CHAPTER 3

ON THE GROWTH RATE OF A PERTURBATION IN

SOME STABILITY PROBLEMS

Essential contents of this chapter have appeared/accepted in the following journals:

(i) Journal of Mathematical and Physical Sciences, 1982, 16, 133-139.
This chapter is largely devoted to the derivation of bounds for the modulus of the complex growth rate of an arbitrary oscillatory perturbation, neutral or unstable, in some hydrodynamic and hydromagnetic stability problems of relevance to oceanography, astrophysics and non-Newtonian fluid mechanics. The results derived are important since the exact solutions for these problems (especially in situations when one or both the boundaries are rigid) in closed form are not obtainable. Instead of presenting a separate analysis for each of these problems we will see subsequently that the governing differential equations and boundary conditions for all these problems are derivable from the simple and elegant equation (2.1.1) and boundary conditions (2.1.2) and its various modifications by the appropriate choice of the matrices $A$, $B$ and $C$. Therefore the results of Chapter 2 when applied to these problems yield results in a unified manner which is one of the aims of the present work as pointed out earlier.

NOTE: All the theorems referred to in this chapter pertain to Chapter 2 unless stated otherwise.

3.1 STABILITY OF SPIRAL FLOWS

The governing equations and boundary conditions of the present problem at the neutral state under the narrow gap approximation involving averaged angular and axial velocities are

$$[(D^2 - a^2) - i(\sigma + R_\lambda a)] (D^2 - a^2)u - 12iR_\lambda au = v, \quad (3.1.1)$$
[(D^2 - a^2) - i(\sigma + R_x a)] v = -\pi a^2 u, \hspace{1cm} (3.1.2)

and \ u = Du = v = 0 \ for \ \ \frac{\phi}{\tau} = \pm \frac{1}{2}, \hspace{1cm} (3.1.3)

where the various symbols occurring in the above equations have the same meaning as in Section 1.5 and \( \sigma \) is a real constant.

THEOREM 1: If (\sigma, u, v) is a solution of equations (3.1.1)-(3.1.3), then

\[ -R_x a < \sigma < \frac{12R_x}{a} - R_x a, \hspace{1cm} (3.1.4) \]

so that \( a > \sqrt{12} \) implies \( \sigma < 0 \).

PROOF: Equation (2.1.1) reduces to the above equations with

\[ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \hspace{1cm} \begin{bmatrix} \frac{i(\sigma + R_x a) + 2a^2}{a} & 0 \\ 0 & -\frac{1}{Ta^2} \end{bmatrix} \]

and

\[ C = \begin{bmatrix} a^2[a^2 + i(\sigma + R_x a)] - 12iR_x a & -1 \\ -1 & \frac{a^2 + i(\sigma + R_x a)}{Ta^2} \end{bmatrix} \]

\[ \text{and} \ X(\xi) = \begin{bmatrix} u(\xi) \\ v(\xi) \end{bmatrix} \]

Further, the boundary conditions on \( X \) conform to those of \( u \) and \( v \). Also \( A_1, B_1 \) and \( C_1 \) come out to be diagonal matrices with diagonal entries:

\[ \text{diag}(A_1) = (0, 0) \]

\[ \text{diag}(B_1) = -(\sigma + R_x a)(1, 0) \]

and

\[ \text{diag}(C_1) = -(\sigma + R_x a)(-a^2 + \frac{12R_x a}{\sigma + R_x a}, -\frac{1}{Ta^2}) \]
Now, with \( l = \sigma + R_x a \), conditions of Theorem 1 are satisfied and hence

\[- R_x a < \sigma < \frac{12R_x}{a} - R_x a.

Clearly \( \alpha > \sqrt{12} \) implies \( \sigma < 0 \) and therefore the neutral state is overstable.

REMARK: Almost identical reasoning shows that if \( \sigma = \sigma_r + i\sigma_i \) then

\[- R_x a < \sigma_r < \frac{12R_x}{a} - R_x a. \quad (3.1.5)

3.2 STABILITY OF THERMAL CONVECTION

(a) Heated from below:

The governing equations and boundary conditions of this problem under Boussinesq approximation are

\[
(D^2 - a^2)(D^2 - a^2 - \frac{p}{\sigma})W = Ra^2 \Theta, \quad (3.2.1)
\]

\[
(D^2 - a^2 - p)\Theta = -W, \quad (3.2.2)
\]

and

\[
W = 0 = \Theta \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1,
\]

and

either \( Dw = 0 \) at \( z = 0 \) and \( z = 1 \) (rigid boundaries),

or \( D^2 W = 0 \) at \( z = 0 \) and \( z = 1 \) (free boundaries).

(3.2.3)

where \( W \) and \( \Theta \) are the z-components of perturbed velocity and the perturbed temperature respectively, \( a \) is the wave
number of the perturbation, \( \sigma = \sigma_r + i\sigma_i \) is the complex growth rate of the perturbation, \( \sigma \) is the thermal Prandtl number and \( R \) is the thermal Rayleigh number.

**THEOREM 2:** If \((p, W, \theta), p = p_r + i\sigma_i,\) is a solution of equations (3.2.1)-(3.2.3), then \(\sigma_i = 0.\)

**PROOF:** Equation (2.1.1) reduces to the above equations with

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -2\sigma_i^2 - \frac{\sigma_i}{\sigma} & 0 \\ 0 & -Ra^2 \end{bmatrix},
\]

\[
C = \begin{bmatrix} a^2(a^2 + \frac{\sigma}{\sigma}) & -Ra^2 \\ -Ra^2 & Ra^2(a^2 + \sigma) \end{bmatrix}, \quad \text{and} \quad X(z) = \begin{bmatrix} W(z) \\ \theta(z) \end{bmatrix}.
\]

Further, the boundary conditions on \(X\) conform to those of \(W\) and \(\theta.\) Also \(A_1, B_1,\) and \(C_1\) come out to be diagonal matrices, with diagonal entries:

\[
dg(A_1) = \sigma_i(0, 0), \quad dg(B_1) = -\sigma_i(\frac{1}{\sigma}, 0), \quad and \quad dg(C_1) = -\sigma_i(-\frac{a^2}{\sigma}, -Ra^2).
\]

Since both the eigenvalues of \(C_2,\) where \(C_2\) is a diagonal matrix with \(dg(C_2) = (-\frac{a^2}{\sigma}, -Ra^2)\) are negative, Theorem 1 implies that

\[
\sigma_i = 0.
\]

This proves the theorem.
(b) **Heated from above:**

The governing equations and boundary conditions of this problem under Boussinesq approximation are

\[(D^2 - a^2)(D^2 - a^2 - \frac{2}{3})W = Ra^2 \Theta , \quad (3.2.4)\]
\[(D^2 - a^2 - p) \Theta = W , \quad (3.2.5)\]

and

\[W = 0 = \Theta \text{ at } z = 0 \text{ and } z = 1 , \]

and either \(DW = 0 \text{ at } z = 0 \text{ and } z = 1 \) (rigid boundaries),

or \(D^2W = 0 \text{ at } z = 0 \text{ and } z = 1 \) (free boundaries),

\[(3.2.6)\]

where the various symbols occurring in the above equations have the same meaning as in part (a) of this section.

**THEOREM 3:** If \((p, W, \Theta)\), \(p = p_r + ip_i\), is a solution of equations (3.2.4)-(3.2.6), then \(p_r < 0\).

**PROOF:** Equation (2.1.1) reduces to the above equations (3.2.4) and (3.2.5) with

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -2a^2 - \frac{2}{3} & 0 \\ 0 & -Ra^2 \end{bmatrix},
\]
\[
C = \begin{bmatrix} a^2(a^2 + \frac{2}{3}) & -Ra^2 \\ -Ra^2 & Ra^2(a^2 + p) \end{bmatrix}
\]

and \(X(z) = \begin{bmatrix} W(z) \\ \Theta(z) \end{bmatrix} \).
Further the boundary conditions on $X$ conform to those of $W$ and $\Theta$. Also $A'_1$, $B'_1$ and $C'_1$ come out to be diagonal matrices with diagonal entries:

$$dg(A'_1) = (1, 0) ,$$
$$dg(B'_1) = - (2a^2 + \frac{p_r}{\sigma}, \text{Ra}^2) ,$$

and $$dg(C'_1) = (a^2 [a^2 + \frac{p_r}{\sigma}], \text{Ra}^2 [a^2 + p_r]) .$$

Thus, with $l = 1$, the conditions of Theorem 2 are satisfied and therefore from (2.2.16), we have

either $$(2a^2 + \frac{p_r}{\sigma}) < 0$$

or $$\text{Inf } [(a^4 + \frac{p_r a^4}{\sigma}), \text{Ra}^2(a^2 + p_r)] < 0 ,$$

which implies that $$p_r < 0 .$$

This proves the theorem.

(c) In presence of magnetic field:

The governing equations and boundary conditions for the present problem under Boussinesq approximation are

$$(D^2 - a^2)(D^2 - a^2 - \frac{p}{\sigma})W + QD(D^2 - a^2)h_z = \text{Ra}^2 \Theta , \quad (3.2.7)$$

$$(D^2 - a^2 - p)\Theta = -W , \quad (3.2.8)$$

$$(D^2 - a^2 - \frac{p \sigma_1}{\sigma})h_z = -DW , \quad (3.2.9)$$

and
\[ W = 0 = \Theta \text{ at } z = 0 \text{ and } z = 1, \]

and

either \[ D^2 W = 0 = h_z \text{ at } z = 0 \text{ and } z = 1 \]

\text{(free perfectly conducting boundaries),}

or \[ D W = 0 = h_z \text{ at } z = 0 \text{ and } z = 1 \]

\text{(rigid perfectly conducting boundaries),} \tag{3.2.10}

where \( h_z \) is the \( z \)-component of perturbed magnetic field, \( \sigma_1 \) is the magnetic Prandtl number and \( Q \) is the Chandrasekhar number.

