Chapter 5

Phase–flip in the amplitude death region

In this Chapter, the mechanism of phase–flip is discussed for delay coupled Landau–Stuart oscillators in the amplitude death regime.

As discussed in Chapter 1, for a dynamical system, the Lyapunov exponents $\lambda_i$ are formally defined as $\ln \Lambda_i$ where the $\Lambda$’s are eigenvalues of the matrix

$$M = \lim_{t \to \infty} [(J^T \cdot J^T)^{1/2}]$$

$J^T$ being the relevant Jacobian matrix that governs the evolution of the dynamics in tangent space over a time interval $t$ starting from initial condition $x$, and the superscript $T$ indicates the transpose of the matrix. When the dynamics is ergodic, the matrix $M$ is independent of initial conditions [35].

The crossings—or avoided crossings—of eigenvalues of matrices as parameters are varied have been studied in a number of contexts. The Wigner–von Neumann theorem [96] relates to the impossibility of eigenvalue degeneracies for a Hermitian matrix as a single parameter is varied [157]. This “non-crossing” rule does not apply to the Lyapunov exponents as system parameters are varied since the Lyapunov exponents are, in some sense, averages over the eigenvalues of products of matrices. However, a number of studies have noted that there is strong repulsion between exponents or avoided crossings as a parameter is varied [2]. Even when there is no repulsion, since the exponents are primarily ordered by rank, there can be interesting consequences when the reordering of exponents takes place. The simplest instance of this is when the largest nontrivial Lyapunov exponent in a nonlinear dynamical system crosses zero; at this crossing of Lyapunov exponents, there is a transition from nonchaotic to chaotic dynamics.

When the system is asymptotically attracted to a fixed point, the Lya-
Punov exponents are the real part of the eigenvalues of the Jacobian matrix. These eigenvalues are complex, and when there is a crossing, there is an abrupt change in the nature of the complex parts (the real parts are equal, of course). The change in the dynamics is reflected in the abrupt change of relative phase of oscillations: this is the phase flip.

These results are discussed in the Section using a system of Landau–Stuart limit cycle oscillators (described in Section 5.1) as an example. Eigenvector analysis is presented in Section 5.3 and we construct an order parameter based on the Gram–Schmidt eigenvectors that detects this transition in Section 5.3.2.

5.1 Delay coupled Landau–Stuart oscillators

We consider the case of delay coupled oscillators where the coupling is diffusive and symmetric,

\[
\begin{align*}
\dot{x} &= f_x(x) + \varepsilon(y(t - \tau), x(t)), \\
\dot{y} &= f_y(y) + \varepsilon(x(t - \tau), y(t)).
\end{align*}
\]

(5.2)

\(x\) and \(y\) denote the variables of the two subsystems and the dynamical equations are specified by \(f_x\) and \(f_y\), respectively, and \(\tau\) represents the time delay in the coupling. We consider the simplest case, namely when the subsystems specified by \(x\) and \(y\) are identical and take each subsystem to be a Landau–Stuart oscillator,

\[
\dot{Z}(t) = (A + i\omega - |Z(t)|^2)Z(t).
\]

(5.3)

This is a well-studied model limit cycle oscillator showing a Hopf bifurcation, with the amplitude of oscillations being directly proportional to \(\sqrt{A}\), \(\omega\) is the frequency of the oscillations and \(Z(t)\) is the complex oscillator variable. The explicit equations of motion for the coupled system are

\[
\begin{align*}
\dot{Z}_1 &= (A_1 + i\omega_1 - |Z_1|^2)Z_1 + \varepsilon(Z_2(t - \tau) - Z_1), \\
\dot{Z}_2 &= (A_2 + i\omega_2 - |Z_2|^2)Z_2 + \varepsilon(Z_1(t - \tau) - Z_2).
\end{align*}
\]

(5.4)

We consider the case of oscillators with \(A_1 = A_2 = 1\), but with different oscillation frequencies \(\omega_1\) and \(\omega_2\). In cartesian coordinates, using \(Z_j = x_j + iy_j\), the equations become
\[ \begin{align*}
\dot{x}_i &= (1 - (x_i^2 + y_i^2))x_i - \omega_1 y_i + \epsilon(x_j(t - \tau) - x_i(t)) \\
\dot{y}_i &= (1 - (x_i^2 + y_i^2))y_i + \omega_2 x_i + \epsilon(y_j(t - \tau) - y_i(t))
\end{align*} \] (5.5)

where \(i,j = 1,2\) and \(j \neq i\). Transforming to polar coordinates \(\phi = \arctan(y/x)\) and \(R = \sqrt{x^2 + y^2}\), we have

\[ \begin{align*}
\dot{R}_i &= R_i(1 - \epsilon - R_i^2) + \epsilon R_j(t - \tau) \cos(\phi_j(t - \tau) - \phi_i) \\
\dot{\phi}_i &= \omega_i + \epsilon(R_j(t - \tau)/R_i) \sin(\phi_j(t - \tau) - \phi_i).
\end{align*} \] (5.6)

