Chapter 3

Weak solution of nonlinear diffusion equation

3.1 Introduction

The aim of this Chapter is to study a nonlinear diffusion models for image denoising. We present proof of the existence and uniqueness theorem of model (3.2.1). We derive some properties of weak solution. We have tested our algorithm on various types of images. To quantify the results, we have used peak signal to noise ratio (PSNR) as metric.

Image denoising is a fundamental problem in both image processing and computer vision with numerous applications. The total variation models [15, 61, 76] and anisotropic diffusion models [13, 56, 78, 83, 86] have been studied as a useful tool to the problem of image denoising and image reconstruction. These partial differential equations based image enhancement techniques have been able to achieve a good edge preservation.

Definition 3.1.1. Let $m > 0$ be an integer and let $1 \leq p \leq \infty$. The Sobolev space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \{ u \in L^p(\Omega) | D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq m \},$$

where $\Omega$ is an open set in $\mathbb{R}^n$.

In other words, $W^{m,p}(\Omega)$ is the collection of all functions in $L^p(\Omega)$ such that all distribution derivatives up to order $m$ are also in $L^p(\Omega)$. Clearly $W^{m,p}(\Omega)$ is a vector space.
In all that follows we will consider functions with values in $\mathbb{R}$ and the corresponding function spaces as vector spaces over $\mathbb{R}$. We provide it with the norm:

$$
||u||_{m,p,\Omega} = \sum_{|\alpha| \leq m} ||D^\alpha u||_{L^p(\Omega)}
$$

(3.1.2)

or, equivalently, for $1 < p < \infty$,

$$
||u||_{m,p,\Omega} = \left( \sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u|^p \right)^{\frac{1}{p}} = \left( \sum_{|\alpha| \leq m} ||D^\alpha u||_{L^p(\Omega)}^p \right)^{\frac{1}{p}}.
$$

(3.1.3)

**Remark 3.1.1.** The case $p=2$ will play a special role in the sequel. These spaces will be denoted by $H^m(\Omega)$. Thus

$$
H^m(\Omega) = W^{m,2}(\Omega)
$$

(3.1.4)

and for $u \in H^m(\Omega)$, we denote its norm by $||u||_{m,\Omega}$. i.e.

$$
||u||_{m,\Omega} = ||u||_{m,2,\Omega}.
$$

(3.1.5)

**Remark 3.1.2.** We will also often use the semi-norms which consist of the $L^p$-norms of the highest order derivatives. We denote these by $| \cdot |_{m,p,\Omega}$. Thus for $u \in W^{m,p}(\Omega)$.

$$
|u|_{m,p,\Omega} = \sum_{|\alpha|=m} ||D^\alpha u||_{L^p(\Omega)},
$$

(3.1.6)

with the obvious modification (when $1 < p < \infty$) if we use equation (3.1.3) to define the norm. Consistent to remark (3.1.1) if $p=2$ we only write $| \cdot |_{m,\Omega}$ instead of $| \cdot |_{m,2,\Omega}$.

**Remark 3.1.3.** We can naturally consider the space $L^p(\Omega)$ as a special case of the Sobolev class, viz. when $m = 0$. i.e. we do not bother about derivatives. In particular we denote the $L^p$-norm of a function by $| \cdot |_{0,p,\Omega}$ (since in this case the semi-norm and norm are the same). Again the $L^2(\Omega)$-norm will be denoted by $| \cdot |_{0,\Omega}$. 
3.2 Nonlinear diffusion model using total variation and Perona-Malik diffusivities

An image can be interpreted as a real function defined on $\Omega$, a bounded and open domain of $\mathbb{R}^2$ (for simplicity we will assume $\Omega$ to be the square domain henceforth). Formation of a noisy image is typically modeled as

$$u_0(x) = u(x) + n(x),$$

where $u(x)$ denote the desired clean image, $u_0(x)$ denote the pixel values of a noisy image for $x \in \Omega$ and $n(x)$ is additive white noise assumed to be close to Gaussian. The values $n(i, j)$ of $n$ at the pixels $(i, j)$ are independent random variables, each with a Gaussian distribution of zero mean and variance $\sigma^2$.