**THEOREM 4**: If \((p, W, \Theta, h_z), p = p_r + ip_i, p_r \geq 0, p_i \neq 0,\) is a solution of equations (3.2.7)-(3.2.10), then

\[ |p| < Q \sigma. \tag{3.2.11} \]

**PROOF**: Equations (3.2.7)-(3.2.9) can be put in the following convenient forms

\[
(D^2 - a^2)(D^2 - a^2 - \frac{p}{\sigma})W + QD\left(\frac{p \sigma_1}{\sigma} h_z - DW\right) - Ra^2 \Theta = 0, \tag{3.2.12}
\]

\[- Ra^2 (D^2 - a^2 - p) \Theta + W = 0, \tag{3.2.13}
\]

\[- \frac{Q p \sigma_1}{\sigma} \left[(D^2 - a^2 - \frac{p \sigma_1}{\sigma})h_z + DW\right] = 0. \tag{3.2.14}
\]

Equation (2.2.37) reduces to the above equations with

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]
\[
B = \begin{bmatrix}
-(2a^2 + Q + \frac{p}{\sigma}) & 0 & 0 \\
0 & -Ra^2 & 0 \\
0 & 0 & -\frac{Qp*}{\sigma}
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
a^2(a^2 + \frac{p}{\sigma}) & -Ra^2 & 0 \\
-Ra^2 & Ra^2(a^2 + p) & 0 \\
0 & 0 & \frac{Qp*}{\sigma}(a^2 + \frac{p}{\sigma})
\end{bmatrix},
\]

\[
E = \begin{bmatrix}
0 & 0 & \frac{Qp}{\sigma} \\
0 & 0 & 0 \\
-\frac{Qp}{\sigma} & 0 & 0
\end{bmatrix},
\]

\[
X(z) = \begin{bmatrix}
W(z) \\
\Theta(z) \\
h_z(z)
\end{bmatrix}
\]

Further the boundary conditions on \( X \) conform to those of \( W \), \( \Theta \) and \( h_z \). Also \( A_1 \), \( B_1 \) and \( C_1 \) come out to be diagonal matrices with diagonal entries:

\[
dg(A_1) = (0, 0, 0),
\]

\[
dg(B_1) = -p_i(\frac{1}{\sigma}, 0, -\frac{Qp}{\sigma}),
\]

and \( dg(C_1) = -p_i(\frac{a^2}{\sigma}, -Ra^2, \frac{Qa^2}{\sigma}) \).

Now expressing \( B_1 \) and \( C_1 \) as
where $B_3$, $B_4$, $C_3$ and $C_4$ are diagonal matrices with diagonal entries:

\[
\begin{align*}
\text{dg}(B_3) &= \left( \frac{i}{\sigma} , 0 , 0 \right), \\
\text{dg}(B_4) &= \left( 0 , 0 , \frac{Q\sigma_1}{\sigma} \right), \\
\text{dg}(C_3) &= \left( 0 , 0 , \frac{Qa^2\sigma_1}{\sigma} \right), \\
\text{dg}(C_4) &= \left( \frac{a^2}{\sigma} , Ra^2 , 0 \right).
\end{align*}
\]

We show that the inequality (2.2.28) is satisfied if $H$ is the null matrix and $G$ is a diagonal matrix with diagonal entries:

\[
\text{dg}(G) = \left( \frac{Q}{|P|} , 0 , 0 \right).
\]

Now

\[
\int_V \left[ (\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 X \right] \, dV \\
= \frac{Q\sigma_1}{\sigma} \int_0^1 \left[ |Dh_z|^2 + a^2 |h_z|^2 \right] \, dz. \tag{3.2.15}
\]

Integrating by parts the right hand side of equation (3.2.15) and using the boundary conditions (3.2.10), we have

\[
\int_V \left[ (\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 X \right] \, dV \\
= -\frac{Q\sigma_1}{\sigma} \int_0^1 h_z^* (D^2 - a^2) h_z \, dz \tag{3.2.16}
\]

Using Schwartz inequality, we get
\[ \int_V [(\nabla X)^+ B_4(\nabla X) + X^+ C_3 X] \, dV \leq \frac{Q \sigma_1}{\sigma} \left[ \int_Q \left| h_z \right|^2 \, dz \right]^{1/2} \left[ \int_Q \left| (D^2 - a^2)h_z \right|^2 \, dz \right]^{1/2} \]  

(3.2.17)

Now from equation (3.2.9), we have

\[ \int_Q \nabla W \cdot \nabla^* \, dz = \int_Q \left[ (D^2 - a^2 - \frac{p \sigma_1}{\sigma})h_z(D^2 - a^2 - \frac{p^* \sigma_1}{\sigma})h_z^* \right] \, dz \]

Integrating by parts and using the boundary conditions (3.2.10), we get

\[ \int_q |\nabla W|^2 \, dz = \int_q \left[ \left| (D^2 - a^2)h_z \right|^2 + \frac{2p \sigma_1}{\sigma} \left( |Dh_z|^2 + a^2 |h_z|^2 \right) \right. \]

\[ + \frac{|p|^2 \sigma_1^2}{\sigma^2} |h_z|^2 \right] \, dz \]  

(3.2.18)

Since \( p \geq 0 \), \( p \neq 0 \), equation (3.2.18) implies that

\[ \int_q \left| (D^2 - a^2)h_z \right|^2 \, dz \leq \int_q |\nabla W|^2 \, dz , \]  

(3.2.19)

\[ \int_q |h_z|^2 \, dz \leq \frac{\sigma_1^2}{|p|^2} \int_q |\nabla W|^2 \, dz . \]  

(3.2.20)

Therefore from inequalities (3.2.17), (3.2.19) and (3.2.20), we have

\[ \int_V [(\nabla X)^+ B_4(\nabla X) + X^+ C_3 X] \, dV \leq \frac{Q}{|p|} \int_Q |\nabla W|^2 \, dz \]

\[ = \int_V (\nabla X)^+ G(\nabla X) \, dV . \]

Thus, with \( l = p_1 \) the conditions of Theorem 7 together with inequality (2.2.28) are satisfied and therefore from (2.2.29),
we have
\[ |p| < Q^\sigma . \]

This proves the theorem.

(d) In presence of rotation:

The governing equations and the boundary conditions for the present problem under Boussinesq approximation are

\[ (D^2 - a^2)(D^2 - a^2 - \frac{p}{\sigma})W = Ra^2 \Phi + TDZ , \quad (3.2.21) \]
\[ (D^2 - a^2 - p)\Phi = -W , \quad (3.2.22) \]
\[ (D^2 - a^2 - \frac{p}{\sigma})Z = -DW , \quad (3.2.23) \]

and

\[ W = 0 = \Phi \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1 , \]

and either
\[ DW = 0 = Z \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1 \quad \text{(rigid boundaries)}, \quad (3.2.24) \]

or
\[ D^2W = 0 = DZ \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1 \quad \text{(free boundaries)}, \quad (3.2.25) \]

where \( Z \) is the \( z \)-component of perturbed vorticity and \( T \) is the Taylor number.

THEOREM 5: If \((p, W, \Phi, Z), p = p_r + ip_i, p_r \gg 0, p_i \neq 0,\) is a solution of equations (3.2.21)-(3.2.24), then

\[ |p|^2 \leq T \sigma^2 . \quad (3.2.26) \]

PROOF: Since \( p_i \neq 0, \) equations (3.2.21)-(3.2.23) can be put in the following convenient forms
\[(D^2 - a^2)(D^2 - a^2 - \frac{p}{\sigma})W - Ra^2 \theta - TDZ = 0, \tag{3.2.27}\]

\[-Ra^2 \left[(D^2 - a^2 - p)\theta + W\right] = 0, \tag{3.2.28}\]

\[T \left[(D^2 - a^2 - \frac{p}{\sigma})Z + D\theta\right] = 0. \tag{3.2.29}\]

Equation (2.2.37) reduces to the above equations with

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
-(2a^2 + \frac{p}{\sigma}) & 0 & 0 \\
0 & -Ra^2 & 0 \\
0 & 0 & T
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
a^4 + \frac{pa^2}{\sigma} & -Ra^2 & 0 \\
-Ra^2 & Ra^2(a^2 + p) & 0 \\
0 & 0 & -T(a^2 + \frac{p}{\sigma})
\end{bmatrix},
\]

\[
E = \begin{bmatrix}
0 & 0 & -T \\
0 & 0 & 0 \\
T & 0 & 0
\end{bmatrix},
\]

and \[X(z) = \begin{bmatrix} W(z) \\ \theta(z) \\ Z(z) \end{bmatrix}.\]
Further the boundary conditions on $X$ conform to those of $W$, \( \Theta \) and $Z$. Also $A_1$, $B_1$ and $C_1$ come out to be diagonal matrices with diagonal entries:

\[
dg(A_1) = p_1(0, 0, 0),
\]
\[
dg(B_1) = -p_1\left(\frac{1}{\sigma}, 0, 0\right),
\]
and \[
dg(C_1) = -p_1\left(-\frac{a^2}{\sigma}, -Ra^2, \frac{T}{\sigma}\right).
\]