When the amplitudes are nearly equal, namely when \(R_i(t - \tau)/R_j = 1, j \neq i\), the dynamical equations for the evolution of oscillator phases reduce to the standard Kuramoto form \([141, 98, 94]\),

\[ \begin{align*}
\dot{\phi}_1 &= \omega_1 + \epsilon \sin(\phi_2(t - \tau) - \phi_1) \\
\dot{\phi}_2 &= \omega_2 + \epsilon \sin(\phi_1(t - \tau) - \phi_2).
\end{align*} \] (5.7)

This condition is typically satisfied once the oscillators are in synchrony. Following Schuster and Wagner (SW) \([141]\) we make the ansatz \(\phi_{1,2} = \Omega t \pm \alpha/2\) to obtain the following transcendental equation for the collective frequency,

\[ \Omega = \bar{\omega} \pm \epsilon \tan(\Omega \tau) \sqrt{\cos^2(\Omega \tau) - \Delta \omega^2/4\epsilon^2}. \] (5.8)

where \(\bar{\omega} = (\omega_1 + \omega_2)/2\) and \(\Delta \omega = \omega_1 - \omega_2\). The system can therefore oscillate at frequencies that are the zeros of the functions \(^1\)

\[ F_\pm(\Omega) = \bar{\omega} - \Omega \pm \epsilon \tan(\Omega \tau) \sqrt{\cos^2(\Omega \tau) - \Delta \omega^2/4\epsilon^2}. \] (5.9)

For identical oscillators, \(A_1 = A_2 = 1\), say, and \(\omega_1 = \omega_2 = \omega\), Eq. (5.9) reduces to

\[ F_\pm(\Omega) = \omega - \Omega \pm \epsilon \sin(\Omega \tau). \] (5.10)

The roots of both \(F_+\) and \(F_-\) (denoted as \(\Omega_\pm\)) are shown along with numerical results in Fig. 5.1(b). The stability criteria for the synchronized states are that \(\cos(\Omega_\tau) > 0\) for in-phase motion (lower branch) and \(\cos(\Omega_\tau) < 0\) for anti-phase motion (the upper branch).

\(^1\) SW \([141]\) showed that the roots of \(F_-\) correspond to stable solutions with frequency \(\Omega\), while the roots of \(F_+\) correspond to unstable oscillations. When the delay \(\tau\) is varied, oscillations at frequencies corresponding to the roots of both \(F_+\) and \(F_-\) can be stabilized.
Figure 5.1: (a) The spectrum of Lyapunov exponents for the coupled system, Eq. (5.5), as a function of the delay. The first two Lyapunov exponents have the same value, as do the third and fourth exponents. (b) The frequency of damped oscillations as a function of $\tau$, the jump at $\tau \sim 0.1745$ corresponding to the crossing in the Lyapunov exponents. Inset figures show transient trajectories for the coupled system before and after the phase flip transition. The curves $\Omega_{\pm}$ are the loci of the zeros of $F_{\pm}$. 
Figure 5.2: (a) Spectrum of Lyapunov exponents for the coupled system in the AD region. The symbols (open circles and triangles) are the real parts of the eigenvalues of the Jacobian. (b) The computed frequency of damped oscillations as a function of $\tau$. Here the symbols (circles and triangles) correspond to the imaginary part of the eigenvalues of the Jacobian; at the phase flip transition point, $\tau \approx 0.1745$, these exchange, leading to the frequency jump.
In the numerical simulations the parameters have been fixed at \( \omega = 9 \) and \( \varepsilon = 10 \). The agreement between the numerics and the analysis is excellent.

### 5.2 Amplitude death and the phase flip

Amplitude death \([17]\) is known to occur in time-delay coupled Landau–Stuart oscillators \([9, 125, 127]\), and we focus on this regime here. Shown in Fig. 5.1(a) is the spectrum of Lyapunov exponents as the time-delay \( \tau \) is varied, and the region marked AD in the figure corresponds to amplitude death.

Note that the largest two Lyapunov exponents are identical in the AD region, as are the next two exponents. Furthermore, there is a prominent "crossing" of the exponents, although since Lyapunov exponents are ordered by rank, these are in fact avoided crossings. Trajectories prior to the avoided crossing are synchronized in-phase while after the crossing point, they are anti-phase synchronized \([110]\). The frequency jump that accompanies the change in relative phase is shown in Fig. 5.1(b).