We propose the following second order - version of the nonlinear diffusion model which is a synthesis of ideas from Catté et al. [13]. Our model is given by:

$$
\begin{cases}
\frac{\partial u}{\partial t} = \nabla \cdot (g_1(|\nabla G_{\sigma} * u|)\nabla u) + \nabla \cdot (g_2(|\nabla G_{\sigma} * u|)\nabla u) - \lambda(u - u_0) \\ 
\text{on } \Omega \times (0, T), \\
u(x, 0) = u_0(x) \text{ in } \Omega, \\
\frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial \Omega \times (0, T),
\end{cases}
$$

(3.2.1)

where $g_1$ and $g_2$ are decreasing function tending to zero at infinity with $g_i(0) = M_i > 0$ and $t \rightarrow g_i(\sqrt{t})$ is smooth, and $G_{\sigma}(x)$ is the Gaussian kernel, namely,

$$G_{\sigma}(x) = \frac{1}{2\pi\sigma^2} e^{-\frac{|x|^2}{2\sigma^2}}. \quad (3.2.2)$$
3.3 Existence and uniqueness of weak solutions

In this Section, we establish the existence and uniqueness of weak solutions of the proposed model following the arguments in [13, 28]. The standard notations are used throughout.

Let Ω denote the square \((0, 1) \times (0, 1)\) of \(\mathbb{R}^2\) and \(H^k(\Omega)\), \(k\) a positive integer, the set of all functions \(u(x)\) defined on \(\Omega\) such that \(u\) and its distributional derivatives \(\frac{\partial^m u}{\partial x^m}\) of order \(|m| = \sum_{j=1}^{N} m_j \leq k\) for all belong to \(L^2(\Omega)\). \(H^k(\Omega)\) is a Hilbert space with the norm

\[
||u||_{H^k(\Omega)} = \left( \sum_{|m| \leq k} \int_{\Omega} \left| \frac{\partial^m u}{\partial x^m} \right|^2 dx \right)^{\frac{1}{2}}. \tag{3.3.1}
\]

The space \(L^\infty(0, T; H^1(\Omega))\) consists of all functions \(u\) such that, for almost every \(t\) in \((0, T)\), \(u\) belongs to \(H^1(\Omega)\). \(L^\infty(0, T; H^1(\Omega))\) is a normed space with the norm

\[
||u||_{L^\infty(0, T; H^1(\Omega))} = \text{ess sup}_{0 \leq t \leq T} ||u(\cdot, t)||_{H^1(\Omega)}. \tag{3.3.2}
\]

**Theorem 3.3.1.** Let \(u_0 \in H^1(\Omega)\) and \(||u_0||_{H^1(\Omega)}\) is appropriately small. Then we have a unique weak solution \(u(x, t)\) such that \(u \in C([0, T]; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))\), and verifying

\[
\begin{cases}
\frac{\partial u}{\partial t} - \nabla \cdot (g_1(\|\nabla G_\sigma * u\|) \nabla u) - \nabla \cdot (g_2(\|\nabla G_\sigma * u\|) \nabla u) = 0 \text{ in } \Omega \times (0, T), \\
u(x, 0) = u_0(x) \text{ in } \Omega, \\
\frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial \Omega \times (0, T),
\end{cases} \tag{3.3.3}
\]

where this system is verified in the distributional sense. Moreover, the unique solution \(u(x, t)\) is in \(C^\infty((0, T) \times \overline{\Omega})\).

**Proof.** Firstly, we consider the proof of the existence of a solution, which is based on the Schauder fixed point argument [35, 53].

We introduce the solution space \(W\) of the problem (3.3.3) as follows:

\[
W(0, T) = \left\{ w \in L^\infty(0, T; H^1(\Omega)), \frac{dw}{dt} \in L^2(0, T; (H^1(\Omega))') \right\},
\]
where \((H^1(\Omega))'\) is the dual space of \(H^1(\Omega)\).

Let \(w \in W\) such that

\[
\|w\|_{L^\infty(0,T;L^2(\Omega))} \leq \|u_0\|_{L^2(\Omega)}, \quad \left\| \frac{\partial w}{\partial t} \right\|_{L^2(0,T; (H^1(\Omega))')} \leq \|u_0\|_{H^1(\Omega)}. \tag{3.3.4}
\]