Now expressing

\[C_1 = -p_1(C_3 - C_4),\]

where $C_3$ and $C_4$ are diagonal matrices with diagonal entries:

\[
dg(C_3) = (0, 0, \frac{T}{\sigma}),
\]
and \[
dg(C_4) = (\frac{a^2}{\sigma}, Ra^2, 0).
\]

We show that the inequality (2.2.28) is satisfied if $H$ is the null matrix and $G$ is the diagonal matrix with diagonal entries:

\[
dg(G) = \left(\frac{T\sigma}{Z}, 0, 0\right).
\]

Since in this case $B_4$ is the null matrix,

\[
\int_{V} \left[ (\text{grad } X)^{\dagger} B_4 (\text{grad } X) + X^{\dagger} C_3 X \right] dv = \int_{V} X^{\dagger} C_3 X dv
\]
\[
= \frac{T}{\sigma} \int_{0}^{1} |Z|^2 dz
\]

(3.2.30)

Now from equation (3.2.23), we have

\[
\int_{0}^{1} DW DW^{*} dz = \int_{0}^{1} \left[ (D^2 - a^2 - \frac{E}{\sigma}) Z (D^2 - a^2 - \frac{E^{*}}{\sigma}) Z^{*} \right] dz.
\]
Integrating by parts and using the boundary conditions (3.2.24), we have

\[
\int_0^1 \left| \frac{\partial W}{\partial z} \right|^2 \, dz = \int_0^1 \left[ \left( D^2 - a^2 \right) \left| \frac{\partial W}{\partial z} \right|^2 + \frac{2p_r}{\sigma} \left( \left| \frac{\partial W}{\partial z} \right|^2 + a^2 |\phi|^2 \right) \right. \\
\left. + \frac{|p|^2 |\phi|^2}{\sigma^2} \right] \, dz \quad (3.2.31)
\]

Since \( p_r \gg 0 \), \( p_1 \neq 0 \), equation (3.2.31) implies that

\[
\int_0^1 |\phi|^2 \, dz \leq \frac{\sigma^2}{|p|^2} \int_0^1 \left| \frac{\partial W}{\partial z} \right|^2 \, dz. \quad (3.2.32)
\]

Equation (3.2.30) and inequality (3.2.32) imply that

\[
\int_\Omega \left| \phi \right|^2 \, dV \leq \frac{T \sigma^2}{|p|^2} \int_0^1 \left| \frac{\partial W}{\partial z} \right|^2 \, dz \\
= \int_\Omega (\text{grad } \phi)^\top G(\text{grad } \phi) \, dV.
\]

Thus, with \( l = p_1 \) the conditions of Theorem 7 together with the inequality (2.2.28) are satisfied and therefore from (2.2.29), we have

\[
|p|^2 \leq T \sigma^2.
\]

This proves the theorem.

3.3 STABILITY OF THERMOHALINE CONVECTION

(a) Veronis' thermohaline configuration:

The governing equations and the boundary conditions for the present problem under Boussinesq approximation are

\[
(D^2 - a^2)(D^2 - a^2 - \frac{p}{\sigma})W = Ra_1^2 \phi - Ra_2^2 \phi, \quad (3.3.1)
\]

\[
(D^2 - a^2 - p)\phi = -W, \quad (3.3.2)
\]
(D^2 - a^2 - \frac{p}{\tau})\varnothing = -\frac{W}{\tau} \quad (3.3.3)

and

W = 0 = \Theta = \varnothing \text{ at } z = 0 \text{ and } z = 1 ,

and

either \text{ } D\varnothing = 0 \text{ at } z = 0 \text{ and } z = 1 \text{ (rigid boundaries)},

or \text{ } D^2\varnothing = 0 \text{ at } z = 0 \text{ and } z = 1 \text{ (free boundaries)},

(3.3.4)

where \( R_s, \tau \) and \( \varnothing \) stand respectively for the salinity Rayleigh number, the ratio of mass diffusivity to heat diffusivity and the perturbed concentration, other symbols have the same meanings as in Section 3.2.

**THEOREM 6:** If \((p, W, \Theta, \varnothing), p = p_r + ip_i, p_r \geq 0, p_i \neq 0,\) is a solution of equations (3.3.1)-(3.3.4), then

\[ |p|^2 \leq R_s \sigma . \quad (3.3.5) \]

**PROOF:** Since \( p_i \neq 0, \) equations (3.3.1'-(3.3.3) can be put in the following convenient forms

\[ (D^2 - a^2)(D^2 - a^2 - \frac{p}{\sigma})W - Ra^2\Theta + R_s a^2 \left[ \frac{\tau}{p} (D^2 - a^2)\varnothing + \frac{W}{p} \right] = 0 , \quad (3.3.6) \]

\[ - Ra^2 \left[ (D^2 - a^2 - p)\Theta + W \right] = 0 \quad (3.3.7) \]

and

\[ \frac{\tau^2 R_s a^2}{p} (D^2 - a^2) \left[ (D^2 - a^2 - \frac{p}{\tau})\varnothing + \frac{W}{\tau} \right] = 0 . \quad (3.3.8) \]

Equation (2.1.1) reduces to the above equations with
Further, the boundary conditions on $X$ conform to those of $W$, $\Theta$ and $\phi$. Also $A_1$, $B_1$ and $C_1$ are diagonal matrices with diagonal entries

$$\text{dg}(A_1) = p_i(0, 0, \frac{\tau R_s a^2}{|p|^2}) ,$$

$$\text{dg}(B_1) = - p_i(\frac{1}{\sigma}, 0, \frac{2 \tau R_s a^2}{|p|^2} [a^2 + \frac{p_k}{\tau}]).$$
and

$$dg(C_1) = -p_i(a^2\frac{R_s}{|p|} - \frac{1}{\sigma}) - Ra^2 - \frac{\tau^2 R_s a^4}{|p|^2} [a^2 + \frac{2p_R}{\tau}] .$$

Thus, with $l = p_i$, conditions of Theorem 1 are satisfied and therefore from (2.2.9), we have

$$|p|^2 < R_s \sigma .$$

This proves the theorem.

(b) In presence of magnetic field:

The governing equations and boundary conditions for the present problem under Boussinesq approximation are

$$\begin{align*}
(D^2 - a^2)(D^2 - a^2 - \frac{p}{\sigma})W &= Ra^2 \theta - R_s a^2 \phi - QD(D^2 - a^2)h_z , \\
(D^2 - a^2 - p)\theta &= -W , \\
(D^2 - a^2 - \frac{p}{\tau})\phi &= -\frac{W}{\tau} , \\
(D^2 - a^2 - \frac{p\sigma}{\tau})h_z &= -DW .
\end{align*}$$

(3.3.9) (3.3.10) (3.3.11) (3.3.12)

and

$$W = \theta = 0 = \phi \text{ at } z = 0 \text{ and } z = 1$$

and

either $DW = 0 = h_z$ at $z = 0$ and $z = 1$

(rigid perfectly conducting boundaries),

or $D^2W = 0 = h_z$ at $z = 0$ and $z = 1$

(free perfectly conducting boundaries).

(3.3.13)
THEOREM 7: If \( (p, W, \Theta, \Phi, h_z) \), \( p = p_r + ip_i \), \( p_r \geq 0 \), \( p_i \neq 0 \), is a solution of equations (3.3.9)-(3.3.13), then

\[
|p|^2 < \text{ greater of } (R_s \sigma, Q^2 \sigma^2). \tag{3.3.14}
\]

PROOF: Since \( p_i \neq 0 \), equations (3.3.9)-(3.3.12) can be put in the following convenient forms

\[
(D^2 - \alpha^2)(D^2 - \alpha^2 - \frac{E}{\sigma})W + Q \left[ \frac{p \sigma_1}{\sigma} Dh_{z} - D^2 W \right] - Ra^2 \Phi
\]

\[
+ R_s a^2 \left[ \frac{\tau}{p} (D^2 - \alpha^2) \Theta + \frac{W}{p} \right] = 0, \tag{3.3.15}
\]

\[
-Ra^2 \left[ (D^2 - \alpha^2 - p) \Theta + W \right] = 0, \tag{3.3.16}
\]

\[
\frac{\tau^2 R_s a^2}{p} (D^2 - \alpha^2) \left[ (D^2 - \alpha^2 - \frac{E}{\tau}) \Phi + \frac{W}{\tau} \right] = 0, \tag{3.3.17}
\]

\[
-\frac{Qp \sigma_1}{\sigma} [(D^2 - \alpha^2 - \frac{p \sigma_1}{\sigma}) h_{z} + Dw] = 0. \tag{3.3.18}
\]

Equation (2.2.37) reduces to the above equations with

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\tau^2 R_s a^2}{p} & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
-(2a^2 + Q + \frac{E}{\sigma}) & 0 & \frac{\tau R_s a^2}{p} & 0 \\
0 & -Ra^2 & 0 & 0 \\
\frac{\tau R_s a^2}{p} & 0 & -\frac{\tau^2 R_s a^2}{p} (2a^2 + \frac{E}{\tau}) & 0 \\
0 & 0 & 0 & -\frac{Qp \sigma_1}{\sigma}
\end{bmatrix}
\]
\[
C = \begin{bmatrix}
    a^4 + \frac{pa^2}{\sigma} + \frac{R_s a^2}{p} & -Ra^2 & -\frac{\tau R_s a^4}{p} & 0 \\
    -Ra^2 & Ra^2(a^2 + p) & 0 & 0 \\
    -\frac{\tau R_s a^4}{p} & 0 & \frac{\tau R_s a^2}{p} & 0 \\
    0 & 0 & 0 & \frac{Qp\sigma_1}{\sigma}(a^2 + \frac{p\sigma_1}{\sigma})
\end{bmatrix}
\]

\[
E = \begin{bmatrix}
    0 & 0 & 0 & \frac{Qp\sigma_1}{\sigma} \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    \frac{Qp\sigma_1}{\sigma} & 0 & 0 & 0
\end{bmatrix},
\]

\[
X(z) = \begin{bmatrix}
    W(z) \\
    \theta(z) \\
    \emptyset(z) \\
    h_2(z)
\end{bmatrix},
\]

