \cdot \left[ 1 + (q-1) \right]^{(j-2,w-2)} - 1 \]  

(6.16)

ways.

(Note that in (6.16) we have considered that all the coefficients \( e_s \) in (6.14) cannot be zero, because in that case the only choices for the coefficients \( d_r \) have been taken care of in (6.7).)

Since the expression (6.16) indicates the number of possible choices for the coefficients \( e_s \) in (6.14) in a single sub-block whereas the coefficients \( e_s \) in (6.14) are among any one of the first \( (s-1) \) sub-blocks, therefore the total number of choices of the coefficients \( e_s \) turns out to be

\[ (s-1) \left[ q^{w-1} \left[ (q-1)(t-w+1) + 1 \right] + (q-1)^2 \right. \]

\[ \left. \cdot \sum_{j=w+1}^{b} (t-j+1) \left[ 1 + (q-1) \right]^{(j-2,w-2)} - 1 \right] \]

\[ \ldots \ldots \]  

(6.17)

Thus, from (6.15) and (6.17), the total number of choices of the coefficients \( d_r \) and \( e_s \) in (6.14) is
\[
[1 + (q-1)]^{(b-1,w-1)} (s-1) \left[ q^{w-1} [(q-1)(t-w+1)+1] \\
+ (q-1)^2 \sum_{j=w+1}^{b} (t-j+1) \left[ 1 + (q-1) \right]^{(j-2,w-2)-1} \right].
\]

\[\cdots \cdots \ (6.18)\]

So, the total possible number of combinations which \(h_j\) cannot be equal is

\[
[1 + (q-1)]^{(j-1,e-1)} + \sum_{p_1,p_2}^{2w-1} \left[ \binom{b-1}{p_1}(q-1)^{p_1} \right. \\
\left. \left. \binom{b-1}{p_1}(q-1)^{p_1} \right] \right.
\]

\[\cdots \cdots \ (6.19)\]

At worst, all possible choices of the coefficients
computed in (6.19) might yield a distinct sum.

Therefore, a column \( h_j \) can be added to \( H \) corresponding to the \( s \)-th sub-block provided that

\[
q^r > \left[ 1 + (q-1)^{(j-1, e-1)} + \sum_{p_1, p_2} \frac{2^{r-1}}{p_1 + p_2} \right. \\
\left. \cdot I(q, j-b; b, p_2) \right] + \sum_{i=d}^{2w-1} \binom{b-1}{i}(q-1)^i
\]

\[
+ \sum_{k=1}^{b-1} \sum_{r_1, r_2, r_3} \binom{b-k-1}{r_1} \binom{b-k-1}{r_2} (q-1)^{r_1 + r_2 + r_3 + 1}
\]

\[
+ \left[ 1 + (q-1)^{(b-l, w-1)} (s-l) \right] q^{w-1} \left[ (q-1)(t-w+1)+1 \right]
\]

\[
+ (q-1)^2 \sum_{j=w+1}^{b} \left[ 1 + (q-1)^{(j-2, w-2)} \right] -1
\]

\[\text{...... (6.20)}\]

But for an \((n,k)\) code to exist, the inequality in (6.20) should hold for \( j = t \), and thus we get (6.1).

Q.E.D.
Discussion. The result just obtained has been proved for \( e < 2w \). It should be noted that

\[
0 \leq p_1 \leq w-1, \quad 0 \leq p_2 \leq w \quad \text{when} \quad e \leq w+1
\]

and

\[
0 \leq p_1 \leq w-1, \quad 1 \leq p_2 \leq w \quad \text{when} \quad w+1 < e < 2w.
\]

Relaxing the random error detection constraint imposed over the code, i.e. setting \( e = 0 \), \( p_1 + p_2 \) shall assume values from 0 to \( 2w-1 \) and the joint summation involving \( p_1, p_2 \) in the inequality (6.1) shall split into two separate summations giving

\[
\sum_{p_1=0}^{w-1} (b-1)(q-1)^{p_1} = [1 + (q-1)](b-1, w-1)
\]

and

\[
\sum_{p_2=0}^{w} I(q, t-b; b, p_2) = \text{number of bursts of length } b \text{ or less with weight } w \text{ or less including the pattern of all zeros in a vector of length } t-b
\]

\[
=q^{w-1}[(q-1)(t-b-w+1)+1] + (q-1)^2 .
\]

\[
\sum_{i=w+1}^{b} (t-b-i+1)[1+(q-1)](i-2, w-2).
\]

Also, we shall have

\[
[1 + (q-1)](t-1, e-1) = 0
\]

and \( d \) will attain value \( w \).
The bound in (6.1) thus reduces to

\[
q^r > [1 + (q-1)]^{(b-1, w-1)} \left[ q^{w-1} \left( (q-1)(t-b-w+1) + 1 \right) \\
+ (q-1)^2 \sum_{i=w+1}^{b} (t-b-i+1) \left[ 1 + (q-1) \right]^{(1-i, w-2)} \right] \\
+ 2^{w-1} \sum_{i=w}^{b-1} (q-1)^i \\
+ \frac{b-1}{r_1} \sum_{k=1}^{r_1} \left( \frac{b-k-1}{r_2} \right) \left( \frac{k}{r_3} \right) (q-1)^{r_1+r_2+r_3+1} \\
+ [1 + (q-1)]^{(b-1, w-1)} (s-1) \left[ q^{w-1} \left( (q-1)(t-w+1) + 1 \right) \\
+ (q-1)^2 \sum_{j=w+1}^{b} (t-j+1) \left[ 1 + (q-1) \right]^{(j-1, w-2)} - 1 \right],
\]

..... (6.21)

which gives an upper bound on the number of parity check digits required for the existence of an \((n,k)\)
linear code over \(GF(q)\), subdivided into \(s\) sub-blocks of length \(t\) each, \(n = st\), \(t > 2b\), that corrects low-density burst errors of length \(b\) or less with weight \(w\) or less occurring within a single sub-block.
Again, relaxing the weight constraint imposed over the bursts to be corrected, i.e. setting \( w = b \), we see that the result proved in Theorem 6.1 is now valid for

\[
\begin{align*}
0 &\leq p_1 \leq b-1, \\
0 &\leq p_2 \leq b, \\
0 &\leq r_1 \leq b-2, \\
1 &\leq r_2 \leq 2b-2, \\
0 &\leq r_3 \leq b-1, \\
r_2 + r_3 &\geq b, \\
e-1 &\leq r_1 + r_2 + r_3 \leq 2b-2 \\
\text{and } d &\text{ = max. } (e, b).
\end{align*}
\]

Since the number of components out of which \( r_2 + r_3 \) can be chosen is \( b-1 \), therefore

\[
\sum_{k=1}^{b-1} \sum_{r_1, r_2, r_3} \binom{b-k-1}{r_1} \binom{k}{r_2} \binom{b-k-1}{r_3} r_1^{r_1} r_2^{r_2} r_3^{r_3+1} = 0.
\]

Also, for \( d = \text{max. } (e, b) \)

\[
2w-1 \sum_{i=d}^{b-1} \binom{b-1}{i} (q-1)^i = 2b-1 \sum_{i=d}^{b-1} \binom{b-1}{i} (q-1)^i,
\]

\[
\text{(since } w = b) \quad = 0.
\]

Moreover, for \( w = b \)
\[ 1 + (q-1)^{b-1}w^{-1} (s-1) \left[ q^{w-1} \left[ (q-1)(t-w+1)+1 \right] \right. \]
\[ + (q-1)^2 \sum_{j=w+1}^{b} (t-j+1) \left[ 1+(q-1)^{(j-2)(w-2)} -1 \right] \]
\[ = q^{b-1} (s-1) \left[ q^{b-1} \left[ (q-1)(t-b+1)+1 \right] -1 \right] \cdot \]

Then the bound obtained in (6.1) thus reduces to
\[ q^r > [1+(q-1)^{(t-1)(s-1)} + \sum_{p_1, p_2}^{2b-1} \left( \frac{b-1}{P_1} \right)^{p_1} (q-1)^{p_2} \]
\[ \cdot I(q, t-b; b, p_2) + q^{b-1}(s-1) \left[ q^{w-1} \left[ (q-1) \right] \cdot \right. \]
\[ \cdot (t-b+1)+1 \left. \right] -1 \], \quad (6.22) \]

which gives an upper bound on the number of parity check digits required for the existence of an \((n,k)\) linear code over GF(q), subdivided into \(s\) sub-blocks of length \(t\) each, \(n = st\), \(t > 2b\), that detects random errors of weight \(e\) or less and corrects burst errors of length \(b\) or less occurring within a single sub-block.