As is clear from Eq. (5.5), the origin \( Z_j = 0, j = 1, 2 \) is always a fixed point of the system. We linearize Eq. (5.5) around this point to obtain the characteristic equation

\[
\text{Det}(J - \lambda I) = 0, \tag{5.11}
\]

where \( I \) is the identity matrix and the \( J \) is the Jacobian matrix. Taking the perturbation with time-dependence proportional to \( e^{\lambda t} \), the characteristic equation corresponding to Eq. (5.5) is

\[
\lambda^2 - 2(a + i \omega) \lambda + (a^2 - \omega^2 + i 2a \omega) - \varepsilon^2 e^{-2\lambda \tau} = 0 \tag{5.12}
\]

where \( a = 1 - \varepsilon \). Substituting \( \lambda = \alpha + i \beta \), where \( \alpha \) and \( \beta \) are the real and imaginary parts of \( \lambda \) gives the equations

\[
\begin{align*}
\alpha^2 - \beta^2 - 2(\alpha \omega - \beta a) + a^2 - \omega^2 - \varepsilon^2 e^{-2\omega \tau} \cos 2\beta \tau = 0 \\
2\alpha \beta - 2(\alpha \omega + a \beta) + 2a \omega + \varepsilon^2 e^{-2\omega \tau} \sin 2\beta \tau = 0.
\end{align*}
\tag{5.13}
\]

These can be solved in a straightforward manner. In the AD region as shown in Fig. 5.2 the real parts of the eigenvalues are identical to the Lyapunov exponents. Note that there are two pairs of complex conjugate eigenvalues, and this results in the spectrum of exponents having the degeneracies seen in Fig. 5.2(a). In the AD region as shown in Fig. 5.2 the real parts of the eigenvalues are identical to the Lyapunov exponents. The imaginary part of the largest eigenvalue is the frequency of the damped oscillation for the coupled system. As the exponents cross, therefore, the pairs of complex
conjugate eigenvalues exchange imaginary parts. This results in the phase and frequency jump that are observed at the phase flip transition.

Also shown in Fig. 5.2 are the frequencies of damped oscillations as a function of $\tau$ and the interchange of imaginary parts of the complex eigenvalues at the crossings is indicated in Fig. 5.2(b) where the solid (green) curve corresponds to the numerically computed frequencies.

5.3 Eigenvector analysis

The significance of the eigenvectors associated with Lyapunov exponents has been a matter of considerable debate [45], although few studies have analyzed these in any detail. The “directions” associated with the largest Lyapunov exponent are those that dominate the dynamics [46], but these are not easy to determine and further there is no standard protocol. In recent years, the procedure for computing covariant Lyapunov eigenvectors has found application in a variety of contexts [45].

Since the largest Lyapunov exponent dominates the dynamics, it is particularly germane to consider the associated eigenvector in the present instance, since the leading Lyapunov exponents change abruptly as a parameter is varied. We now examine the phase flip transition using the GS eigenvectors for the delay coupled system.

5.3.1 Gram Schmidt eigenvectors

Given the special structure of the coupled system, when the dynamics is synchronized, there are special symmetries in the eigenvectors and eigenvalues. Although the system is, in principle, infinite dimensional since time-delay is involved, in numerical computations [39] it is customary to use a large but finite dimensional representation by discretization. Recall that the Q-R method for computation of the Lyapunov spectrum proceeds as follows. Let $D(t_1)$ be the $n \times k$ matrix having as its columns $k$ perturbations initialized at time $t_i$. Then at time $t_f$, we have

$$D(t_f) = O(t_i, t_f)D(t_1)$$

(5.14)

where $O(t_i, t_f)$ is the resolvent of the tangent dynamics. Decomposition of $D(t_f)$ as $Q(t_f)R(t_f)$, where the columns of $Q$ contain the $k$ orthonormal vectors, and $R$ is a $k \times k$ upper triangular matrix gives, for almost all initial perturbations, when $(t_f - t_i)$ is large enough, the Lyapunov vectors, as columns of the matrix $R$, while the diagonal elements of
Figure 5.3: Comparative behavior of the eigenvectors of coupled Kuramoto oscillators (a), and the Landau Stuart system (b). In (a), there is a change from in-phase to anti-phase on variation of delay due to the multistability in the system [141] but the order parameter shows no variation. In (b), in the Landau Stuart system the order parameter becomes zero above the phase flip point ($\tau \sim 0.1745$). Averages are taken over 100 different initial conditions for each value of $\tau$. In (a) the parameter values for the phase oscillators have been fixed at $\omega = 4$ and $\varepsilon = 0.4$. 
give the \( k \) algebraically largest Lyapunov exponents. Following the process for upto \( t_f \) for \( t_f \) much larger than \( t_i \), the Gram Schmidt (GS) basis converges to an orthogonal set of GS vectors which solely depend on the coordinates of the system in the phase space [38]. The eigenvalues occur as complex conjugate pairs, their real parts corresponding to the Lyapunov exponents in the amplitude death region. In the present case, the individual subsystems have dimension \( d = 2 \), the total system is 4-dimensional, but computations are done in a 4N dimensional basis\(^2\). We observe that the pairwise identical structure of the first pair and the second pair of GS eigenvectors gets exchanged at the value of \( \tau \) where the Lyapunov exponents cross.