We consider the following linear problem \([E_w]\):

\[
\left\langle \frac{\partial u(t)}{\partial t}, v \right\rangle_{(H^1(\Omega))' \times H^1(\Omega)} + \int_\Omega g_1(|\nabla G_{\sigma} \ast w(t)|) \nabla u(t) \nabla v(t) dx
\]

\[
\quad + \int_\Omega g_2(|\nabla G_{\sigma} \ast w(t)|) \nabla u(t) \nabla v(t) dx = 0
\]

for all \(v \in H^1(\Omega)\), a.e. \(t \in [0,T]\). Since \(w\) and \(\partial w/\partial t\) satisfy (3.3.4), then \(|\nabla G_{\sigma} \ast w|\) and \(|\nabla G_{\sigma} \ast (\partial w/\partial t)|\) belong to \(L^\infty((0,T); C^\infty(\Omega))\) and there exists a constant \(M = M(G_{\sigma}, ||u_0||_{H^1(\Omega)})\) such that \(|\nabla G_{\sigma} \ast w| \leq M\) and \(|\nabla G_{\sigma} \ast (\partial w/\partial t)| \leq M\) a.e. \(t\), for all \(x \in \Omega\). Since \(g_1(s)\) and \(g_2(s)\) are decreasing and positive, it follows that a.e. in \((0,T) \times \Omega:\)

\[
0 < \lambda_1 \leq g_1(|\nabla G_{\sigma} \ast w|) \quad \text{and} \quad 0 < \lambda_2 \leq g_2(|\nabla G_{\sigma} \ast w|).
\]

By classical results on the parabolic equations \([12, 28]\), the problem \([E_w]\) has a unique solution \(U_w \in W\) \([1, 12]\), satisfying the estimates

\[
\|U_w\|_{L^\infty(0,T; H^1(\Omega))} \leq c_1, \tag{3.3.5}
\]

\[
\|U_w\|_{L^\infty(0,T; L^2(\Omega))} \leq \|u_0\|_{L^2(\Omega)}, \tag{3.3.6}
\]

\[
\left\| \frac{\partial U_w}{\partial t} \right\|_{L^2(0,T; (H^1(\Omega))')} \leq \|u_0\|_{H^1(\Omega)}, \tag{3.3.7}
\]

where \(c_1\) is a constant which only depends on \(G_{\sigma}\), \(g_1\), \(g_2\) and \(||u_0||_{H^1(\Omega)}\). Choosing \(v = U_w\) in \(E_w\), integrating over the interval \((0,t)\), we arrive to the inequality

\[
\frac{1}{2} \int_\Omega U_w^2 dx + (\lambda_1 + \lambda_2) \int_0^t \int_\Omega |\nabla U_w|^2 dx \, ds \leq \frac{1}{2} \int_\Omega u_0^2 dx, \tag{3.3.8}
\]
which implies (3.3.6). Choosing \( v = \partial U_w / \partial t \) in \( E_w \), integrating by parts yields

\[
\int_{\Omega} \left( \frac{\partial U_w}{\partial t} \right)^2 \, dx + \frac{1}{2} \int_{\Omega} g_1(|\nabla G_\sigma \ast w|) \frac{\partial|\nabla U_w|^2}{\partial t} \, dx + \frac{1}{2} \int_{\Omega} g_2(|\nabla G_\sigma \ast w|) \frac{\partial|\nabla U_w|^2}{\partial t} \, dx.
\]

Integrating over the interval \((0, t)\) we arrive to that

\[
\int_{0}^{t} \int_{\Omega} \left( \frac{\partial U_w}{\partial s} \right)^2 \, dx \, ds + \frac{1}{2} \int_{\Omega} g_1(|\nabla G_\sigma \ast w|) |\nabla U_w|^2 \, dx \, ds + \frac{1}{2} \int_{\Omega} g_2(|\nabla G_\sigma \ast w|) |\nabla U_w|^2 \, dx \, ds
\]

\[
= \frac{1}{2} \int_{\Omega} g_1(|\nabla G_\sigma \ast w|) |\nabla u_0|^2 \, dx \, ds + \frac{1}{2} \int_{0}^{t} \int_{\Omega} g'_1(|\nabla G_\sigma \ast w|) \times \nabla \left( G_\sigma \ast \frac{\partial w}{\partial s} \right) |\nabla U_w|^2 \, dx \, ds + \frac{1}{2} \int_{0}^{t} \int_{\Omega} g'_2(|\nabla G_\sigma \ast w|) \times \nabla \left( G_\sigma \ast \frac{\partial w}{\partial s} \right) |\nabla U_w|^2 \, dx \, ds.
\]

(3.3.9)

When \( u \in W \) and (3.3.8), noticing that \(|g_1(s)| \leq k_1\) and \(|g_2(s)| \leq k_2\), we can deduce that

\[
\int_{0}^{t} \int_{\Omega} \left( \frac{\partial U_w}{\partial s} \right)^2 \, dx \, ds + \frac{\lambda_1 + \lambda_2}{2} \int_{\Omega} |\nabla u|^2 \, dx \leq \int_{\Omega} |\nabla u_0|^2 \, dx + \frac{M(k_1 + k_2)}{4(\lambda_1 + \lambda_2)} \int_{\Omega} u_0^2 \, dx \, ds.
\]

(3.3.10)

Since \(||u_0||_{H^1(\Omega)}\) is small, letting \((M(k_1 + k_2)/4(\lambda_1 + \lambda_2)) \leq 1\) yields (3.3.5) and (3.3.7).