Further the boundary conditions on \( X \) conform to those of \( W \), \( \theta \), \( \emptyset \) and \( h_2 \). Also \( A_1 \), \( B_1 \) and \( C_1 \) come out to be diagonal matrices with diagonal entries

\[
dq(A_1) = p_i(0, 0, \frac{\tau R_s a^2}{|p|^2}, 0),
\]

\[
dq(B_1) = -p_i\left(\frac{1}{\sigma}, 0, \frac{2\tau R_s a^2}{|p|^2}(a^2 + \frac{p\sigma_1}{\sigma}), -\frac{Q\sigma_1}{\sigma}\right),
\]

and
\[ \text{dg}(C_1) = - p_1 \left( a^2 \left[ \frac{R_s}{|p|^2} - \frac{1}{\sigma} \right], - Ra^2, - \frac{\tau^2 R_s a^4}{|p|^2} \left( a^2 + \frac{2p_T}{\tau} \right), \frac{Qa^2 \sigma_1}{\sigma} \right). \]

Now expressing \( B_1 \) and \( C_1 \) as
\[ B_1 = - p_i (B_3 - B_4), \]
\[ C_1 = - p_i (C_3 - C_4), \]
where \( B_3, B_4, C_3 \) and \( C_4 \) are diagonal matrices with diagonal entries
\[ \text{dg}(B_3) = \left( \frac{1}{\sigma}, 0, \frac{2\tau^2 R_s a^2}{|p|^2} \left( a^2 + \frac{p_T}{\tau} \right), 0 \right), \]
\[ \text{dg}(B_4) = (0, 0, \sigma_1), \]
\[ \text{dg}(C_3) = (0, 0, 0, \sigma_1), \]
and
\[ \text{dg}(C_4) = \left( - a^2 \left[ \frac{R_s}{|p|^2} - \frac{1}{\sigma} \right], Ra^2, \frac{\tau^2 R_s a^4}{|p|^2} \left( a^2 + \frac{2p_T}{\tau} \right), 0 \right). \]

We show that the inequality (2.2.28) is satisfied if \( H \) is the null matrix and \( G \) is a diagonal matrix with diagonal entries
\[ \text{dg}(G) = \left( \frac{Q}{|p|}, 0, 0, 0 \right). \]

Now
\[ \int_V \left[ (\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 X \right] \text{d}V = \]
\[ = \frac{Q \sigma_1}{\sigma} \int_0^1 \left[ |Dh_x|^2 + a^2 |h_z|^2 \right] \text{d}z. \quad (3.3.19) \]

Proceeding exactly as in Theorem 3 of the present chapter, we have
\[
\int_V \left[ (\text{grad } x)^\top \text{B}_4(\text{grad } x) + x^\top C_3 x \right] \, dV < \frac{Q}{|p|} \int_0^1 |DW|^2 \, dz
\]

Thus, with \( l = p_1 \), the conditions of Theorem 7 together with inequality (2.2.28) are satisfied and therefore from (2.2.29), we have

either \( \frac{R_s}{|p|} - \frac{3}{\sigma} > 0 \) or \( \frac{Q}{|p|} - \frac{3}{\sigma} > 0 \),

which gives

\[ |p|^2 < \text{greater of } (R_s \sigma, \sigma^2 \sigma^2) \]

This proves the theorem.

(c) In presence of rotation:

The governing equations and boundary conditions for the present problem under Boussinesq approximation are

\[
(D^2 - a^2)(D^2 - a^2 - \frac{D}{\sigma})W = Ra^2 \theta - R_s \sigma^2 \theta + TDZ, \quad (3.3.20)
\]

\[
(D^2 - a^2 - p)\theta = -W, \quad (3.3.21)
\]

\[
(D^2 - a^2 - \frac{D}{\tau})\phi = -\frac{W}{\tau}, \quad (3.3.22)
\]

\[
(D^2 - a^2 - \frac{D}{\sigma})Z = -DW, \quad (3.3.23)
\]

and

\[
W = 0 = \theta = \phi \text{ at } z = 0 \text{ and } z = 1,
\]

and

\[
\text{either } DW = 0 = Z \text{ at } z = 0 \text{ and } z = 1, \text{ (rigid boundaries)}
\]

(3.3.24)
or \( D^2 W = 0 = DZ \) at \( z = 0 \) and \( z = 1 \), (free boundaries).

(3.3.25)

**THEOREM 8:** If \((p, W, \Theta, \Phi, Z)\), \(p = p_r + ip_i, p_r \geq 0, p_i \neq 0, \) is a solution of equations (3.3.20)-(3.3.24), then

\[
|p|^2 < \text{greater of } (R_s \sigma, T \sigma^2). \tag{3.3.26}
\]

**PROOF:** Since \( p_i \neq 0, \) equations (3.3.20)-(3.3.23) can be put in the following convenient forms

\[
(D^2 - a^2)(D^2 - a^2 - \frac{B}{\sigma})W - Ra^2 \Theta + R_s a^2 \left[ \frac{\tau}{p} (D^2 - a^2)\Phi + \frac{W}{\tau} \right] - TDZ = 0, \tag{3.3.27}
\]

\[
-Ra^2 \left[ (D^2 - a^2 - p)\Theta + W \right] = 0, \tag{3.3.28}
\]

\[
\frac{\tau^2 R_s a^2}{p} \left[ (D^2 - a^2 - \frac{B}{\sigma})\Theta + \frac{W}{\tau} \right] = 0, \tag{3.3.29}
\]

\[
T \left[ (D^2 - a^2 - \frac{B}{\sigma})Z + DW \right] = 0. \tag{3.3.30}
\]

Equation (2.2.37) reduces to the above equations with

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\tau^2 R_s a^2}{p} & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
\[ B = \begin{bmatrix}
-\left(2a^2 + \frac{p}{\sigma}\right) & 0 & \frac{\tau R_s a^2}{p} & 0 \\
0 & -R a^2 & 0 & 0 \\
\frac{\tau R_s a^2}{p} & 0 & \frac{\tau^2 R_s a^2}{p} & 0 \\
0 & 0 & 0 & T
\end{bmatrix} \]

\[ C = \begin{bmatrix}
a^4 + \frac{p a^2}{\sigma} + \frac{R_s a^2}{p} & -R a^2 & -\frac{\tau R_s a^2}{p} & 0 \\
-R a^2 & R a^2(a^2+p) & 0 & 0 \\
-\frac{\tau R_s a^2}{p} & 0 & \frac{\tau^2 R_s a^2}{p}(a^4 + \frac{p a^2}{\tau}) & 0 \\
0 & 0 & 0 & -T(a^2 + \frac{p}{\sigma})
\end{bmatrix} \]

\[ E = \begin{bmatrix}
0 & 0 & 0 & -T \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
T & 0 & 0 & 0
\end{bmatrix} \]

and \[ X(z) = \begin{bmatrix}
W(z) \\
\Theta(z) \\
\bar{\Theta}(z) \\
Z(z)
\end{bmatrix}. \]

Further the boundary conditions on \( X \) conform to those of \( W, \Theta, \bar{\Theta} \) and \( Z \). Also \( A_1, B_1 \) and \( C_1 \) come out to be diagonal matrices with diagonal entries

\[ \text{deg}(A_1) = p_1(0, 0, \frac{\tau^2 R_s a^2}{|p|^2}, 0), \]
\[ \text{dg}(B_1) = -p_i \left( \frac{1}{\sigma}, 0, \frac{2 \tau^2 R_s a^2}{|p|^2} \right) (a^2 + \frac{p_i}{r}), 0 \),
\]
\[ \text{dg}(C_1) = -p_i \left( a^2 \left[ \frac{R_s}{|p|^2} - \frac{1}{\sigma} \right], -Ra^2, -\frac{\tau^2 R_s a^4}{|p|^2} (a^2 + \frac{2p_i}{\tau}), \frac{T}{\sigma} \right) . \]

Now expressing
\[ C_1 = -p_i (C_3 - C_4) , \]
where \( C_3 \) and \( C_4 \) are diagonal matrices with diagonal entries
\[ \text{dg}(C_3) = (0, 0, 0, \frac{T}{\sigma}) , \]
\[ \text{dg}(C_4) = \left( a^2 \left[ \frac{1}{\sigma} - \frac{R_s}{|p|^2} \right], Ra^2, \frac{\tau^2 R_s a^4}{|p|^2} (a^2 + \frac{2p_i}{\tau}), 0 \right) . \]

We show that the inequality (2.2.28) is satisfied if \( H \) is the null matrix and \( G \) is the diagonal matrix with diagonal entries
\[ \text{dg}(G) = \left( \frac{T_0}{|p|^2}, 0, 0, 0 \right) . \]

Since in this case \( B_4 = (0)_{4 \times 4} \),
\[ \int_V \left[ (\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 X \right] dV = \int_V X^\dagger C_3 X dV \]
\[ = \frac{T}{\sigma} \int_0^1 |z|^2 dz \ . \]
(3.3.31)

Proceeding exactly as in Theorem 4 of the present chapter, we have
\[ \int_V \left[ (\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 X \right] dV < \frac{T_0}{|p|^2} \int_0^1 |dw|^2 dz \]
\[ = \int_V (\text{grad } X)^\dagger G(\text{grad } X) dV \]
Thus, with $l = p$, the conditions of Theorem 7 together with inequality (2.2.28) are satisfied and therefore from (2.2.29), we have

either $\frac{R_s}{|p|} - \frac{1}{\sigma} > 0$ or $\frac{T\sigma}{|p|^2} - \frac{1}{\sigma} > 0$,

which implies

$|p|^2 < \text{greater of } (R_s\sigma, T\sigma^2)$.