Next, relaxing the random error detection constraint as well as weight constraint imposed over the bursts to be corrected, \(i.e.\), setting \(c = 0\) and \(w = b\), the bound
obtained in Theorem 6.1 reduces to a sufficient condition for the existence of an \((n, k)\) linear code over \(GF(q)\), subdivided into \(s\) sub-blocks of length \(t\) each, \(n = st\), \(t > 2b\), that corrects burst errors of length \(b\) or less occurring within a single sub-block, reducing the inequality in (6.1) to

\[
q^r > q^{2(b-1)} \left[ (q-1)(t-2b+1) + 1 \right] \\
+ q^{b-1}(s-1) \left[ q^{b-1} \left[ (q-1)(t-b+1) + 1 \right] - 1 \right]. \quad (6.23)
\]

Simple deductions in special cases viz. by considering a single sub-block, i.e. by taking \(s = 1\); by relaxing the condition of the correction of bursts in a sub-block, which shall give an analogue of the Varshamov-Gilbert bound for codes having sub-block structure; and some others can also be derived.

6.2. Bounds for Codes Correcting Random/Low-Density Burst Errors within a Single Sub-Block

A linear code capable of correcting either random errors of weight \(e\) or less or low-density bursts of length \(b\) or less with weight \(w\) or less within a single sub-block must satisfy the following conditions:
(a) The syndrome resulting from the occurrence of an error of weight $e$ or less within a single sub-block must be distinct from the syndrome resulting from any other error of weight $e$ or less within the same sub-block.

(b) The syndrome resulting from the occurrence of an error of weight $e$ or less within a single sub-block must be distinct from the syndrome resulting from any other error of weight $e$ or less within any other sub-block.

(c) The syndrome resulting from the occurrence of a low-density burst error of length $b$ or less with weight $w$ or less within a single sub-block must be distinct from the syndrome resulting from any other low-density burst of length $b$ or less with weight $w$ or less within the same sub-block.

(d) The syndrome resulting from the occurrence of a low-density burst error of length $b$ or less with weight $w$ or less within a single sub-block must be distinct from the syndrome resulting from any other low-density burst of length $b$ or less with weight $w$ or less within any other sub-block.

In the following, we shall derive two results. The first result gives a lower bound on the number of check
digits required for the existence of a linear code capable of correcting either random errors or low-density burst errors occurring within a single sub-block. In the second result, we derive an upper bound on the number of check digits, which assures the existence of such a code. The bursts considered are 'open-loop bursts' and the block of n digits, consisting of r check digits, and k = n - r information digits, is considered to be subdivided into s mutually exclusive sub-blocks. Each sub-block contains t = \frac{n}{s} digits.

Theorem 6.2. The number of parity check digits \( r \) in an \((n, k)\) linear code over \( \text{GF}(q) \), subdivided into \( s \) sub-blocks of length \( t \) each, that corrects either random errors of weight \( e \) or less or low-density bursts of length \( b \) or less with weight \( w \) or less, \( e < w < b \), is bounded from below by

\[
\begin{align*}
r &\geq \log_q \left\{ 1 + \max \left[ s \sum_{i=1}^{e} (q-1)^i, \right. \right. \\
&\left.\left. s \left[ q^{w-1} \left(1 - \frac{1}{q^b} \right) \right], \right. \right. \\
&\left.\left. (t-b-w+1) + 1 \right] + (q-1)^b \sum_{i=w+1}^{b} (t-b-i+1) \right. \right. \\
&\left.\left. \left[ 1 + (q-1) \right]^{-1} \right] \right. \right. \\
&\left.\left. (1-2, w-2) - 1 \right] \right. \right. \\
\end{align*}
\]

(6.24)
Proof. Since the code is capable of correcting all errors of weight \( e \) or less within a single sub-block, any syndrome produced by \( e \) or fewer errors in a given sub-block must be distinct from any such syndrome likewise resulting from another set of up to \( e \) errors in the same sub-block (refer to condition (a)). Moreover, syndromes produced by combinations of up to \( e \) errors in different sub-blocks must be distinct by condition (b).

Thus, syndromes produced by combinations of up to \( e \) errors, whether in the same sub-block or in different sub-blocks, be distinct. Since there are

\[
{t \choose i}(q-1)^i
\]  

(6.25)

combinations of \( i \) errors out of \( t \) places in a given sub-block, and there are \( s \) sub-blocks in all, there must be

\[
1 + s \sum_{i=1}^{c} {t \choose i}(q-1)^i
\]  

(6.26)

distinct syndromes, counting the all-zero syndrome.

Again, since the code is capable of correcting all low-density bursts of length \( b \) or less with weight \( w \) or less within a single sub-block, by the conditions (c) and (d) and arguments in the preceding paragraph,
syndromes produced by low-density bursts of length \( b \) or less with weight \( w \) or less, whether in the same sub-block or in different sub-blocks, be distinct. Since there are

\[
q^{w-1} \left( (q-1)(t-b-w+1) + 1 \right) + (q-1)^2 \sum_{i=w+1}^{b} (t-b+i+1).
\]

\[
\cdot \left[ 1 + (q-1)^{(i-2,w-2)} \right]^{-1}
\]

(6.27)

low-density bursts of length \( b \) or less with weight \( w \) or less \((w \geq 1)\) out of the \( t \) places in a given sub-block, and there are \( s \) sub-blocks in all, there must be

\[
1 + s \left[ q^{w-1} \left( (q-1)(t-b-w+1) + 1 \right) + (q-1)^2 \sum_{i=w+1}^{b} (t-b+i+1). \right.

\]

\[
\cdot \left[ 1 + (q-1)^{(i-2,w-2)} \right]^{-1}\]

(6.28)

distinct syndromes, counting the all-zero syndrome.

Since the code is to be used to correct either random errors or low-density burst errors within a single sub-block, the maximum of (6.26) and (6.28) shall give us a lower bound on the number of parity check digits for the required code, i.e. we must have
\[
q^r > \left[ (q-1)^{\frac{t}{2}} \right] \left( q \right)^{s-1} + \left[ (q-1)^{\frac{t}{2}} \right] \left( q \right)^{s-1} \left( q \right)^{s-1} \\
. \sum_{i=1}^{\frac{t}{2}} \left( \begin{array}{c} t \cr i \end{array} \right) \left( q-1 \right)^i + \sum_{p_1, p_2}^{p_1 + p_2 \geq 2e} \left( \begin{array}{c} q-1 \cr p_1 \end{array} \right) \left( q-1 \right)^{p_1} \\
. \sum_{i=d}^{2w-1} \left( \begin{array}{c} q-1 \cr i \end{array} \right) \left( q-1 \right)^i \\
. \sum_{k=1}^{b-1} \sum_{r_1, r_2, r_3} \left( \begin{array}{c} b-k-1 \cr r_1 \end{array} \right) \left( \begin{array}{c} k \cr r_2 \end{array} \right) \left( \begin{array}{c} b-k-1 \cr r_3 \end{array} \right) \left( q-1 \right)^{r_1 + r_2 + r_3 + 1}
\]

which on simplification is the required expression (6.24).