From (3.3.5)-(3.3.7), we introduce the subspace \(W_0\) of \(W(0, T)\) defined by

\[
W_0 = \{ w \in W(0, T), \ w(0) = u_0, \||w||_{L^\infty((0,T);H^1(\Omega))} \leq c_1, \ ||w||_{L^\infty((0,T);L^2(\Omega))} \leq ||u_0||_{L^2(\Omega)}, \\
\||dw/dt||_{L^2((0,T);H^1(\Omega))'} \leq ||u_0||_{H^1(\Omega)} \}. \]

(3.3.12)

By construction, \( w \rightarrow f(w) \equiv U_w \) is a mapping from \( W_0 \) into \( W_0 \). Moreover, \( W_0 \) is a not empty, convex and weakly compact in \( W(0, T) \).

In order to apply the Schauder fixed point theorem, we need to prove that the mapping \( f : w \rightarrow U_w \) is weakly continuous from \( W_0 \) into \( W_0 \). Let \( \{w_j\} \) be a sequence in \( W_0 \) which converges weakly to some \( w \) in \( W_0 \) and let \( u_j = U_{w_j} \). The sequence \( \{w_j\} \)
of $W_0$ contains a subsequence $\{w_j\}$ such that

$$\frac{du_j}{dt} \to \frac{du}{dt} \text{ weakly in } L^2(0,T;(H^1(\Omega))'),$$

$$u_j \to u \text{ in } L^\infty(0,T;L^2(\Omega)),$$

$$\frac{\partial u_j}{\partial x_i} \to \frac{\partial u}{\partial x_i} \text{ weakly in } L^\infty(0,T;L^2(\Omega)),$$

$$w_j \to w \text{ in } L^\infty(0,T;(L^2(\Omega)),$$

$$\frac{\partial G_\sigma}{\partial x_i} * w_j \to \frac{\partial G_\sigma}{\partial x_i} * w \text{ in } L^2(\Omega), \text{ a.e on } (0,T) \times \Omega,$$

$$g_1(\nabla G_\sigma * w_j) \to g_1(\nabla G_\sigma * w) \text{ in } L^2(0,T;L^2(\Omega)),$$

$$g_2(\nabla G_\sigma * w_j) \to g_2(\nabla G_\sigma * w) \text{ in } L^2(0,T;L^2(\Omega)),$$

$$u_j(0) \to u(0) \text{ in } L^2(\Omega).$$

Passing to the limit in the relation

$$\left\langle \frac{du_j(t)}{dt}, v \right\rangle + \int_\Omega g_1(|\nabla G_\sigma * w_j(t)|) \nabla u_j(t) \nabla v(t) dx + \int_\Omega g_2(|\nabla G_\sigma * w_j(t)|) \nabla u_j(t) \nabla v(t) dx = 0,$$

The above convergence allows us to pass to the limit in the problem $(E_{w_j})$ and obtain $u = U_w = f(w)$. Moreover, since the solution is unique, the whole sequence $u_j = f(w_j)$ converges weakly in $W_0$ to $u = f(w)$; that is, $f$ is weakly continuous.

Second, the regularity of the solution using the general theory of parabolic equations and the bootstrap argument [13, 40], we can deduce that $u$ is a strong solution of (3.3.3) and $u \in C^\infty((0,T) \times \Omega)$.

Finally, we shall proof that the uniqueness, following the idea in [28]. Let $u_1$ and $u_2$ be two solutions of (3.3.3). For almost every $t$ in $[0,T]$, we have

$$\begin{cases} \frac{d}{dt}(u_1 - u_2)(t) - \nabla \cdot (\beta_1(t) \nabla (u_1 - u_2)(t)) = \nabla \cdot (\beta_2(t) \nabla (u_1 - u_2)(t)) \\ \frac{\partial (u_1 - u_2)}{\partial n} = 0 \text{ on } \partial \Omega \times (0,T), \end{cases} \quad (3.3.13)$$

in the distribution sense, where $\alpha(t) = g_1(|\nabla G_\sigma * u_1(t)|)$, $\alpha_2(t) = g_2(|\nabla G_\sigma * u_1(t)|)$,
\[ \beta_1(t) = g_1(|(\nabla G_{\sigma} \ast u_2)(t)|) \quad \text{and} \quad \beta_2(t) = g_2(|(\nabla G_{\sigma} \ast u_2)(t)|). \]