This proves the theorem.

(d) For a visco-elastic fluid:

The governing equations of this problem under Boussinesq approximation when the visco-elastic fluid is described by Maxwell's constitutive relation are easily seen to be

$$(D^2 - a^2) \left[(D^2 - a^2 - \frac{p(1 + \Gamma p)}{\sigma}\right] W = Ra^2(1 + \Gamma p)\theta \tag{3.3.32}$$

$$-(D^2 - a^2 - p)\theta = -W \tag{3.3.33}$$

$$-(D^2 - a^2 - \frac{p}{t})\phi = -\frac{W}{t} \tag{3.3.34}$$

and

$W = 0 = \theta = \phi$ at $z = 0$ and $z = 1$,

and

either $DW = 0$ at $z = 0$ and $z = 1$ (rigid boundaries),

or $D^2W = 0$ at $z = 0$ and $z = 1$ (free boundaries).

where $\Gamma$ is an elastic parameter.
THEOREM 9: If \((p, W, \theta, \phi)\), \(p = p_r + ip_\perp\), \(p_r \geq 0, p_\perp \neq 0\), is a solution of equations (3.3.32)-(3.3.35), then
\[
|p|^2 < \left( \frac{R_s \sigma + \sqrt{R_s^2 \sigma^2 + 4R_s \sigma^2}}{2} \right) .
\] (3.3.36)

PROOF: Since \(p_\perp \neq 0\), equations (3.3.32)-(3.3.34) can be put in the following convenient forms
\[
(D^2 - a^2) [D^2 - a^2 - \frac{p(1 + \tau p)}{\sigma}] W - R a^2 (1 + \tau p) \theta
- R_s a^2 (1 + \tau p) \left[ \frac{W}{p} + \frac{\tau}{p} (D^2 - a^2) \phi \right] = 0 ,
\] (3.3.37)
\[
- R a^2 (1 + \tau p^*) \left[ (D^2 - a^2 - p) \theta + W \right] = 0 ,
\] (3.3.38)
and
\[
\frac{\tau^2 R_s a^2 (1 + \tau p^*)}{p^*} \left[ (D^2 - a^2) (D^2 - a^2 - \frac{p}{\tau} \phi + \frac{W}{\tau} \right] = 0 .
\] (3.3.39)

Equation (2.1.1) reduces to the above equations with
\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{\tau^2 R_s a^2 (1 + \tau p^*)}{p^*}
\end{bmatrix}
\]
\[
B = \begin{bmatrix}
-2 a^2 + \frac{B(1 + \tau p)}{\sigma} & 0 & \frac{\tau R_s a^2 (1 + \tau p)}{p} \\
0 & -R a^2 (1 + \tau p^*) & 0 \\
\frac{\tau R_s a^2 (1 + \tau p^*)}{p^*} & 0 & -\frac{\tau^2 R_s a^2 [2 a^2 (1 + \tau p^*)]}{p^*}
\end{bmatrix}
+ \frac{p(1 + \tau p^*)}{\tau p}.
\[
C = \begin{bmatrix}
\frac{a^4 + pa^2(1+\gamma p)}{\sigma} & -Ra^2(1+\gamma p) & \frac{\tau Ra}{p}
\frac{R_s a^2(1+\gamma p)}{p} & -Ra^2(1+\gamma p) & \frac{\tau R_s a^4(1+\gamma p)}{p}
\frac{-R_s a^2(1+\gamma p^*)}{p} & Ra^2(1+\gamma p^*) & 0
(a^2+p)
\frac{-\tau R_s a^4(1+\gamma p^*)}{p} & 0 & \frac{\tau R_s a^4 a^2(1+\gamma p^*)}{p}
\frac{-\tau R_s a^4 a^2(1+\gamma p^*)}{p} & 0 & \frac{\tau R_s a^4 a^2(1+\gamma p^*)}{p}
\frac{\tau R_s a^4 a^2(1+\gamma p^*)}{p} & 0 & \frac{\tau R_s a^4 a^2(1+\gamma p^*)}{p}
\end{bmatrix}
\]

and \[X(z) = \begin{bmatrix}
W(z)
\Theta(z)
\emptyset(z)
\end{bmatrix} \]

Further the boundary conditions on \(X\) conform to those of \(W\), \(\Theta\) and \(\emptyset\). Also \(A_1\), \(B_1\) and \(C_1\) come out to be diagonal matrices with diagonal entries

\[
dg(A_1) = \frac{\tau^2 R_s a^2}{|p|^2}, \frac{R_s a^2}{|p|^2}.
\]

\[
dg(B_1) = -p \left( \frac{1}{\sigma} \left[ 1 + 2ip \right], -Ra^2 \gamma, \tau^2 R_s a^2 \right)
\]

\[
dg(C_1) = -p \left( \frac{R_s}{|p|^2} - \frac{1}{\sigma} - \frac{2ip \gamma}{\sigma} \right) a^2, a^2[Ra^2 \gamma - R],
- \tau^2 R_s a^4 \left[ \frac{a^2}{|p|^2} \frac{2p \gamma}{|p|^2} + \frac{\gamma}{|p|^2} \right].
\]
Now expressing $B_1$ and $C_1$ as

$$B_1 = -P_i(B_3 - B_4),$$

and

$$C_1 = -P_i(C_3 - C_4),$$

where $B_3$, $B_4$, $C_3$ and $C_4$ are diagonal matrices with diagonal entries

$$\text{dg}(B_3) = \left(1 + 2\eta \rho \tau \right), 0, \tau^2 R_s a^2 \left(\frac{2a^2}{|p|^2} + \frac{1}{\tau} \left(\frac{2\rho}{|p|^2} + \frac{2\rho}{|p|^2}\right)\right),$$

$$\text{dg}(B_4) = \left(0, Ra^2 \eta \rho \tau, 0\right),$$

$$\text{dg}(C_3) = \left(0, Ra^4 \eta \rho \tau, 0\right),$$

and

$$\text{dg}(C_4) = \left(a^2 \left(\frac{2a^2}{|p|^2} + \frac{2\rho}{|p|^2}\right), Ra^2, \tau^2 R_s a^4 \left(\frac{a^2}{|p|^2} + \frac{1}{\tau} \left(\frac{2\rho}{|p|^2} + \frac{2\rho}{|p|^2}\right)\right)\right).$$

We show that the inequality (2.2.22) is satisfied if $H$ is a diagonal matrix with diagonal entries

$$\text{dg}(H) = \left(\frac{Ra^2 \eta \rho}{|p|}, 0, 0\right).$$

Now

$$\int_V \left[(\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 (x) X\right] \, dV$$

$$= Ra^2 \eta \int_0^\frac{1}{a^2} \frac{1}{\tau} \left[|\text{Dz}|^2 + a^2 |\Theta|^2\right] \, dz, \quad (3.3.40)$$

Integrating by parts the right hand side of equation (3.3.40) and using the boundary conditions (3.3.35), we have
\[ \int [ (\text{grad } X) \uparrow B_4 (\text{grad } X) + X \uparrow C_3 (x) X ] \, dV \]

\[ = - Ra^2 \int_0^1 \Theta^* (D^2 - a^2) \Theta \, dz . \tag{3.3.41} \]

Using Schwartz inequality, we get

\[ \int [ (\text{grad } X) \uparrow B_4 (\text{grad } X) + X \uparrow C_3 X ] \, dV < Ra^2 \pi [ \int_0^1 | \Theta |^2 \, dz ]^{1/2} \]

\[ \cdot [ \int_0^1 | (D^2 - a^2) \Theta |^2 \, dz ]^{1/2} . \tag{3.3.42} \]

Now from equation (3.3.33), we have

\[ \frac{1}{\sigma} \int \Theta \Theta^* \, dz = \frac{1}{\sigma} [ (D^2 - a^2 - p) \Theta (D^2 - a^2 - p^*) \Theta^* ] \, dz \tag{3.3.43} \]

Integrating by parts and using the boundary conditions (3.3.35), we have

\[ \frac{1}{\sigma} \int |W|^2 \, dz = \frac{1}{\sigma} [ |(D^2 - a^2) \Theta|^2 + 2p_r (|D \Theta|^2 + a^2 | \Theta |^2) ] \]

\[ + |p|^2 | \Theta |^2 ] \, dz \tag{3.3.44} \]

Since \( p_r \gg 0, p_i \neq 0 \), equation (3.3.44) implies that

\[ \frac{1}{\sigma} \int |(D^2 - a^2) \Theta|^2 \, dz < \frac{1}{\sigma} \int |W|^2 \, dz \tag{3.3.45} \]

and

\[ \frac{1}{\sigma} \int | \Theta |^2 \, dz < \frac{1}{|p_r|} \int |W|^2 \, dz \tag{3.3.46} \]

Therefore from inequalities (3.3.42), (3.3.45) and (3.3.46), we have
\[
\int_{V} \left[ (\text{grad } X)^{\dagger} B_{4}(\text{grad } X) + x^{\dagger} C_{3}X \right] \, \text{d}V < \frac{Ra^{2} \mathcal{R}}{|p|} \int_{0}^{1} |W|^{2} \, \text{d}z = \int_{V} x^{\dagger} HX \, \text{d}V.
\]

Thus, with \( l = p_{1} \), the conditions of Theorem 4 are satisfied, therefore

\[
\frac{Ra^{2} \mathcal{R}}{|p|} > a^{2} \left[ \frac{1}{\sigma} - \frac{R_{s}}{|p|^{2}} + \frac{2 \mathcal{R} p_{r}}{\sigma} \right].
\]

Since \( p_{r} \geq 0 \), we have

\[
\frac{R_{s}}{|p|^{2}} + \frac{\mathcal{R} p_{r}}{|p|} - \frac{1}{\sigma} > 0,
\]

This implies that

\[
|p| < \frac{\mathcal{R} \sigma + \sqrt{\mathcal{R}^{2} \sigma^{2} + 4R_{s} \sigma}}{2}.
\]

This proves the theorem.