Q.E.D.

Theorem 6.3. An \((n, k)\) linear code over GF(q) capable of correcting either random errors of weight \(e\) or less or low-density bursts of length \(b\) or less with weight \(w\) or less, \(e < w \leq b\), occurring within a single sub-block can always be constructed using \(r\) check digits where \(r\) is the smallest integer satisfying the inequality

\[
q^r > \left[ (q-1)^{\frac{t}{2}} \right] \left( q \right)^{s-1} + \left[ (q-1)^{\frac{t}{2}} \right] \left( q \right)^{s-1} \left( q \right)^{s-1} \\
. \sum_{i=1}^{\frac{t}{2}} \left( \begin{array}{c} t \cr i \end{array} \right) \left( q-1 \right)^i + \sum_{p_1, p_2}^{p_1 + p_2 \geq 2e} \left( \begin{array}{c} q-1 \cr p_1 \end{array} \right) \left( q-1 \right)^{p_1} \\
. \sum_{i=d}^{2w-1} \left( \begin{array}{c} q-1 \cr i \end{array} \right) \left( q-1 \right)^i \\
. \sum_{k=1}^{b-1} \sum_{r_1, r_2, r_3} \left( \begin{array}{c} b-k-1 \cr r_1 \end{array} \right) \left( \begin{array}{c} k \cr r_2 \end{array} \right) \left( \begin{array}{c} b-k-1 \cr r_3 \end{array} \right) \left( q-1 \right)^{r_1 + r_2 + r_3 + 1}
\]

(expr. contd.)
\[ + \left[ \sum_{i=0}^{w-1} \frac{(b-1)(q-1)}{(t-i+1)} \right] \left[ q^{e-1} \left[ (q-1)(t-e+1) + 1 \right] - 1 \right] \]

\[ + (q-1)^2 \sum_{i=e+1}^{b} \left[ (q-1) \right]^{i-2} \left[ (q-1) \right]^{w-2} \]

\[ + [1 + (q-1)]^{(b-1,w-1)} (s-1)(q-1)^2 \sum_{i=e+1}^{b} (t-i+1) \]

\[ \sum_{j=e-1}^{w-2} \frac{(i-2)}{(q-1)^j} \]

(6.29)

where

\[ 0 \leq p_1 \leq w-1, \]
\[ 0 \leq p_2 \leq w, \]
\[ 0 \leq r_1 \leq w-2, \]
\[ 1 \leq r_2 \leq 2w-2, \]
\[ 0 \leq r_3 \leq w-1, \]
\[ r_2 + r_3 \geq w, \]
\[ 2e-1 \leq r_1 + r_2 + r_3 \leq 2w-2, \]
\[ d = \text{max} \left( 2e, w \right), \]

and \([1 + x]^{(m,r)}\) and \(I(q,n;b,p)\) stand for the symbols stated in Theorem 6.1 of the preceding section.

**Proof.** As in Theorem 6.1, after having selected \((s-1)t\) columns corresponding to the first \((s-1)\) sub-blocks and the first \((j-1)\) columns \(h_1, h_2, ..., h_{j-1}\) corresponding to the \(s\)-th sub-block of the parity-check matrix \(H\), a \(j\)-th
column $h_j$ can be added if it fulfills four requirements laid down below.

As a first requirement, since the code is to correct all combinations of weight $e$ or less within a single sub-block therefore by condition (a), the column $h_j$ to be added to the $s$-th sub-block should be such that it is not a linear combination of any $2e-1$ or fewer columns among $h_1, h_2, \ldots, h_{j-1}$. Any $2e-1$ or fewer columns out of $j-1$, including the pattern of all zeros, can be chosen in

$$[1 + (q-1)](j-1,2e-1)$$

ways.

Further, by condition (b), the syndromes of any two error patterns of weight $e$ or less, each belonging to different sub-blocks, must not be same. So, the second requirement is that the $j$-th column $h_j$ to be added to the $s$-th sub-block should be such that it is not a linear combination of any $(e-1)$ or fewer columns out of the $(j-1)$ columns $h_1, h_2, \ldots, h_{j-1}$ of the $s$-th sub-block together with any set of $e$ or fewer columns out of any one of the first $(s-1)$ sub-blocks.

This condition gives rise to

$$[1 + (q-1)](j-1,e-1) (s-1) \sum_{i=1}^{e} \binom{t}{i}(q-1)^i$$  \hspace{1cm} (6.31)
possible linear combinations which cannot be equal to \(h_j\).

By condition (c), the syndromes of any two error patterns, each of which is a low-density burst of length \(b\) or less with weight \(w\) or less, belonging to the same sub-block, should be different. So, the third requirement is that the \(j\)-th column \(h_j\) to be added to the \(s\)-th sub-block should be such that it is not a linear combination of any \((w-1)\) or fewer columns out of the immediately preceding \((b-1)\) columns \(h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1}\) together with a linear combination of any \(w\) columns out of any \(b\) or fewer consecutive columns among all the \((j-1)\) columns \(h_1, h_2, \ldots, h_{j-1}\) of the \(s\)-th sub-block.

As all possible combinations of \((2e-1)\) or fewer columns within the \(s\)-th sub-block are included in (6.30) therefore in order to compute additional possible linear combinations that cannot equal \(h_j\), we use the arguments given for a similar case in Theorem 6.1 of the preceding section. We see that the additional possible linear combinations that cannot equal \(h_j\), are

\[
\begin{align*}
\sum_{P_1, P_2 \mid 0} &\binom{b-1}{P_1} \binom{q-1}{P_2} I(q, j-b, b, p_2) + \sum_{i=d}^{2w-1} \binom{b-1}{i} (q-1)^i \\
+ &\sum_{k=1}^{b-1} \sum_{r_1, r_2, r_3} \binom{b-k-1}{r_1} \binom{k}{r_2} \binom{b-k-1}{r_3} (q-1)^{r_1+r_2+r_3+1}
\end{align*}
\]

...... (6.32)
where

\[ 0 \leq p_1 \leq w-1, \]
\[ 0 \leq p_2 \leq w, \]
\[ 0 \leq r_1 \leq w-2, \]
\[ 1 \leq r_2 \leq 2w-2, \]
\[ 0 \leq r_3 \leq w-1, \]
\[ r_2 + r_3 \geq w, \]
\[ 2e-1 \leq r_1 + r_2 + r_3 \leq 2w-2 \]

and \( d = \max. (2e, w). \)

The fourth requirement, by condition (d), is that the syndrome of any low-density burst of length \( b \) or less with weight \( w \) or less within a sub-block must be different from the syndrome resulting from any other low-density burst of length \( b \) or less with weight \( w \) or less within any other sub-block. So, the \( j \)-th column \( h_j \) to be added to the \( s \)-th sub-block should be such that it is not a linear combination of any \( (w-1) \) or fewer columns out of the immediately preceding \( (b-1) \) columns \( h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1} \) of the \( s \)-th sub-block together with a linear combination of any \( w \) or fewer out of any \( b \) or fewer consecutive columns from among any one of the first \( (s-1) \) sub-blocks.