Then multiplying the above equality by \( v(t) \), integrating over \( \Omega \), and using the Neumann boundary conditions, we get a.e. \( t \in [0, T] \),

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u_1 - u_2)(t)v(t) + \int_{\Omega} \beta_1(t) \nabla v(t) \cdot (\nabla u_1(t) - \nabla u_2(t)) + \int_{\Omega} \beta_2(t) \nabla v(t) \cdot (\nabla u_1(t) - \nabla u_2(t)) = -\int_{\Omega} (\alpha_1(t) - \beta_1(t)) \nabla u_1(t) \cdot \nabla v(t) - \int_{\Omega} (\alpha_2(t) - \beta_2(t)) \nabla u_1(t) \cdot \nabla v(t).
\]

Taking \( v(t) = u_1(t) - u_2(t) \) and using the bounds \( \lambda_1, \lambda_2 \) of \( g_1, g_2 \) respectively, we get

\[
\frac{1}{2} \frac{d}{dt} \|u_1(t) - u_2(t)\|_{L^2(\Omega)}^2 + \lambda_1 \|\nabla u_1(t) - \nabla u_2(t)\|_{L^2(\Omega)}^2 + \lambda_2 \|\nabla u_1(t) - \nabla u_2(t)\|_{L^2(\Omega)}^2 \leq ||\alpha_1(t) - \beta_1(t)||_{L^\infty(\Omega)} \|\nabla u_1(t)\|_{L^2(\Omega)} \|\nabla u_1(t) - \nabla u_2(t)\|_{L^2(\Omega)} + ||\alpha_2(t) - \beta_2(t)||_{L^\infty(\Omega)} \|\nabla u_1(t)\|_{L^2(\Omega)} \|\nabla u_1(t) - \nabla u_2(t)\|_{L^2(\Omega)}.
\]

Moreover, since \( g_1, g_2 \) and \( G_{\sigma} \) are smooth, we have

\[
||\alpha_1(t) - \beta_1(t)||_{L^\infty(\Omega)} \leq C_3 \|u_1(t) - u_2(t)\|_{L^2(\Omega)},
\]

\[
||\alpha_2(t) - \beta_2(t)||_{L^\infty(\Omega)} \leq C_4 \|u_1(t) - u_2(t)\|_{L^2(\Omega)},
\]

where \( C_3 \) and \( C_4 \) are constants which depends only on \( g_1, g_2, \lambda_1, \lambda_2 \) and \( G_{\sigma} \). Combining these inequalities and using Schwarz inequality, we obtain

\[
\frac{d}{dt} \|u_1(t) - u_2(t)\|_{L^2(\Omega)}^2 \leq C \|\nabla u_1(t)\|_{L^2(\Omega)}^2 \|u_1(t) - u_2(t)\|_{L^2(\Omega)}^2,
\]

for a.e., \( 0 \leq t \leq T \), where \( C \) is a constant which depends only on \( g_i \), \( i = 1, 2, G_{\sigma} \) and \( u_0 \).

Since \( u_1(0) - u_2(0) = u_0 \), using Gronwall’s inequality yields [28]

\[
\|u_1(t) - u_2(t)\|_{L^2(\Omega)}^2 \leq 0;
\]

that is, \( u_1 = u_2 \).
3.4 Some properties of weak solution

This section deals with two stages:

(a) Investigate the continuity with respect to initial data of the weak solution for (3.3.3).

(b) Investigate the stability of weak solution and the maximum principle. According to the uniqueness proof in Theorem 3.3.1.

**Theorem 3.4.1.** Assume $u$ is the weak solutions of problem (3.3.3) with the initial data $u_0$. Then

\[
\int_{\Omega} (u - u_0) dx = 0,
\]

\[
||u(.,t) - u_\Omega||_{L^2(\Omega)} \leq e^{-\left(\frac{\lambda_1 + \lambda_2}{\rho}\right)t} ||u_0 - u_\Omega||_{L^2(\Omega)},
\]

a.e. $t \in [0, \infty)$, where $u_\Omega = \left(\frac{1}{|\Omega|}\right) \int_\Omega u_0 \, dx$, and $|\Omega|$ is Lebesgue measure of $\Omega$.