(e) For a visco-elastic fluid in presence of magnetic field:

The governing equations and boundary conditions of this problem under Boussinesq approximation when the visco-elastic fluid is described by Maxwell's constitutive relation are

\[
(D^{2} - a^{2}) \left[ D^{2} - a^{2} - \frac{p(1 + \mathcal{R} p)}{\sigma} \right] \Theta = Ra^{2} (1 + \mathcal{R} p) \Theta - R_{s} a^{2} (1 + \mathcal{R} p) \varnothing - Q(1 + \mathcal{R} p) D (D^{2} - a^{2}) h_{z},
\]

\[
(D^{2} - a^{2} - \nu) \Theta = -W, \quad (3.3.47)
\]

\[
(D^{2} - a^{2} - \frac{p}{\tau}) \varnothing = -\frac{W}{\tau}, \quad (3.3.48)
\]

\[
(D^{2} - a^{2}) \left[ D^{2} - a^{2} - \frac{p(1 + \mathcal{R} p)}{\tau} \right] \Theta = Ra^{2} (1 + \mathcal{R} p) \Theta - R_{s} a^{2} (1 + \mathcal{R} p) \varnothing - Q(1 + \mathcal{R} p) D (D^{2} - a^{2}) h_{z},
\]

\[
(D^{2} - a^{2} - \frac{p}{\tau}) \varnothing = -\frac{W}{\tau}, \quad (3.3.49)
\]
\[ (D^2 - a^2 - \frac{p \sigma_1}{\sigma}) h_z = -DW, \] (3.3.50)

and

\[ W = 0 = \Phi = \phi \text{ at } z = 0 \text{ and } z = 1, \]

and

either

\[ DW = 0 = h_z \text{ at } z = 0 \text{ and } z = 1 \]

(rigid perfectly conducting boundaries)

or

\[ D^2 W = 0 = h_z \text{ at } z = 0 \text{ and } z = 1 \]

(free perfectly conducting boundaries) \quad (3.3.51)

THEOREM 10: If \( (p, W, \Phi, \phi, h_z), \ p = p_r + ip_i, \ p_r \geq 0, \ p_i \neq 0 \), is a solution of equations (3.3.47)-(3.3.51), then

\[
|p|^2 < \text{ greater of } \frac{R \Gamma \sigma + \sqrt{R^2 \Gamma^2 \sigma^2 + 4 R s \sigma}}{2}, \ Q^2 \sigma^2].
\] (3.3.52)

PROOF: Since \( p_i \neq 0 \), we can write equations (3.3.47)-(3.3.50)
in the following convenient forms:

\[
(D^2 - a^2) [D^2 - a^2 - \frac{p(1 + \Gamma p)}{\sigma}] W - R a^2 (1 + \Gamma p) \Phi
+ R_s a^2 (1 + \Gamma p) \left[ \frac{W}{p} + \frac{\tau}{p} (D^2 - a^2) \Phi \right] + Q(1 + \Gamma p) D \left[ \frac{p \sigma_1}{\sigma} h_z - DW \right] = 0,
\] (3.3.53)

\[
- R a^2 (1 + \Gamma p^*) \left[ (D^2 - a^2 - p) \Phi + W \right] = 0, \quad (3.3.54)
\]

\[
\frac{\tau^2 R_s a^2}{p^*} (1 + \Gamma p^*) (D^2 - a^2) [(D^2 - a^2 - \frac{p}{\tau}) \Phi + \frac{W}{\tau}] = 0, \quad (3.3.55)
\]

\[
- \frac{Q \sigma_1}{\sigma} (1 + \Gamma p^*) [(D^2 - a^2 - \frac{p \sigma_1}{\sigma}) h_z + DW] = 0. \quad (3.3.56)
\]
Equation (2.2.37) reduces to the above equations with

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{\tau R_s a^2}{p} (1 + \eta p^*) & 0 \\
0 & 0 & 0 & 0 
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
\left[-\frac{2a^2 + \frac{p(1 + \eta p)}{\sigma}}{\sigma}\right] & 0 & \frac{\tau R_s a^2}{p} (1 + \eta p) & 0 \\
0 & \left[-R a^2 (1 + \eta p^*)\right] & 0 & 0 \\
\frac{\tau R_s a^2}{p} (1 + \eta p^*) & 0 & -\frac{2R_s a^2 [2a^2 (1 + \eta p^*)]}{p^*} & 0 \\
0 & 0 & 0 & -(1 + \eta p^*) \cdot \frac{Q \sigma_1}{\sigma} 
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
a^4 + \frac{p a^2 (1 + \eta p)}{\sigma} & \frac{-R a^2 (1 + \eta p)}{\sigma} & \frac{-\tau R_s a^4 (1 + \eta p)}{p} & 0 \\
\frac{R_s a^2 (1 + \eta p)}{p} & 0 & 0 & 0 \\
-\frac{R a^2 (1 + \eta p^*)}{\sigma} & \frac{R a^2 (1 + \eta p^*)}{\sigma} & 0 & 0 \\
(a^2 + p) & \frac{-\tau R_s a^4 (1 + \eta p^*)}{p^*} & 0 & \frac{\tau R_s a^4 [2a^2 (1 + \eta p^*)]}{p^*} \\
-\frac{p (1 + \eta p^*)}{\sigma} & 0 & 0 & \frac{Q \sigma_1}{\sigma} \\
\end{bmatrix}
\]
\[
E = \begin{bmatrix}
0 & 0 & 0 & \frac{Q \rho_1}{\sigma} \left(1 + \gamma p^*\right) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\frac{Q \rho_1}{\sigma} \left(1 + \gamma p^*\right) & 0 & 0 & 0
\end{bmatrix},
\]
and
\[
X(z) = \begin{bmatrix}
W(z) \\
\Theta(z) \\
\Phi(z) \\
h_z(z)
\end{bmatrix}.
\]

Further the boundary conditions on \(X\) conform to those of \(W, \Theta, \Phi\) and \(h_z\). Also \(A_1, B_1\) and \(C_1\) come out to be diagonal matrices with diagonal entries:

\[
dg(A_1) = p_1(0, 0, \frac{\tau^2_R \sigma a^2}{\|p\|^2}, 0)
\]

\[
dg(B_1) = -p_1(\frac{1}{\sigma} \gamma + \Theta p, -\Lambda a^2, \tau_2^2 R_s a^2 \left[2a^2 - \frac{2\gamma p^*}{\|p\|^2}\right] + \frac{1}{\tau} \left(\gamma^2 + \frac{2p^2}{\|p\|^2}\right), -\frac{Y \sigma_1}{\sigma}),
\]

\text{where } \gamma = 1 + 2\tau p^*;
\]

\[
dg(C_1) = -p_1(a^2 \left[\frac{R^2}{\|p\|^2} - \frac{1}{\sigma} - \frac{2[p^*]}{\sigma}\right], a^2 \left[R a^2 - R\right],
\]

\[
-\tau^2_R a^4 \left[\frac{a^2}{\|p\|^2} + \frac{1}{\tau} \left(\frac{2p^2}{\|p\|^2} + \gamma^2\right)\right], \frac{Q a^2 \sigma_1}{\sigma} \left[\gamma + \frac{\gamma^2}{a^2 \sigma}\right]).
\]

Now expressing \(B_1\) and \(C_1\) as

\[
B_1 = -p_1(B_3 - B_4),
\]
and \( C_1 = -p_1(C_3 - C_4) \),

where \( B_3, B_4, C_3 \) and \( C_4 \) are diagonal matrices with diagonal entries:

\[
dg(B_3) = \left( \frac{1}{\sigma} \gamma + Q^2, 0, \tau^2 R_s a^2 \left[ \frac{2a^2}{|p|^2} + \frac{1}{\tau} (\gamma^2 + \frac{2p^2}{|p|^2}) \right], 0 \right),
\]

\[
dg(B_4) = \left( 0, Ra^2 \gamma, 0, \frac{\gamma Q \sigma_1}{\sigma} \right),
\]

\[
dg(C_3) = \left( 0, Ra^2 \gamma, 0, \frac{Q a^2 \sigma_1}{\sigma} \right) \left\{ \gamma + \frac{Ra^2 \gamma^2}{a^2 \sigma} \right\},
\]

and

\[
dg(C_4) = \left( a^2 \left[ \frac{1}{\sigma} + \frac{2 \gamma}{\sigma} \frac{p}{|p|^2} - \frac{Ra^2}{|p|^2} \right], Ra^2, \tau^2 R_s a^2 \left[ \frac{a^2}{|p|^2} + \frac{1}{\gamma} \left( \frac{2p^2}{|p|^2} + \gamma^2 \right) \right], 0 \right).
\]

We show that the inequality (2.2.28) is satisfied if \( G \) and \( H \) are diagonal matrices with diagonal entries:

\[
dg(G) = \left( \frac{Q \gamma}{|p|^2} + Q \gamma, 0, 0, 0 \right),
\]

and \( \text{dg}(H) = \left( \frac{Ra^2 \gamma}{|p|^2}, 0, 0, 0 \right) \).