As all possible combinations of \( (e-1) \) or fewer columns within the \( s \)-th sub-block together with linear combinations
of any e or fewer columns from among any one of the first (s-1) sub-blocks are included in (6.31), therefore the additional possible linear combinations that cannot equal \( h_j \), are (refer (5.40), (5.41) and (5.42) of the fifth chapter)

\[
\begin{align*}
\left[ \sum_{i=e}^{w-1} (b-1)(q-1)^i \right] (s-1) \left[ q^{e-1} [(q-1)(t-e+1)+1] -1 \\
+ (q-1)^2 \sum_{i=e+1}^{b} (t-i+1) [1+(q-1)](i-2,e-2) \right] \\
+ \left[ \sum_{i=e}^{w-1} (b-1)(q-1)^i \right] (s-1) \left[ (q-1)^2 \sum_{i=e+1}^{b} (t-i+1) \right] \\
\cdot \left[ \sum_{j=e-1}^{w-2} (i-2)(q-1)^j \right] + [1+(q-1)](b-1,e-1) (s-1) \\
\cdot (q-1)^2 \sum_{i=e+1}^{b} (t-i+1) \sum_{j=e-1}^{w-2} (i-2)(q-1)^j
\end{align*}
\]

which on simplification equals

\[
\begin{align*}
\left[ \sum_{i=e}^{w-1} (b-1)(q-1)^i \right] (s-1) \left[ q^{e-1} [(q-1)(t-e+1)+1] -1 \\
+ (q-1)^2 \sum_{i=e+1}^{b} (t-i+1) [1+(q-1)](i-2,w-2) \right]
\end{align*}
\]

(expr. contd.)
\[ + \left[ 1 + (q-1) \right]^{(b-1,w-1)} (s-1)(q-1)^2 \sum_{i=e+1}^{b} (t-i+1) . \]

\[ \cdot \sum_{j=e-1}^{w-2} \binom{i-2}{j}(q-1)^j. \quad (6.33) \]

Thus, from (6.30), (6.31), (6.32) and (6.33), the total possible number of combinations which \( h_j \) cannot be equal is

\[ [1 + (q-1)]^{(j-1,2e-1)} + [1 + (q-1)]^{(j-1,e-1)(s-1)} . \]

\[ \cdot \sum_{i=1}^{e} \binom{t}{i}(q-1)i + \sum_{p_1,p_2}^{2w-1} \left[ \binom{b-1}{p_1} (q-1)^{p_1} \right] . \]

\[ \cdot \sum_{i=d}^{2w-1} \binom{b-1}{i}(q-1)i \]

\[ + \sum_{k=1}^{b-1} \sum_{r_1,r_2,r_3}^{r_1+r_2+r_3+1} \binom{b-k-1}{r_1} \binom{k}{r_2} \binom{b-k-1}{q-1} \binom{r_1+r_2+r_3+1}{r_1} \]

\[ + \sum_{i=e}^{w-1} \binom{b-1}{i}(q-1)i \] (s-1) \[ q^{e-1} [(q-1)(t-e+1)+1] -1 \]

\[ + [1 + (q-1)]^{(i-2,w-2)} \sum_{i=e+1}^{b} (t-i+1) \]

\[ \cdot \sum_{j=e-1}^{w-2} \binom{i-2}{j}(q-1)^j. \quad (6.34) \]
The result now follows as in Theorem 6.1. Q.E.D.

Discussion. The result just obtained has been proved for \( e < w \). For \( e \geq w \), the expressions obtained in (6.32) and (6.33) both vanish and the bound obtained in inequality (6.29) reduces to

\[
q^r > \left[ 1 + (q-1) \right] (t-1) 2^{e-1} + \left[ 1 + (q-1) \right] (t-1) 2^{e-1} (s-1) + \sum_{i=1}^{e} \binom{i}{t}(q-1)^i,
\]

which gives an upper bound on the number of parity check digits required for the existence of an \((n, k)\) linear code over GF\((q)\), subdivided into \( s \) sub-blocks of length \( t \) each, \( n = st \), that corrects random errors of weight \( e \) or less occurring within a single sub-block.

Again, if we drop random error correction consideration, i.e. if we take \( e = 1 \), the bound obtained in Theorem 6.3 gives a sufficient number of parity check digits required for the existence of an \((n, k)\) linear code that corrects low-density bursts of length \( b \) or less with weight \( w \) or less occurring within a single sub-block.
6.3. Bounds for Codes Correcting Simultaneously Random and Low-Density Burst Errors within a Single Sub-Block

In the previous section, we studied codes that correct either random errors or low-density burst errors lying within a single sub-block. Such codes correct either of the two types of the given error patterns, but may not correct both in general. The failure to correct random errors together with low-density burst errors is so because while formulating the four conditions (a)-(d) at the beginning of the previous section, we have not taken care of the fact that the syndromes of these two types of error patterns be different. This study is made in this section.

A linear code capable of correcting random errors of weight e or less simultaneously with low-density bursts of length b or less with weight w or less within a single sub-block must satisfy conditions (a) - (d) stated in the preceding section of this chapter alongwith the following two conditions:

(e) The syndrome resulting from the occurrence of an error of weight e or less within a single sub-block must be distinct from the syndrome resulting from a low-density burst of length b or less with weight
w or less occurring within the same sub-block.

(f) The syndrome resulting from the occurrence of an error of weight e or less within a single sub-block must be distinct from the syndrome resulting from a low-density burst of length b or less with weight w or less occurring within some other sub-block.

In the following, we shall derive two results. The first result gives a lower bound on the number of parity check digits required for the existence of a linear code capable of correcting random errors simultaneously with low-density burst errors occurring within a single sub-block. In the second result, we derive an upper bound on the number of check digits, which assures the existence of such a code.

In what follows, by a burst we shall mean an 'open-loop burst', and the block of \( n \) digits consisting of \( r \) check digits and \( k = n - r \) information digits is considered to be subdivided into \( s \) mutually exclusive sub-blocks, each of length \( t = \frac{n}{s} \) digits.

**Theorem 6.4** The number of parity check digits in an \((n, k)\) linear code over \( \text{GF}(q) \), subdivided into \( s \) sub-blocks of length \( t \) each, that corrects all
combinations of \( e \) or fewer errors and all low-density bursts of length \( b \) or less with weight \( w \) or less, \( 1 \leq e < w \leq b \), occurring within a single sub-block is at least

\[
\log_q \left[ 1 + s \sum_{i=1}^{e} \frac{(t)(q-1)^i}{i} + s \sum_{i=e+1}^{b} \frac{(t-1+i)}{i} \right.
\]

\[
\cdot \left. \sum_{j=e+1}^{w} \frac{(1-2)(q-1)^j}{j-2} \right].
\] (6.35)

**Proof.** The maximum number of distinct syndromes available using \( r \) check digits is \( q^r \). The proof proceeds by first counting the number of syndromes that are required to be distinct by conditions (a) - (f) and then setting this number less than or equal to \( q^r \).

Since the code is capable of correcting all errors of weight \( e \) or less within a single sub-block, any syndrome produced by \( e \) or fewer errors in a given sub-block must be distinct from any such syndrome likewise resulting from another set of up to \( e \) errors in the same sub-block (refer to condition (a)). Moreover, syndromes produced by combinations of up to \( e \) errors in different sub-blocks must be distinct by condition (b). Thus, syndromes produced by combinations of up to \( e \) errors, whether in the same sub-block or in different sub-blocks, should be
distinct. Since there are

\[ \binom{t}{i} (q-1)^i \]

combinations of \( i \) errors out of \( t \) places in a sub-block, and there are \( s \) sub-blocks in all, there must be

\[ 1 + s \sum_{i=1}^{t} \binom{t}{i} (q-1)^i \]

(6.36)
distinct syndromes, counting the all-zero syndrome.

Also, since the code corrects all those low-density bursts of length \( b \) or less which are of weight \( w \) or less occurring within a single sub-block, any syndrome produced by a low-density burst of length \( b \) or less with weight \( w \) or less within any sub-block must be distinct from any such syndrome likewise resulting from another low-density burst of length \( b \) or less with weight \( w \) or less within the same sub-block (refer to condition (c)). Moreover, syndromes produced by low-density bursts of length \( b \) or less with weight \( w \) or less in different sub-blocks must be distinct by condition (d). Thus, syndromes produced by low-density bursts of length \( b \) or less with weight \( w \) or less, whether in the same sub-block or in different sub-blocks, should be distinct.
Again, since the code corrects random errors of weight \( e \) or less simultaneously with low-density bursts of length \( b \) or less with weight \( w \) or less occurring within a single sub-block, in view of conditions (e) and (f), the syndromes produced by low-density bursts of length \( b \) or less with weight \( w \) or less, whether in the same sub-block or in different sub-blocks, should be distinct except for those which are random errors of weight \( e \) or less.