**Proof.** Let $u$ be the solutions for problem (3.3.3) with the initial data $u_0$. For almost every $t$ in $[0,T]$, we have

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \nabla \cdot (g_1(|\nabla G_\sigma * u|)\nabla u) + \nabla \cdot (g_2(|\nabla G_\sigma * u|)\nabla u) = 0 \text{ on } \Omega \times (0,T), \\
u(x,0) &= u_0(x) \text{ in } \Omega, \\
\frac{\partial u}{\partial \mathbf{n}} &= 0 \text{ on } \partial \Omega \times (0,T),
\end{aligned}
\]

in the distribution sense. Integrating over the interval $(0,T)$ and using the Neumann boundary conditions yield

\[
\int_\Omega (u - u_0) dx = 0.
\]

Then, multiplying the equation (3.4.2) by $(u - u_\Omega)$, and integrating by parts over $\Omega$ yields

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega (u - u_\Omega)^2 dx + \int_\Omega g_1(|\nabla G_\sigma * u|)|\nabla u|^2 dx + \int_\Omega g_2(|\nabla G_\sigma * u|)|\nabla u|^2 dx = 0.
\]
Using the following Poincare-Wirtinger inequality \([4], \text{ page 148}\), we have
\[
\left\| u - \frac{1}{|\Omega|} \int_{\Omega} u \, dx \right\|^2_{L^2(\Omega)} = \left\| u - u_\Omega \right\|^2_{L^2(\Omega)} \leq \mu \int_{\Omega} |\nabla u|^2 \, dx,
\]
with the constant \(\mu \equiv \mu(\Omega)\). Substituting (3.4.5) to (3.4.4) yields
\[
\frac{d}{dt} \int_{\Omega} (u - u_\Omega)^2 \, dx \leq - \frac{2(\lambda_1 + \lambda_2)}{\mu} \int_{\Omega} (u - u_\Omega)^2 \, dx.
\]
(3.4.6)

Multiplying this inequality by \(e^{\frac{2(\lambda_1 + \lambda_2)t}{\mu}}\) and integrating over the interval \((0,t)\) we arrive to the inequality
\[
\int_{\Omega} (u - u_\Omega)^2 \, dx \leq e^{-\frac{2(\lambda_1 + \lambda_2)}{\mu} t} \int_{\Omega} (u_0 - u_\Omega)^2 \, dx.
\]
(3.4.7)
Hence, we obtain the assertion of the theorem.

Next, let us build upon the maximum principle as follows.

**Theorem 3.4.2.** Let \(u\) be the weak solutions of problem (3.3.3) with the initial data \(u_0 \in L^\infty(\Omega)\). Then
\[
\inf_{x \in \Omega} u_0 \leq u \leq \sup_{x \in \Omega} u_0.
\]
(3.4.8)

**Proof.** Let \(I := \sup_{x \in \Omega} u_0\) and \(j := \inf_{x \in \Omega} u_0\). Multiply (3.3.3) by \((u - I)_+\), where
\[
(u - I)_+ = \begin{cases} u - I, & \text{if } u - M > 0, \\ 0, & \text{otherwise}, \end{cases}
\]
(3.4.9)
and integrate over \(\Omega\), we get
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u(t) - I)^2_+ \, dx + \int_{\Omega} g_1(\nabla G \ast u_i) \nabla (u(t) - I)_+ \, dx \\
+ \int_{\Omega} g_2(\nabla G \ast u_i) \nabla (u(t) - I)_+ \, dx = 0.
\]
(3.4.10)
Then
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u(t) - I)^2_+ \, dx \leq 0.
\]
(3.4.11)
Therefore, \((1/2)(d/dt) \int_{\Omega} (u(t) - I)^2 \, dx\) is decreasing in \(t\), and since
\[
\int_{\Omega} (u(t) - I)^2 \, dx \geq 0, \quad \int_{\Omega} (u(t) - I)^2 \, dx|_{t=0} = 0, \quad (3.4.12)
\]
we have that
\[
\int_{\Omega} (u(t) - I)^2 \, dx = 0, \quad \forall \ t \in [0, \infty), \quad (3.4.13)
\]
and so
\[
u(t) \leq \sup_{x \in \Omega} u_0 \ a. \ e. \ on \ \Omega, \ \forall \ t > 0. \quad (3.4.14)
\]
Multiplying (3.3.3) by \((u - j)_-\), a similar argument yields that \(u \geq j\) for all \(t \in [0, \infty)\).
Equation (3.4.8) is followed directly.