Now

\[
\int [(\text{grad } X)^\top B_4 (\text{grad } X) + X^\top C_3 X] \, dV = \frac{Q \gamma^2 \sigma_1^2}{\sigma^2} \int_0^1 |h_z|^2 \, dz + \frac{Q \sigma_1}{\sigma} \int_0^1 \left( |D\varphi|^2 + a^2 |\varphi|^2 \right) \, dz + \frac{Q \sigma_1}{\sigma} \gamma \int_0^1 \left( |Dh_z|^2 + a^2 |h_z|^2 \right) \, dz.
\]

Integrating by parts and using boundary conditions (3.3.51), we have
\[ V \left[ (\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 X \right] \, dV = \frac{Q \sigma_1^2}{\sigma^2} |p|^2 \int_0^1 |h_z|^2 \, dz \]

\[ - Ra^2 \int_0^1 (D^2 - a^2) \Theta^2 \, dz - \frac{Q \sigma_1}{\sigma} \int_0^1 h_z (D^2 - a^2) h_z^* \, dz \]

Using the Schwartz inequality, we have

\[ V \left[ (\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 X \right] \, dV \leq \frac{Q \sigma_1^2}{\sigma^2} |p|^2 \int_0^1 |h_z|^2 \, dz \]

\[ + Ra^2 \left[ \int_0^1 |\Theta|^2 \, dz \right]^{1/2} \left[ \int_0^1 (D^2 - a^2) \Theta^2 \, dz \right]^{1/2} \]

\[ + \frac{Q \sigma_1}{\sigma} \left[ \int_0^1 |h_z|^2 \, dz \right]^{1/2} \left[ \int_0^1 (D^2 - a^2) h_z^2 \, dz \right]^{1/2} \]

(3.3.57)

Since \( p_r \geq 0, p_\perp \neq 0 \), inequalities (3.2.19), (3.2.20), (3.3.45) and (3.3.46) hold here also and we have

\[ V \left[ (\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 X \right] \, dV \]

\[ \leq \frac{Ra^2}{|p|} \int_0^1 |W|^2 \, dz + \frac{Q}{|p|} \int_0^1 |DN|^2 \, dz + Q \int_0^1 \frac{1}{\sigma} \frac{1}{\sigma} \, dz \]

\[ = V \left[ (\text{grad } X)^\dagger G(\text{grad } X) + X^\dagger H X \right] \, dV \]

Thus, with \( l = p_\perp \), the conditions of Theorem 7 together with inequality (2.2.28) are satisfied and therefore from (2.2.29), we have

either \( \frac{R_s}{|p|^2} + \frac{R^n}{|p|} - \frac{1}{\sigma} > 0 \), or \( \frac{Q}{|p|} - \frac{1}{\sigma} > 0 \)

This implies that

\[ |p|^2 \leq \text{greater of } \left[ \frac{R^n}{} \sqrt{\frac{2}{\sigma} - 4R_s^2} \right], Q^2 \sigma^2 \]

This proves the theorem.
(f) For a visco-elastic fluid in presence of rotation:

The governing equations of this problem under Boussinesq approximation when the visco-elastic fluid is described by Maxwell's constitutive relation are

\[(D^2 - a^2) (D^2 - a^2 - \frac{p(1 + \gamma p)}{\sigma}) W = Ra^2(1 + \gamma p) \Theta\]

\[= - R_a a^2(1 + \gamma p) \Theta + T(1 + \gamma p) DZ , \quad (3.3.58)\]

\[(D^2 - a^2 - p) \Theta = - W , \quad (3.3.59)\]

\[(D^2 - a^2 - \frac{p}{\tau}) \Theta = - \frac{W}{\tau} , \quad (3.3.60)\]

\[\left[D^2 - a^2 - \frac{p(1 + \gamma p)}{\sigma}\right] Z = - (1 + \gamma p) DW , \quad (3.3.61)\]

and

\[W = 0 = \Theta = \Theta = DW = Z \text{ at } z = 0 \text{ and } z = 1 . \quad (3.3.62)\]

**THEOREM 11:** If \((p, W, \Theta, \Theta, Z)\), \(p = p_R + ip_i\), \(p_R \geq 0\), \(p_i \neq 0\) is a solution of equations (3.3.58)-(3.3.62), then

\[|p|^2 < \text{greater of } \left[\frac{R(\gamma \sigma + \sqrt{R^2 \gamma^2 \sigma^2 + 4R_s \sigma^2}}{2}, T\sigma^2\right]. \quad (3.3.63)\]

**PROOF:** Since \(p_i \neq 0\), equations (3.3.58)-(3.3.61) can be put in the following convenient forms:

\[(D^2 - a^2) \left[D^2 - a^2 - \frac{p(1 + \gamma p)}{\sigma}\right] W - Ra^2(1 + \gamma p) \Theta\]

\[+ R_s a^2(1 + \gamma p) \left[\frac{W}{p} + \frac{T}{p} (D^2 - a^2) \Theta\right] - T(1 + \gamma p) DZ = 0 , \quad (3.3.64)\]

\[- Ra^2(1 + \gamma p^*) \left[(D^2 - a^2 - p) \Theta + W\right] = 0 . \quad (3.3.65)\]
\[
\frac{\tau^2 R_s a^2}{p^*} (1 + \gamma_p^*) (D^2 - a^2) \left[ (D^2 - a^2 - \frac{B}{\tau}) \varnothing + \frac{W}{\tau} \right] = 0,
\]
(3.3.66)

\[
\frac{T(1 + \gamma p^*)}{(1 + \gamma p^*)} \left[ \left\{ \frac{D^2 - a^2 - \frac{p(1 + \gamma p)}{\sigma} - \tau^2 R_s a^2 (1 + \gamma p^* )}{p^*} \right\} \varnothing + (1 + \gamma p^*) DW \right] = 0.
\]
(3.3.67)

Equation (2.2.37) reduces to the above equations with

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\tau^2 R_s a^2 (1 + \gamma p^* )}{p^*} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
-\left[ 2 a^2 + p \left( \frac{1 + \gamma p^*}{\sigma} \right) \right] & 0 & \frac{\tau^2 R_s a^2 (1 + \gamma p^* )}{p^*} & 0 & 0 \\
0 & -R a^2 (1 + \gamma p^* ) & 0 & 0 & 0 \\
\frac{\tau R_s a^2 (1 + \gamma p^* )}{p^*} & 0 & -\frac{\tau^2 R_s a^2 [2 a^2 (1 + \gamma p^* )]}{p^*} & 0 & 0 \\
-\frac{p (1 + \gamma p^* )}{\gamma p^*} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{T (1 + \gamma p^* )}{(1 + \gamma p)}
\end{bmatrix}
\]
Further the boundary conditions on $X$ conform to those of $W$, $\Theta$, $\Phi$ and $Z$. Also $A_1$, $B_1$ and $C_1$ come out to be diagonal matrices with diagonal entries $\frac{a^2 + p}{\sigma}$.
\[ \text{dg}(A^1) = p^1(0, 0, \frac{\tau^2 R_s a^2}{|p|^2}, 0) \]

\[ \text{dg}(B^1) = -p^1\left(\frac{1}{\sigma} [1 + 2\gamma p^2], -Ra^2 \gamma, \frac{\tau^2 R_s a^2}{|p|^2} \left[\frac{2a^2}{2} + 1 \cdot \left(\frac{2p^2}{p^2} + \gamma^2\right)\right], \right. \]

\[ + \frac{1}{\tau} \left(\frac{2p^2}{p^2} + \gamma^2\right), \frac{2\gamma p^2}{(1 + \gamma p)^2} \left[1 + \gamma p^2\right] \right) , \]

\[ \text{dg}(C^1) = -p^1\left(\frac{a^2}{|p|^2} - \frac{1}{\sigma} - \frac{2\gamma p^2}{\sigma}, a^2 [Ra^2 \gamma - R], \right. \]

\[ - \tau^2 R_s a^4 \left[\frac{a^2}{|p|^2} + \frac{1}{\tau} \left(\frac{2p^2}{p^2} + \gamma^2\right)\right], \]

\[ \left. - T \left[\frac{2\gamma a^2 (1 + \gamma p^2)}{|1 + \gamma p|^2} - \frac{1}{\sigma} \right] \right) . \]