Since the bursts of length \( e \) or less have weight \( e \) or less and hence are included in \((6,36)\), we need only compute the number of bursts of length \( e+1, e+2, \ldots, b \) with weight greater than \( e \) but of weight less than or equal to \( w \) occurring within a single sub-block. The number of bursts of length \( i(>e) \) with weight greater than \( e \) but of weight less than or equal to \( w \) in a vector of length \( t \) is

\[
\sum_{j=e+1}^{w} \binom{t-2}{j-2}(t-i+1)(q-1)^j.
\]

Therefore, the number of bursts of length \( e+1, e+2, \ldots, b \) with weight greater than \( e \) but of weight less than or equal to \( w \) in a vector of length \( t \) is

\[
\sum_{i=e+1}^{b} \binom{t-1+1}{i+1} \sum_{j=e+1}^{w} \binom{i-2}{j-2}(q-1)^j.
\]
Since there are $s$ sub-blocks in all, there must be
\[ s \sum_{i=e+1}^{b} (t-i+1) \sum_{j=e+1}^{r} (i-2)(q-1)^j \] (6.37)
distinct syndromes.

Thus, from (6.36) and (6.37), the total number of distinct syndromes is
\[ 1 + s \sum_{i=1}^{e} \binom{t}{i}(q-1)^i + s \sum_{i=e+1}^{b} (t-i+1) \sum_{j=e+1}^{r} (i-2)(q-1)^j. \]

Since the maximum number of distinct syndromes available using $r$ check digits is $q^r$, we must have
\[ q^r \geq 1 + s \sum_{i=1}^{e} \binom{t}{i}(q-1)^i + s \sum_{i=e+1}^{b} (t-i+1) \sum_{j=e+1}^{r} (i-2)(q-1)^j. \] (6.38)

The result now follows by taking logarithm on both the sides. Q.E.D.

Incidentally, it can be shown that the result just obtained applies for block codes also which do not necessarily form a linear subspace.

Discussion. The result just obtained has been proved for $e < w$. However, if we take $e \geq w$ then the code is
capable of correcting all random errors of weight \( e \) or less and in particular all low-density bursts of length \( b \) or less with weight \( w \) or less, occurring within a single sub-block. Therefore, the low-density burst consideration becomes redundant and the corresponding expression in (6.37) vanishes. The result of Theorem 6.4 thus reduces to that the number of check digits is bounded from below by

\[
\log_q \left[ 1 + s \sum_{i=1}^{e} \binom{t}{i} (q-1)^i \right],
\]

which is an analogue of the Hamming's sphere-packing bound for codes correcting random errors of weight \( e \) or less occurring within a single sub-block of the code.

For \( w = b \), i.e. relaxing the weight constraint over the bursts to be corrected, the lower bound on the number of parity checks becomes

\[
\log_q \left[ 1 + s \sum_{i=1}^{e} \binom{t}{i} (q-1)^i + s \sum_{i=e+1}^{b} \binom{t-1}{i} (q-1)^i \right].
\]

\[
. \sum_{j=e+1}^{b} \binom{i-2}{j-2} (q-1)^j
\]

which gives a necessary number of parity check digits required for a code that corrects all combinations of
weight \( e \) or less and all bursts of length \( b \) or less occurring within a single sub-block of the code.

Next, setting \( e = 1 \), i.e. relaxing the constraint of random error correction, the bound obtained in (6.35) reduces to

\[
\log_q \left[ 1 + s \sum_{i=2}^{b} (t-i+1) \sum_{j=2}^{w} \frac{1}{j} (q-1)^{j-1} \right]
\]

\[
= \log_q \left[ 1 + s \left( q^{w-1} \sum_{i=2}^{b} (t-i+1) \sum_{j=2}^{w} \frac{1}{j} (q-1)^{j-1} \right) \right]
\]

which gives a lower bound on the number of parity check digits required for a code that corrects all bursts of length \( b \) or less with weight \( w \) or less occurring within a sub-block of the code.

Further, for \( e = 1 \) and \( w = b \), i.e. relaxing the constraints of random error correction and that of weight imposed over the bursts to be corrected, the bound in (6.35) reduces to

\[
\log_q \left[ 1 + s \sum_{i=2}^{b} (t-i+1) \sum_{j=2}^{b} \frac{1}{j} (q-1)^{j-1} \right]
\]
\[
= \log_q \left[ 1 + s \left[ q^{b-1} \left[ (q-1)(t-b+1) + 1 \right] - 1 \right] \right],
\]

which gives a lower bound on the number of parity check digits required for a code that corrects all bursts of length \( b \) or less occurring within a sub-block of the code. This result is an analogue of the result proved by Fire (1959) in the binary case for codes having no sub-block structure.

Lastly, for \( s = 1 \) (which means \( t = n \)), i.e., considering the case of usual codes having no sub-block structure, the bound obtained in (6.35) reduces to

\[
\log_q \left[ 1 + \sum_{i=1}^{e} \binom{n}{i} (q-1)^i + \sum_{i=e+1}^{b} (n-i+1) \cdot \sum_{j=e+1}^{w} \binom{i-2}{j-2} (q-1)^j \right],
\]

which coincides with a result due to Sharma and Dass (1977a, Theorem 2).

Several other deductions in special cases viz., by taking \( s = 1, w = b; s = 1, e = 1; s = 1, e = 1, \) \( w = b \), may also be derived for codes having no sub-block structure.
Theorem 6.5. Given positive integers e, w and b such that $1 \leq e < w \leq b$, a sufficient condition for there to exist an $(n, k)$ linear code over $GF(q)$, $n > 2b$, subdivided into $s$ sub-blocks of length $t$ each, that corrects all combinations of $e$ or fewer errors and all low-density bursts of length $b$ or less with weight $w$ or less, occurring within a single sub-block, is

$$q^r > [1 + (q-1)]^{(t-1, 2e-1)} + [1 + (q-1)]^{(t-1, e-1)}(s-1) .$$

$$\cdot \left[ \sum_{i=1}^{e} (q-1)^i + (q-1)^2 \sum_{i=e+1}^{b} (t-i+1) \sum_{j=e-1}^{w-2} (q-1)^j \right]$$

$$+ \frac{2w-1}{p_1 p_2} \frac{p_1}{p_2} \frac{p_1}{p_2} I(q, t-b; b, p_2)$$

$$+ \frac{2w-1}{p_1} (q-1)^i$$

$$+ \sum_{i=d}^{b-1}$$

$$+ \sum_{k=1}^{b-1} \sum_{r_1, r_2, r_3} (q-1)^k (r_1 \cdot r_2 \cdot r_3)$$

$$+ \left[ \sum_{i=e}^{w-1} (q-1)^i \right] (s-1) \sum_{i=1}^{e} (q-1)^i + \frac{t-b-1}{e} (q-1)^e$$

$$+ (s-1)(q-1)^2 \sum_{i=e+1}^{w-2} (q-1)^j$$

(expr. contd.)
where

\[ \begin{align*}
0 & \leq p_1 \leq w-1, \\
0 & \leq p_2 \leq w, \\
0 & \leq r_1 \leq w-2, \\
1 & \leq r_2 \leq 2w-2, \\
0 & \leq r_3 \leq w-1, \\
r_2 + r_3 & \geq w, \\
2e-1 & \leq r_1 + r_2 + r_3 \leq 2w-2, \\
d & = \max(2e, w), \\
2 & \leq q_1 \leq w, \\
0 & \leq q_2 \leq e-1,
\end{align*} \]

and \([1 + x]^{(m, r)}\) and \(I(q, n; b, p)\) stand for the symbols stated in Theorem 6.1 of this chapter.

