ABSTRACT

WAVELETS, DIFFUSION EQUATIONS AND THEIR APPLICATIONS

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Abstract

Acquisition, storing, and processing of digital images are still becoming more and more important applications in our daily life. Denoising, smoothing, and simplification are central problems in digital image processing. All three of them aim at enhancing the quality of an image either to a human observer or as preprocessing step for a computer vision system. The crucial point in these approaches is to distinguish between important image features that should be kept or even enhanced, and those parts of the image content that are considered as noise and should be removed. Mathematically very different ways have been used to model how a smooth image should look like: For example, a certain smoothness can be formalised in terms of differentiability orders and small modulus of derivatives. This idea leads to regularisation methods and related partial differential equations. Another kind of smoothness assumption is that certain wavelet coefficients should be small, leading to the popular wavelet shrinkage methods. While the preceding ideas already use formulations with the help of functions, adaptive averaging approaches usually start directly on the level of digital data sampled at discrete pixels. Comparing the differences of the grey value helps to find important features such as edges in this case.

In 1986, Stephane Mallat and Yves Meyer developed a multiresolution analysis using wavelets. They mentioned the scaling function of wavelets for the first time; it allowed researchers and mathematicians to construct their own family of wavelets using the derived criteria. Around 1998, Ingrid Daubechies used the theory of multiresolution wavelet analysis to construct her own family of wavelets. Since then the field has grown enormously.

Total variation is good for quantifying the simplicity of an image since it measures oscillations without unduly punishing discontinuities. For this reason, blocky images (consisting only of a few almost piecewise constant segments) reveal very small total variation. In order to restore noisy blocky images, Rudin, Osher and Fatemi [25] have proposed to minimize the total variation under constraints which reflect assumptions
about noise. To fix ideas, let us study an example. Given an image \( u_0 \) with additive noise of zero mean and known variance \( \sigma^2 \), we seek a restoration \( u \) satisfying

\[
\minimize \int_{\Omega} |\nabla u| \, dx \, dy = \int_{\Omega} \sqrt{u_x^2 + u_y^2} \, dx \, dy,
\]

subject to constraints

\[
\int_{\Omega} u \, dx \, dy = \int_{\Omega} u_0 \, dx \, dy,
\]

and

\[
\int_{\Omega} \frac{1}{2} (u - u_0)^2 \, dx \, dy = \sigma^2.
\]

In order to solve this constrained variational problem, PDE methods can be applied. A solution of (1)-(3) verifies necessarily the Euler-Lagrange equation

\[
0 = \nabla \cdot \left( \frac{\nabla u}{|\nabla u|} \right) - \lambda_1 - \lambda_2 (u - u_0),
\]

with homogeneous Neumann boundary conditions. The unknown Lagrange multipliers \( \lambda_1 \) and \( \lambda_2 \) have to be determined in such a way that the constraints are fulfilled. In [25] a gradient descent method is proposed to solve (4). It uses an explicit finite difference scheme with central and one-sided spatial differences and adapts the Lagrange multiplier by means of the gradient projection method of Rosen [23].

One may also formulate the constrained TV minimization problem as an unconstrained problem [9]: The penalised least square problem

\[
\min_u \int_{\Omega} |\nabla u| \, dx \, dy + \frac{\lambda}{2} \int_{\Omega} (u - u_0)^2 \, dx \, dy.
\]

is equivalent to the constrained TV minimization.

In recent years, problems of type (5) have attracted much interest from mathematicians working on inverse problems, optimization, or numerical analysis [1, 8]. To overcome the problem that the TV integral contains the nondifferentiable argument \( |\nabla(u)| \), one applies regularization strategies from nonsmooth optimization. Much research is done in order to find efficient numerical methods for which convergence can be established.

It has already been mentioned that numerical schemes may provide implicit regularizations which stabilize the Perona-Malik process [22]. Hence, it has been suggested to introduce the regularization directly into the continuous equation in order to become more independent of the numerical implementation [7].
A mathematically sound formulation of this idea is given by Catté, Lions, Morel and Coll [7]. By replacing the diffusivity $g(|\nabla u|^2)$ of the Perona-Malik model by a Gaussian-smoothed version $g(|\nabla u_\sigma|^2)$ with $u_\sigma := G_\sigma * u$ they end up with

$$\frac{\partial u}{\partial t} = \text{div}(g(|\nabla u|)\nabla u).$$

(6)

In 2003, Mrázek et al. [21] studied the connection between discrete one-dimensional schemes for nonlinear diffusion and shift-invariant Haar wavelet shrinkage. They derived new wavelet shrinkage functions from existing diffusivities functions and identified some previously used shrinkage functions as corresponding to well known diffusivities.