Now let us consider the following problem:
\[
\begin{align*}
\nabla \cdot (g_1(|\nabla G_\sigma * u|) \nabla u) + \nabla \cdot (g_2(|\nabla G_\sigma * u|) \nabla u) &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \vec{n}} &= 0 \quad \text{on } \partial \Omega, \\
\int_{\Omega} (u - u_0) &= 0.
\end{align*}
\]

**Theorem 3.4.3.** Assume \(u_0 \in L^1(\Omega)\). Then the problem (3.4.15) admits a unique weak solution \(u \in H^1(\Omega)\) such that
\[
\begin{align*}
\int_{\Omega} g_1(|\nabla G_\sigma * u|) \nabla u \nabla v + \int_{\Omega} g_2(|\nabla G_\sigma * u|) \nabla u \nabla v &= 0 \quad \forall \ v \in C^\infty(\Omega), \\
\int_{\Omega} (u - u_0) &= 0.
\end{align*}
\]

**Proof.** It is clear that \(u = u_\Omega\) is a unique solution for the problem (3.4.15).

Next we turn to the proof of the uniqueness of the solution for the problem (3.4.15).
Let \(u_1\) and \(u_2\) be two weak solutions (3.4.15). Multiplying (3.4.15) by \(u\), integrating over \(\Omega\), and using the Neumann boundary conditions, we get
\[
\int_{\Omega} g_1(|\nabla G_\sigma * u|) |\nabla u|^2 + \int_{\Omega} g_2(|\nabla G_\sigma * u|) |\nabla u|^2 = 0. \quad (3.4.17)
\]
Using the following Poincaré-Wirtinger inequality, we have

\[ \left\| u - \frac{1}{|\Omega|} \int_\Omega u \, dx \right\|_{L^2(\Omega)}^2 = \| u - u_\Omega \|^2_{L^2(\Omega)} \leq \mu \int_\Omega |\nabla u|^2 \, dx, \]  

(3.4.18)

with the constant \( \mu \equiv \mu(\Omega) \). Substituting (3.4.15) and (3.4.18) to (3.4.17) yields

\[ \int_\Omega (u - u_\Omega)^2 \, dx = 0. \]  

(3.4.19)

Then

\[ \int_\Omega (u_1 - u_2)^2 \, dx \leq \int_\Omega (u_1 - u_\Omega)^2 \, dx + \int_\Omega (u_2 - u_\Omega)^2 \, dx = 0. \]  

(3.4.20)

That is, \( u_1 = u_2 \).

### 3.5 Convergent iterative scheme

**Theorem 3.5.1.** Let \( u_0 \in H^1(\Omega) \). The sequence \( \{u^n\} \) defined by solving the iterative scheme

\[
\begin{aligned}
\frac{\partial u^{n+1}(t)}{\partial t} &= \nabla \cdot (g_1(|\nabla G_{\sigma} * u^n(t)|) \nabla u^{n+1}(t)) \\
&\quad + \nabla \cdot (g_2(|\nabla G_{\sigma} * u^n(t)|) \nabla u^{n+1}(t)) \quad \text{in } (0, T) \times \Omega, \\
\end{aligned}
\]

\[
\begin{aligned}
u^{n+1}(x, 0) &= u_0 \quad \text{in } \Omega, \\
\frac{\partial u^{n+1}(t)}{\partial \eta} &= 0 \quad \text{on } \partial (0, T) \times \Omega,
\end{aligned}
\]

(3.5.1)

converges in \( C([0, T]; L^2(\Omega)) \) to the strong solution of (3.3.3).

**Proof.** We denote by \( \alpha^n = g_1(|\nabla G_{\sigma} * u^n|) \) and \( \beta^n = g_2(|\nabla G_{\sigma} * u^n|) \). The problem (3.5.1) has a unique solution \( u^{n+1} \) by a classical theory on parabolic equations [12, 28]. It is clear that

\[
\alpha^n \geq g_1(|\nabla G_{\sigma} * u^n|_{L^\infty(\Omega)}), \quad \beta^n \geq g_2(|\nabla G_{\sigma} * u^n|_{L^\infty(\Omega)}) \quad \text{a.e on } (0, T) \times \Omega.
\]