**Proof.** The steps carried out in the proof of Theorem 6.3 stand as it is in this case, since they cover the conditions (a) - (d) for adding the \(j\)-th column \(h_j\) to the \(s\)-th sub-block of the parity check matrix, \(H\). Therefore, we need to compute additional possible linear combinations that \(h_j\) cannot be equal to in view of the conditions (e) and (f).
Since a sub-block may contain random errors and low-density burst errors, therefore by condition (e),
the syndrome of any random error of weight e or less within a sub-block must be different from the syndrome
resulting from a low-density burst of length b or less with weight w or less within the same sub-block. So,
the j-th column $h_j$ to be added to the s-th sub-block should be such that

(1) $h_j$ is not a linear combination of any (e-1) or fewer columns out of the (j-1) columns $h_1, h_2, \ldots, h_{j-1}$,
the s-th sub-block together with a linear combination of any w or fewer columns out of b or fewer consecutive columns from among the (j-1) columns $h_1, h_2, \ldots, h_{j-1}$ of the s-th sub-block; in other words,

$$h_j \neq (a_1 h_{11} + a_2 h_{12} + \ldots + a_w h_{1w})$$
$$+ (b_{k_1} h_{k_1} + b_{k_2} h_{k_2} + \ldots + b_{k_{e-1}} h_{k_{e-1}}),$$

(6.40)

where the $h_i$ are any w columns from a set of b consecutive columns and the $h_k$ are any e-1 columns among the previously chosen (j-1) columns $h_1, h_2, \ldots, h_{j-1}$ of the s-th sub-block.
and

$$(ii) \ h_j \text{ is not a linear combination of any } w-1 \text{ or fewer columns out of the immediately preceding} (b-1) \text{ columns } h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1} \text{ together with any set of } e \text{ or fewer columns out of the} j-1 \text{ columns } h_1, h_2, \ldots, h_{j-1} \text{ of the } s-\text{th sub-block; in other words}$$

$$h_j \notin (c_{s_1} h_{s_1} + c_{s_2} h_{s_2} + \ldots + c_{s_{w-1}} h_{s_{w-1}}) + (d_{t_1} h_{t_1} + d_{t_2} h_{t_2} + \ldots + d_{t_e} h_{t_e}), \quad (6.41)$$

where the $h_s$ are any $w-1$ columns among $h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1}$, and the $h_t$ are any $e$ columns among $h_1, h_2, \ldots, h_{j-b-1}$ with all the coefficients $d_t$ non-zero.

It is clear that condition (i) assures that the syndrome of any random error pattern of weight $e$ or less occurring within the $s$-th sub-block is different from the syndrome of any pattern which is a burst of length $b$ or less with weight $w$ or less out of $j$ components except when the low-density burst pattern include the last component (i.e., the $j$-th); and the exact $e$-weight pattern is selected from the first $j-b-1$ components in a way that is now specified by the inequality (6.41).
Since all possible linear combinations of 2e-1 or fewer columns are included in (6.30), we should choose the coefficients in (6.40) and (6.41) so that at least 2e of the a\(_i\) and the b\(_k\) taken together and at least o of the c\(_s\) are non-zero. In order to do so, choose q\(_1\) of the a\(_i\) and q\(_2\) of the b\(_k\) such that q\(_1\) + q\(_2\) \geq 2e. Obviously, 2 \leq q\(_1\) \leq w, 0 \leq q\(_2\) \leq e-1. Now q\(_1\) of the a\(_i\) which form a burst of length b or less having weight q\(_1\) in a vector of length j-1, can be selected in

\[ I(q, j-1; b, q_1) \]  

(6.42) 

ways; and q\(_2\) of the b\(_k\) can be selected in

\[ \binom{j-1}{q_2}(q-1)^{q_2} \]  

(6.43) 

ways. Further, at least o of the c\(_s\) can be chosen in

\[ \sum_{i=o}^{w-1} \binom{b-1}{i}(q-1)^i \]  

(6.44) 

ways, whereas the number of choices in which all the nonzero d\(_t\) can be selected is

\[ \binom{j-b-1}{e}(q-1)^{q_2}. \]  

(6.45) 

Thus, from (6.42), (6.43), (6.44) and (6.45), the total number of choices of the coefficients in this case turns
out to be

\[ e + w - 1 \sum_{q_1, q_2} \left[ \left( \frac{j-1}{q_2} \right) (q-1)^{q_2} I(q, j-1; b, q_1) \right] \]
\[ q_1 + q_2 = 2e \]

\[ + \left[ \sum_{i=e}^{w-1} \left( \frac{b-1}{1} \right) (q-1)^i \right] \left[ \left( \frac{j-b-1}{0} \right) (q-1)^e \right]. \]  \quad (6.46)

Finally, by condition (f), the syndrome of any random error of weight \( e \) or less within a sub-block must be different from the syndrome resulting from a low-density burst of length \( b \) or less with weight \( w \) or less within some other sub-block. So, the \( j \)-th column \( h_j \) to be added to the \( s \)-th sub-block should be such that

(iii) \( h_j \) is not a linear combination of any \((e-1)\) or fewer columns out of the \( (j-1) \) columns \( h_1, h_2, \ldots, h_{j-1} \) of the \( s \)-th sub-block together with a linear combination of any \( w \) or fewer columns out of \( b \) or fewer consecutive columns from among the columns corresponding to any one of the first \((s-1)\) sub-blocks,

and

(iv) \( h_j \) is not a linear combination of any \( w-1 \) or fewer columns out of the immediately preceding
(b-1) columns $h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1}$ of the s-th sub-block together with any set of e or fewer columns out of any one of the first (s-1) sub-blocks.

It is clear that condition (iii) assures that the syndrome of any random error pattern occurring within the s-th sub-block will be different from the syndrome of any low-density burst error pattern occurring within any one of the first (s-1) sub-blocks; whereas condition (iv) assures that the syndrome of any low-density burst error pattern occurring within the s-th sub-block will be different from the syndrome of any random error pattern occurring within any one of the first (s-1) sub-blocks.

Keeping in view the cases considered earlier, in order to compute the additional possible linear combinations arising out of (iii), we observe that when (e-1) or fewer columns to be considered out of the j-1 columns $h_1, h_2, \ldots, h_{j-1}$ lie within the immediately preceding b-1 columns $h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1}$, this case has been covered by condition (d). Again, when w or fewer columns out of b or fewer consecutive columns to be considered out of any one of the first (s-1) sub-blocks are e or less in number, this case has also been covered by condition (b). So, to choose additional possible linear combinations arising out of (iii), we may restate (iii) as follows:
\( h_j \) is not a linear combination of any \((e-1)\) or fewer columns out of the \(j-1\) columns \(h_1, h_2, \ldots, h_{j-1}\) but not confining to \(h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1}\) together with a linear combination of any \(w\) or fewer but \(e+1\) or more out of \(b\) or fewer consecutive columns from among any one of the first \((s-1)\) sub-blocks.