### 5.3.2 Order Parameter

We can use this feature of the eigenvectors to construct an order parameter. Consider the scalar product

\[
\gamma = \langle e_i^{\dagger}(\tau')|e_i^\dagger(\tau) \rangle
\]

where \( \tau' \) is a delay time on one side of the flip transition, and \( i = 1 \) is the eigenvector associated with the leading eigenvalue (one could consider other \( i \) as well). We take \( \gamma^2 \) as the order parameter, and fix \( \tau' = 0.14 \). Shown in Fig. 5.3(b) is the manner in which this quantity oscillates prior to the transition, while after the transition, the scalar product vanishes identically.

This behaviour can be contrasted with the dynamics of delay-coupled Kuramoto oscillators [141]

\[
\begin{align*}
\dot{\theta}_1 &= \omega + \varepsilon \sin(\theta_2(t - \tau) - \theta_1) \\
\dot{\theta}_2 &= \omega + \varepsilon \sin(\theta_1(t - \tau) - \theta_2)
\end{align*}
\]

wherein multistability occurs [141]. Starting from a specific set of initial conditions can lead to any of these attractors depending upon the location in the various basins of attraction. The order parameter defined above is depicted in Fig. 5.3(a) and contrasted with the delay coupled Landau Stuart system. Even when the dynamics for the simple phase oscillators changes—the frequency of the attractors is very different—the corresponding basis remains fixed, and there is no change in the order parameter on varying \( \tau \)

\^2 If \( D \) and \( d \) are the dimensions of the coupled system and the individual subsystems \((D = 2d)\), note that the \( ND \) dimensional vectors \( e_i^\dagger(\tau) \) have the property,

a) \( e_i^\dagger = e_{i + d}^\dagger \), for \( i = 1, 2 \) before transition and for \( i = 3, 4 \) after the transition and,

b) \( e_i^\dagger = -e_{i + d}^\dagger \), for \( i = 3, 4 \) before transition and for \( i = 1, 2 \) after the transition. where \( j = k + (n - 1)D, \ n = 1, 2, ..., N \) and \( k = 1, 2, ... d \). For our case, \( D = 4 \) and \( d = 2 \)
Figure 5.4: (Color online) Spectrum of Lyapunov exponents for the delay coupled Rössler system Eq. (5.17). At the phase flip transition point in the death region, for $\tau \approx 1.8$, the complex conjugate pairs of eigenvalues exchange their imaginary parts at the avoided crossing in the Lyapunov spectrum.
since there are no crossings in the Lyapunov spectrum of the system [141]. These results hold more generally. For example Fig. 5.4 shows the Lyapunov spectrum for a system of delay coupled Rössler oscillators. The equations are

\[
\begin{align*}
\dot{x}_i &= -\omega_i y_i - z_i + \varepsilon (x_j(t - \tau) - x_i(t)) \\
\dot{y}_i &= \omega_i x_i + ay_i \\
\dot{z}_i &= f + z_i(x_i - c),
\end{align*}
\]

(5.17)

where \(i, j = 1, 2\) and \(i \neq j\). We can clearly see that we have a crossing of Lyapunov exponents for this system as well, and at this point (here \(\tau \sim 1.8\)), we get a frequency and a phase jump similar to the previous case (results not shown here). The order parameter behaves the same way as it does in the Landau Stuart system [119].

5.4 Summary

This Chapter analyzed the mechanism behind the phase–flip transition in the amplitude death regime for delay coupled Landau–Stuart systems. Since the asymptotic dynamics is on a fixed point, this facilitates in the eigenvalue analysis of the system. The study suggests that the occurrence of the phase flip can be traced to an avoided crossing in the spectrum of Lyapunov exponents as a parameter of the system is varied.

The Chapter also introduced an order parameter based on Gram–Schmidt eigenvectors which was used to spot the transition in the system. For generalization, result on the crossing of the Lyapunov exponents for time–delay coupled chaotic Rössler oscillators was also presented. For a detailed analysis of phase–flip in the Rössler system, see Ref. [119].