Now we show that the sequence \( \{u^n\} \) converges in \( C([0, T]; L^2(\Omega)) \) to \( u \), the strong
solution of (3.3.3). We observe, from the estimate (3.3.15), that
\[ \frac{d}{dt} ||u^{n+1}(t) - u(t)||^2_{L^2(\Omega)} \leq a(t) ||u^n(t) - u(t)||^2_{L^2(\Omega)}, \] 
(3.5.2)
where \( a(t) = C \langle ||\nabla u(t)||^2_{L^2(\Omega)} \rangle \).
Moreover, we have
\[ ||u^0 - u(t)||^2_{L^2(\Omega)} \leq C_0 \quad \forall \quad t \in [0, T], \]
where \( C_0 \) is a constant which only depends on \( ||u_0||_{H^1(\Omega)} \). Then Gronwall’s inequality yields, for any \( t \in [0, T] \):
\[ ||u^1(t) - u(t)||^2_{L^2(\Omega)} \leq C_0 \left( \int_0^T a(s)ds \right), \]
and, by iteration,
\[ ||u^{n+1}(t) - u(t)||^2_{L^2(\Omega)} \leq C_0 \frac{1}{(n+1)!} \left( \int_0^T a(s)ds \right)^{n+1}, \]
which implies that the sequence \( \{u^n\} \) converges in \( C([0, T]; L^2(\Omega)) \) to the strong solution of (3.3.3).

### 3.6 Numerical experiments

In this Section, we perform two numerical experiments in 2D. The original images Lena and Boat shown in figure 2.5.1 have 256 × 256 pixels and each pixels has a value in \([0, 255]\), and the Gaussian white noise is added by the normal imnoise function imnoise (I,'Gaussian', M, \( \sigma^2 \)) (i.e., the mean M and variance \( \sigma^2 \)) in Matlab. We first scale the intensities of the images into the range between zero and one before we begin our experiments. In our tests, we will use the PSNR as a criteria for the quality of restoration:
\[ \text{PSNR} = 10 \log_{10} \left( \frac{R^2}{\frac{1}{mn} \sum_{i,j}^n (u(i,j) - u_{new}(i,j))^2} \right), \] 
(3.6.1)
where \( \{u(i, j) - u_{new}(i, j)\} \) are the differences of the pixel values between the restored and original images.
Now we demonstrate the numerical performance of our proposed second order model. The forward-backward difference scheme based on, see reference [82] with the Perona-Malik (PM) diffusivity $g_1(s) = \frac{1}{\sqrt{1 + \alpha^2 s^2}}$ with $\alpha = 5$, see the references [18, 56, 82], and the Total variation (TV) diffusivity $g_2(s) = \frac{1}{s}$, in its regularized form $g_2(s) = \frac{1}{\sqrt{s^2 + \epsilon^2}}$ with $\epsilon = 0.01$, is a popular choice, see the references [7, 16, 46]. It enforces piecewise constant results and therefore encourages sharp edges in the image. Throughout the experiments we have taken $\Delta t = 0.1$ [46] and Lagrange multiplier $\lambda = 0.85$ as in [15] and [17].

![Figure 3.6.1: Noisy Lena images with different levels of Gaussian noise (a)-(b), $\sigma^2 = 0.002, 0.004$, respectively; (c)-(d) corresponding denoised images by Catté model with Perona-Malik (PM) diffusivity; (e)-(f) by our model (3.2.1).](image-url)
Figure 3.6.2: Noisy Boat images with different levels of Gaussian noise (a)-(b), $\sigma^2 = 0.002, 0.004$, respectively; (c)-(d) corresponding denoised images by Catté model with Perona-Malik (PM) diffusivity; (e)-(f) by our model (3.2.1).

Table 3.6.1: Results obtained by using models (3.2.1) and Catté et al. [13] applied to the images in Figure 3.6.1 with two different levels of Gaussian noise ($\sigma^2 = 0.002$ and 0.004).

<table>
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<th>Images</th>
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<th>Images</th>
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<tr>
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<td>Fig. 3.6.1(c)</td>
<td>30.31</td>
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<tr>
<td>Fig. 3.6.1(b)</td>
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<td>Fig. 3.6.1(d)</td>
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<td>No. of</td>
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Table 3.6.2: Results obtained by using models (3.2.1) and Catté et al. [13] applied to the images in Figure 3.6.2 with two different levels of Gaussian noise ($\sigma^2 = 0.002$ and 0.004).

<table>
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<td>(3.2.1)</td>
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3.7 Conclusion

We have presented a new second order partial differential equation based nonlinear diffusion model for image denoising. The main idea is to apply a priori smoothness on the solution image. The forward-backward difference schemes are used to discretize model (3.2.1) and Catté model. The model (3.2.1) gives larger PSNR values than that of Catté model, at the same iteration numbers.