Now \((e-1)\) or fewer columns out of \(j-1\) columns but not confining to \(h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1}\) can be chosen in

\[
[1 + (q-1)]^{(j-1, e-1)} - \left[1 + (q-1)\right]^{(b-1, e-1)}
\]

ways; whereas to choose \(w\) or fewer but \(e+1\) or more out of \(b\) or fewer consecutive columns out of \(t\) columns is equivalent to compute the number of bursts of length \(b\) or less with weight \(e+1\) or more upto at most \(w\) in a vector of length \(t\), whose number is (refer Sharma, Dass and Gupta (1977))

\[
(q-1)^2 \sum_{i=e+1}^{b} \sum_{j=e-1}^{t-i+1} (i-2)(q-1)^j.
\]

Thus, the total number of combinations to which \(h_j\) cannot be equal in view of \((iii^*)\) is
\[
\left[ 1 + (q-1)^{j-1}, e-1 \right] - \left[ 1 + (q-1)^{(b-1)}(e-1) \right] (s-1).
\]

\[
\left[ (q-1)^2 \sum_{i=e+1}^{b} (t-1+1) \sum_{j=e-1}^{w-2} (i-2)(q-1)^j \right].
\]

(6.47)

Lastly, in order to compute the additional possible linear combinations arising out of (iv), we observe that when \((w-1)\) or fewer columns out of the immediately preceding \((b-1)\) columns to be considered are \(e-1\) or less in number, this case has been covered by condition (b). Again, when any \(e\) or less columns out of any one of the first \((s-1)\) sub-blocks to be considered lie within a burst of length \(b\) or less, this case has also been covered by condition (d). So, to choose additional possible linear combinations arising out of (iv), we may restate (iv) as follows:

(iv*) \(h_j\) is not a linear combination of any \(e\) or more upto at most \((w-1)\) out of the immediately preceding \((b-1)\) columns \(h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1}\) of the \(s\)-th sub-block together with any \(e\) or fewer columns not lying within a burst of length \(b\) or less out of any one of the first \((s-1)\) sub-blocks.
Now e or more up to at most \((w-1)\) columns out of the \(b-1\) columns \(h_{j-b+1}, h_{j-b+2}, \ldots, h_{j-1}\) can be chosen in

\[
\sum_{i=e}^{w-1} \binom{b-1}{i}(q-1)^i
\]  \hspace{1cm} (6.48)

ways; whereas \(e\) or fewer columns not lying within a burst of length \(b\) or less out of a vector of length \(t\) can be selected in

\[
\begin{align*}
\sum_{i=1}^{\min(e, t)} \binom{t}{i}(q-1)^i & - \left[ q^{e-1} \left( (q-1)(t-e+1) + 1 \right) - 1 \right] \\
+ (q-1)^2 \sum_{i=e+1}^{b} \binom{t}{i+1} \left[ 1 + (q-1)^{t-i} \right]^{(i-2, e-2)}
\end{align*}
\]

ways. This has been done by subtracting all bursts of length \(b\) or less with weight \(e\) or less from the total number of vectors of weight \(e\) or less in a vector of length \(t\).

Thus, the total number of combinations to which \(h_j\) cannot be equal in view of \((iv^*)\) is

\[
\begin{align*}
\sum_{i=e}^{\min(e, t)} \binom{b-1}{i}(q-1)^i & \left( (q-1)(t-e+1) + 1 \right) - 1 \\
+ (q-1)^2 \sum_{i=e+1}^{b} \binom{t}{i+1} \left[ 1 + (q-1)^{t-i} \right]^{(i-2, e-2)}
\end{align*}
\]

\hspace{1cm} (6.49)
At worst, all the linear combinations computed in expressions (6.34), (6.46), (6.47) and (6.49) might yield a distinct sum. Thus, a column $h_j$ can be added to the $s$-th sub-block of $H$ provided that all the $r$-tuples are not exhausted by these linear combinations, i.e. the $j$-th column $h_j$ can be added to the $s$-th sub-block if

\[
q^r > [1 + (q-1)]^{(j-1,2e-1)} + [1 + (q-1)]^{(j-1,e-1)(s-1)}.
\]

\[
\sum_{i=1}^{e} \binom{t}{i} (q-1)^i + \sum_{p_1,p_2}^{p_{1},p_{2}=2e} \left( \frac{b-1}{p_1} (q-1)^{p_1} \right).
\]

\[
I(q,j-b;b,p_2) + \sum_{i=d}^{2w-1} \binom{b-1}{i} (q-1)^i
\]

\[
+ \sum_{k=1}^{b-1} \sum_{r_1,r_2,r_3}^{b-k-1} \binom{k}{r_1} \binom{b-k-1}{r_2} (q-1)^{r_1+r_2+r_3+1}
\]

\[
+ \left[ \sum_{i=e}^{w-1} \binom{b-1}{i} (q-1)^i \right] (s-1) \left[ q^{e-1} [(q-1).\right]
\]

\[
(t-e+1 + 1) (q-1)^2 \sum_{i=e+1}^{b} (t-i+1).
\]

\[
[1 + (q-1)]^{(1-2,w-2)} + [1 + (q-1)]^{(b-1,w-1)}.
\]

(expr. contd.)
\[ (s-1)(q-1)^2 \sum_{i=e+1}^{b} (t-i+1) \sum_{j=e-1}^{w-1} (i-j)(q-1)^j \]

\[ + \sum_{q_1, q_2}^{e+w-1} (q_2(q-1)^{q_2} I(q_2, j-l, b, q_1) \]

\[ q_1 + q_2 = 2e \]

\[ + \left[ \sum_{i=e}^{w-1} (b-1)(q-1)^i \right] \left[ \sum_{j=e}^{j-b-1} (q-1)^j \right] \]

\[ + \left[ 1 + (q-1)^{(j-1, e-1)} - [1 + (q-1)]^{(b-1, e-1)} \right] (s-1). \]

\[ (q-1)^2 \sum_{i=e+1}^{b} (t-i+1) \sum_{j=e-1}^{w-1} (i-j)(q-1)^j \]

\[ + \left[ \sum_{i=e}^{w-1} (b-1)(q-1)^i \right] (s-1) \left[ \sum_{i=1}^{e} (q-1)^i \right] \]

\[ q^{e-1} - \]

\[ [(q-1)(t-e+1) + 1] - 1 + (q-1)^2 \sum_{i=e+1}^{b} (t-i+1) \]

\[ [1 + (q-1)]^{(i-2, e-2)} \]

which on simplification gives

\[ q^r > [1 + (q-1)]^{(j-1, 2e-1)} + [1 + (q-1)]^{(j-1, e-1)}. \]

\[ (s-1) \left[ \sum_{i=1}^{e} (q-1)^i + (q-1)^2 \sum_{i=e+1}^{b} (t-i+1) \right]. \]

(expr. contd.)
\[ \sum_{j=e-1}^{w-2} \binom{j-2}{1} (q-1)^j + \sum_{p_1, p_2: p_1 + p_2 = 2e}^{2w-1} \binom{b-1}{p_1} (q-1)^{p_1} \]

\[ \sum_{i=d}^{2w-1} \binom{b-1}{1} (q-1)^i \]

\[ \sum_{k=1}^{b-1} \sum_{r_1, r_2, r_3} (q-1)^{r_1+r_2+r_3+1} \]

\[ \sum_{i=e}^{w-1} \binom{b-1}{i} (q-1)^i \]

\[ (s-1) \sum_{i=1}^e \binom{s}{i} (q-1)^i + \binom{j-b-1}{e} (q-1)^e \]

\[ (s-1) (q-1)^2 \sum_{i=e+1}^{b} \binom{t-i+1}{i} \sum_{j=e-1}^{w-2} \binom{j-2}{1} (q-1)^j \]

\[ \sum_{q_{1}, q_{2}: q_{1} + q_{2} = 2e}^{e+w-1} \binom{j-1}{q_{1}} (q-1)^{q_{2}} I(q, j-1, b, q_{1}) \]

(6.50)

But for an \((n, k)\) linear code, the above inequality should hold for \(j = t\) and thus we get expression (6.39) stated in the theorem. \(Q.E.D.\)

Discussion. The result just obtained has been proved for \(e < w\). However, if we take \(e \geq w\), the code is then capable to correct all random errors of weight \(e\) or less and in particular all low-density bursts of length \(b\) or less with weight \(w\) or less, occurring within a single
sub-block. Therefore, the low-density burst consideration becomes superfluous and the bound obtained in (6.39) reduces to

\[ q^r > [1 + (q-1)]^{(t-1,2e-1)} + [1 + (q-1)]^{(t-1,e-1)} \]

\[ \times (s-1) \sum_{i=1}^{e} (t)(q-1)^i, \]

which is an analogue of the well-known Varshamov-Gilbert bound for codes correcting random errors of weight \(e\) or less occurring within a single sub-block of the code.