In 2007, Mrázek and Weickert [20] established a connection between discrete two-dimensional schemes for shift-invariant Haar wavelet shrinkage on one hand and nonlinear diffusion on the other. By using a single iteration on a single scale, they proved that two methods can be made equivalent by the choice of the nonlinearity which controls each method: the shrinkage function, or the diffusivity function, respectively.

In 2008, Welk et al. [28] studied a class of numerical schemes for nonlinear diffusion filtering that offers insight on the design of novel wavelet shrinkage rules for isotropic and anisotropic image enhancement.

The present thesis entitled “Wavelets, diffusion equations and their Applications” is based on my study and research in the course of last four years, as a research scholar in the Department of Mathematics, Aligarh Muslim University, Aligarh. The present thesis comprises to six chapters and each chapter is further divided into various sections. The first section of each chapter provides an introduction to its contents.

Chapter 1 contains preliminary notions, basic definitions, examples and relevant well known results related to our study which are required for the development of the subject in subsequent chapters.

Chapter 2 is divided into six sections. The first section is introductory; the second section deals with the concept of new nonlinear diffusion model for image restoration; the third section contains the result which guarantees the existence, uniqueness and stability of viscosity solution of our model defined in previous section. In fourth section, we have defined a discrete scheme for incorporating numerical experiments which are given in fifth section. The last section deals with the conclusion.

Chapter 3 is divided into seven sections. The first section is introductory. In second section, we have defined a new nonlinear anisotropic diffusion model for image denoising. In section three, we have proved a result which establishes the existence and uniqueness of weak solutions obtained by our new model defined in previous section. In the fourth
section, we discuss about some properties of weak solution. In fifth section, we have defined a convergent iterative scheme and with the help of which we have exorted some numerical experiments in sixth section. The last section concludes the chapter.

Chapter 4 is divided into nine sections. The first section is introductory. Second and third sections deal with the physical background and mathematical consideration of anisotropic diffusion model for image denoising respectively; fifth and sixth sections deal with the study of concept of new time dependent models for 2D and 1D respectively. We have compared our models with the existing models by means of standard metrics in fourth, seventh and eighth sections. The last section deals with the conclusion.

Chapter 5 is divided into five sections. The first section is introductory. In second section, we have defined a new PDE based deconvolution model. In third section, we have used two different types of diffusivity in our model defined in previous section. Numerical experiments has been done in fourth section. The last section deals with the conclusion.

Chapter 6 is devoted to the study of reconstruction of wavelet coefficients using nonlinear diffusion reaction equation. The first section of this chapter is introductory; the second section deals with the study of discrete wavelet transform. In the third section, we have defined a new diffusion equation based on wavelet coefficients. In fourth section, we have defined a discrete scheme for incorporating numerical experiments which are given in fifth section. The last section deals with the conclusion.

In the end, a bibliography is given which by no means is exhaustive one but lists only those books and papers which have been referred to in the thesis.
The published/accepted/communicated research papers based on the work of this thesis are as follows:


(2) An efficient PDE-based model for image restoration, Accepted for publication in *Indian Journal of Industrial and Applied Mathematics*.

(3) A time dependent model for image restoration with forward-backward Diffusivities, Communicated in *International Journal of Advances in Engineering Sciences and Applied Mathematics*.

(4) PDE-based nonlinear diffusion model for image denoising, Communicated in *Applied and Computational Mathematics an International Journal*.

(5) An efficient PDE-based nonlinear anisotropic diffusion model for image denoising, Communicated in *Neural, Parallel, and Scientific Computations*.


Some results of this thesis have been presented in the following international conferences:

(1) International Conference on The Occasion of Silver Jubilee of the Indian Society of Industrial and Applied Mathematics (ISIAM) at Sharda University, Greater Noida, India, Jan 29-31, 2016.

(2) International Conference on Analysis and its Applications, at Department of Mathematics, Aligarh Muslim University, Aligarh, India, Dec 19-21, 2015.
Bibliography