For \(w = b, i.e.\) relaxing the weight constraint over the bursts to be corrected, the upper bound on the number of parity checks obtained in (6.39) becomes

\[ q^r > [1 + (q-1)]^{(t-1,2e-1)} + [1 + (q-1)]^{(t-1,e-1)} \]

\[ \times (s-1) \left[ \sum_{i=1}^{e} (t)(q-1)^i + (q-1)^2 \sum_{i=s+1}^{b} (t-i+1) \right]. \]

\[ \sum_{j=e-1}^{b-2} (1-2)(q-1)^j \right] + \sum_{p_1,p_2: p_1+p_2=2e}^{2b-1} \left[ (b-1)(q-1)^{p_1} \right]. \]

\[ I(q,t-b; b_1, b_2) + \left[ \sum_{i=e}^{b-1} (b-1)(q-1)^i \right]. \]

(expr. contd.)
\[
\begin{align*}
&\left[ (s-1) \sum_{i=1}^{e} (t)(q-1)^i + (t-b-1)(q-1)^0 + (s-1)(q-1)^2 \right. \\
&\left. + \sum_{i=e+1}^{b} (t-1+1) \sum_{j=e-1}^{b-2} (1-2)(q-1)^j \right] \\
&+ \sum_{q_1, q_2}^{e+b-1} \left[ (t-1)(q-1)^{q_2} I(q, t-1; b, q_1) \right], \\
&\quad q_1 + q_2 = 2e
\end{align*}
\]

which gives a sufficient condition for the existence of an \((n, k)\) linear code that corrects all combinations of weight \(e\) or less and all bursts of length \(b\) or less occurring within a single sub-block of the code.

Next, setting \(e = 1\), i.e. relaxing the constraint of random error correction, the bound obtained in (6.39) then gives an upper bound on the number of parity-check digits required for a code that corrects all bursts of length \(b\) or less with weight \(w\) or less occurring within a sub-block of the code. This result is an analogue of the result proved by Dass (1975).

Further, for \(e = 1\) and \(w = b\), i.e. relaxing the constraints of random error correction and that of the weight imposed over the bursts to be corrected, the bound in (6.39) reduces to an upper bound on the number of check digits required for the existence of a code.
that corrects all bursts of length b or less occurring within a sub-block of the code. This result is an analogue of the result proved by C.N. Campopiano given in Peterson and Weldon, Jr. (1972, Theorem 4.17).

Lastly, for \( s = 1 \) (which means \( t = n \), i.e., considering the case of usual codes having no sub-block structure, the bound obtained in (6.39) reduces to an upper bound on the number of parity check digits required for the existence of a code that corrects all combinations of weight \( e \) or less and all low-density bursts of length \( b \) or less with weight \( w \) or less. This result is an analogue of the result proved by Sharma and Dass (1977a, Theorem 3).

Several other deductions in special cases viz. by taking \( s = 1, w = b; s = 1, e = 1; s = 1, e = 1, w=b \) may also be derived for codes having no sub-block structure.

**Alternative Form.** If \( B \) is taken to be the largest value of \( b \) for which the bound derived in Theorem 6.5 holds, then for \( b = B + 1 \), the inequality in (6.39) gets reversed and we get

\[
q^r \leq [1 + (q-1)](t-1,2e-1) + [1 + (q-1)](t-1,e-1)(s-1).
\]

(expr. contd.)
Another alternative form of the bound obtained in (6.39) can be obtained by taking \( t \) to be the largest value of \( j \) satisfying (6.50) and then replacing \( j \) by \( t+1 \).
Remark. It may be pointed out here that the codes discussed in this section correct all low-density bursts of length b or less with weight w or less within a sub-block and, being capable of correcting all errors of weight e or less within a sub-block also, possess minimum weight at least $2e + 1$ within a sub-block. However, the converse is not true in general.
CONCLUSIONS AND FURTHER SCOPE

The second chapter of this thesis deals with codes correcting and locating burst errors. A study of multiple-burst locating codes on the lines similar to those of multiple-burst correcting codes may be of interest from applications point of view.

The third chapter deals with codes correcting and locating low-density burst errors. As a more general case, it is possible to consider similar situations when weights of bursts lie between \( w_1 \) and \( w_2 \) where \( w_1 \leq w_2 \leq b \). Such bursts may be called 'moderate-density bursts'. The results naturally would involve additional combinatorial complexities.

The fourth chapter deals with codes locating random errors as well as burst errors. It may be pointed out that several types of analytically-constructed codes possess the property of correcting certain burst errors along with the correctable random errors. The most powerful codes which fall in this category are the BCH codes (refer to Bose and Chaudhuri (1960, 1960a) and Hocquenghem (1959)) and, in particular, the RS codes (refer to Reed and Solomon (1960)). A t-error correcting RS code over \( \text{GF}(q^m) \) can correct all bursts of length
mt-m+1 when regarded as a code over GF(q), refer to Peterson and Weldon, Jr. (1972, p.371). Product codes also have this property. The product of an \((n_1,k_1)\) code with error-correcting ability \(t_1 = \left\lfloor \frac{d_1-1}{2} \right\rfloor\) and an \((n_2,k_2)\) code with error-correcting ability \(t_2 = \left\lfloor \frac{d_2-1}{2} \right\rfloor\) can correct all patterns of weight \(t = \left\lfloor \frac{d_1 d_2 - 1}{2} \right\rfloor\) or less and all bursts of length up to \(b = \max(n_1 t_2, n_2 t_1)\). Proof of this assertion can be found in Burton and Weldon, Jr. (1965) and Theorem 5.4 of Peterson and Weldon, Jr. (1972). Other works on constructing burst and random error correcting codes can be found in Bahl and Chien (1969), Hsu (1974), Hsu, Kasam and Chien (1968), Posner (1965), Stone (1961), Tavares and Shiva (1970), Tong (1968) and Wolf (1965a).

In the paper by Hsu, Kasam and Chien (1968), an extensive list of computer-generated binary codes capable of burst error and random error correcting codes is given. It will be worth to carry out a similar study for error locating codes discussed in this chapter.

The fifth chapter deals with codes correcting and locating random errors simultaneously with low-density burst errors. The burst errors for the purposes of correction are taken in the sense of Chien and Tang (1965). It may be pointed out that for a similar class of codes
dealing with open-loop burst errors, it has been shown by Retter (1976) and Manov (1977) that Goppa codes and Generalized BCH codes exist arbitrarily close to the extended Varshamov-Gilbert bound obtained by Sharma and Dass (1974). It will be interesting to examine the relationship between Srivastava Codes (1967) and the extended Varshamov-Gilbert bounds studied in this chapter. The relationship between generalized Srivastava codes due to Helgert (1972, 1972a, 1974, 1975, 1975a) and some of the upper bounds on codes correcting burst errors studied in this thesis may also be worth investigating.

The sixth chapter deals with the correction of random and low-density burst errors w.r.t codes having sub-block structure. Quasi-cyclic codes also follow a similar sub-block structure in the sense that these codes are cyclic within a sub-block and not over the whole word length. By imposing an additional constraint of being cyclic in a sub-block on codes studied in this chapter, it may be fruitful to study the capabilities of these codes.

Finally, the studies made in this thesis on error locating codes can be generalized to 'Extended Error Locating Codes' on the lines similar to those of Wolf (1965). The asymptotic behaviour of such codes may also be studied.