Now expressing \( B_1 \) and \( C_1 \) as

\[ B_1 = -p^1(B_3 - B_4) , \]

and \( C_1 = -p^1(C_3 - C_4) , \]

where \( B_3, B_4, C_3 \) and \( C_4 \) are diagonal matrices with diagonal entries:

\[ \text{dg}(B_3) = (\frac{1}{\sigma} [1 + 2\gamma p^2], 0, \frac{\tau^2 R_s a^2}{|p|^2} \left[\frac{2a^2}{2} + 1 \cdot \left(\frac{2p^2}{p^2} + \gamma^2\right)\right], \]

\[ \frac{2\gamma p^2}{(1 + \gamma p)^2} \left[1 + \gamma p^2\right] ) , \]

\[ \text{dg}(B_4) = (0, Ra^2 \gamma, 0, - \frac{2\gamma p^2}{(1 + \gamma p)^2} ) , \]

\[ \text{dg}(C_3) = (0, Ra^4 \gamma, 0, \frac{T^2}{\sigma} - \frac{2\gamma p^2}{|1 + \gamma p|^2} ) , \]

and
We show that the inequality (2.2.28) is satisfied if \( G \) and \( H \) are diagonal matrices with diagonal entries:

\[
\begin{align*}
\text{dg}(G) &= \left( \frac{T_\sigma}{|p|}, 0, 0, 0 \right), \\
\text{and} \quad \text{dg}(H) &= \left( \frac{\text{Ra}^2 r}{|p|}, 0, 0, 0 \right).
\end{align*}
\]

Now

\[
\int_V \left[ (\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 X \right] \, dv
= \text{Ra}^2 \left. \int_0^1 \left( |D\phi|^2 + a^2 |\phi|^2 \right) \, dz + \frac{T}{\sigma} \int_0^1 |Z|^2 \, dz \right.
- \frac{2T \text{Ra}^2 r}{|1 + r\sigma|} \int_0^1 \left( |DZ|^2 + a^2 |Z|^2 \right) \, dz . \quad (3.3.68)
\]

Integrating by parts the first term on the right hand side and using the boundary conditions (3.3.62), we get

\[
\int_V \left[ (\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 X \right] \, dv
= - \text{Ra}^2 \left. \int_0^1 \phi (D^2 - a^2) \phi^* \, dz + \frac{T}{\sigma} \int_0^1 |Z|^2 \, dz \right.
- \frac{2T \text{Ra}^2 r}{|1 + r\sigma|} \int_0^1 \left( |DZ|^2 + a^2 |Z|^2 \right) \, dz . \quad (3.3.69)
\]

Using Schwartz inequality, we get
\[
\int_V \left[ (\text{grad} \, X)^\top B_4 (\text{grad} \, X) + X^\top C_3 X \right] \, dV = \\
Ra^2 \tau \left[ \int_0^1 |\Theta|^2 \, dz \right]^{1/2} \left[ \int_0^1 |(D^2 - a^2)\Theta|^2 \, dz \right]^{1/2} \\
+ \frac{T}{\sigma} \int_0^1 |Z|^2 \, dz - \frac{2\tau}{|1 + \tau p|^2} \int_0^1 (|DZ|^2 + a^2 |Z|^2) \, dz.
\]

(3.3.70)

From equation (3.3.61), we have

\[
\int_0^1 (1 + \tau p) \, \text{DN}(1 + \tau p^*) \, \text{DN}^* \, dz \\
= \int_0^1 [D^2 - a^2 - \frac{p(1 + \tau p)}{\sigma}]Z \cdot [D^2 - a^2 - \frac{p^*(1 + \tau p^*)}{\sigma}] Z^* \, dz
\]

Integrating by parts and using the boundary conditions (3.3.62), we have

\[
\int_0^1 |\text{DN}|^2 \, dz = \frac{1}{|1 + \tau p|^2} \int_0^1 (D^2 - a^2)Z^2 \, dz + \frac{|p|^2}{\sigma^2} \int_0^1 |Z|^2 \, dz \\
+ \frac{2[p_x + \tau (p_x^2 - p_i^2)]}{\sigma |1 + \tau p|^2} \int_0^1 (|DZ|^2 + a^2 |Z|^2) \, dz,
\]

which implies that

\[
\frac{T}{|p|^2} \int_0^1 |Z|^2 \, dz - \frac{2\tau p_i^2 \tau}{|p|^2 |1 + \tau p|^2} \int_0^1 (|DZ|^2 + a^2 |Z|^2) \, dz \\
< \frac{T \sigma}{|p|^2} \int_0^1 |\text{DN}|^2 \, dz
\]

Since \( p_i^2 \leq |p|^2 \), therefore
Since $p_1 \geq 0$, $p_1 \neq 0$, inequalities (3.3.45) and (3.3.46) hold here also. Therefore using these and (3.3.71) in (3.3.70), we have

\[
\iint_V \left[ \left( \begin{array}{c} \nabla X \vphantom{\frac{1}{2}} \\ \nabla X \end{array} \right)^\top B_4 \left( \begin{array}{c} \nabla X \vphantom{\frac{1}{2}} \\ \nabla X \end{array} \right) + X^\top C_3 X \right] \, dV
\leq \frac{2\tau_\nu}{1 + \tau_\nu |p|^2} \iint_V \left| \nabla X \right|^2 \, dV + \frac{T_\sigma}{|p|^2} \iint_V \left| \nabla W \right|^2 \, dV
\]

Thus, with $l = p_1$, the conditions of Theorem 7 together with the inequality (2.2.28) are satisfied and therefore from (2.2.29), we have

\[
\text{either } \frac{R_s}{|p|^2} + \frac{R_\nu}{|p|^2} - \frac{1}{\sigma} > 0 \quad \text{or} \quad \frac{T_\sigma}{|p|^2} - \frac{1}{\sigma} > 0.
\]

This implies that

\[
|p|^2 < \text{greater of } \left[ \frac{R_\nu \sigma + \sqrt{R_\nu^2 - 2R_s^2 \sigma^2} + 4R_s \sigma^2}{2}, T_\sigma^2 \right].
\]

This proves the theorem.

(g) **Stern's thermohaline configuration:**

The governing equations and the boundary conditions for the present problem under Boussinesq approximation are given by equations (3.3.1)-(3.3.4) with $R = -\overline{R}$, $R_s = -\overline{R}_s$, where $\overline{R} > 0$, $\overline{R}_s > 0$. 
THEOREM 12: If \((p, \omega, \theta, \phi), p = p_r + i p_i, p_r \geq 0, p_i \neq 0,\)
is a solution of the governing equations and boundary conditions for the problem under consideration, then

\[ |p|^2 = \text{Re} \sigma. \] (3.3.72)

PROOF: Since \(p_i \neq 0,\) we can write the governing equations for the present problem in the following convenient forms:

\[
(D^2 - a^2)(D^2 - a^2 - \frac{p}{\sigma})W + \overline{Ra}^2 \left[ \frac{W}{p} + \frac{1}{p} (D^2 - a^2)\theta \right] - \overline{R_s} a^2 \phi = 0,
\] (3.3.73)

\[
\frac{\overline{Ra}^2}{p} (D^2 - a^2) \left[ (D^2 - a^2 - p)\theta + W \right] = 0,
\] (3.3.74)

\[- \tau \overline{R_s} a^2 \left[ (D^2 - a^2 - \frac{p}{\sigma})\phi + \frac{W}{\tau} \right] = 0.
\] (3.3.75)

Equation (2.1.1) reduces to the above equations with

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & \overline{Ra}^2 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
-(2a^2 + \frac{p}{\sigma}) & \overline{Ra}^2 & 0 \\
\overline{Ra}^2 & -(2a^2 + \frac{p}{\sigma}) & 0 \\
0 & 0 & - \tau \overline{R_s} a^2
\end{bmatrix},
\]
Further, the boundary conditions on $X$ conform to those of $W$, $\Theta$ and $\varnothing$. Also $A_1$, $B_1$ and $C_1$ come out to be diagonal matrices with diagonal entries

$$dg(A_1) = p_1(0, \frac{Ra^2}{p}, 0),$$

$$dg(B_1) = -p_1\left(\frac{1}{\sigma}, \frac{2Ra^2}{p^2} [a^2 + p_\tau], 0\right),$$

and

$$dg(C_1) = -p_1\left(a^2 \left[\frac{Ra}{|p|^2} - \frac{1}{\sigma}\right], -\frac{Ra^4}{|p|^2} (a^2 + 2p_\tau), -R_s a^2\right).$$

Now, with $l = p_1$, conditions of Theorem 1 are satisfied and therefore from (2.2.9), we have

$$|p|^2 < -R_\sigma.$$

This proves the theorem.
(h) In presence of magnetic field:

The governing equations and the boundary conditions for the present problem under Boussinesq approximation are given by equations (3.3.9)-(3.3.13) with $R = -\overline{R}$ and $R_s = -\overline{R}_s$, where $\overline{R} > 0$, $\overline{R}_s > 0$.

**THEOREM 13:** If $(p, W, \Theta, \varnothing, h_z)$, $p = p_r + ip_i$, $p_r \geq 0$, $p_i \neq 0$, is a solution of the governing equations and boundary conditions for the problem under consideration, then

$$|p|^2 \leq \text{greater of} (-R\sigma, \sigma^2 \sigma^2).$$

**PROOF:** Since $p_i \neq 0$, we can write the governing equations for the present problem in the following convenient forms:

$$\begin{align*}
(D^2 - a^2)(D^2 - a^2 - \frac{D}{\sigma})W + \overline{Ra}^2 \left[ \frac{W}{p} + \frac{1}{p} (D^2 - a^2)\Theta \right] - \overline{R}_s a^2 \Theta \\
+ Q \left[ \frac{p\sigma}{\sigma} Dz - D^2 W \right] &= 0, \quad (3.3.77) \\
\frac{\overline{Ra}^2}{p} (D^2 - a^2) \left[ (D^2 - a^2 - \frac{D}{\tau})\Theta + W \right] &= 0, \quad (3.3.78) \\
- \overline{R}_s a^2 \left[ (D^2 - a^2 - \frac{D}{\tau})\Theta + \frac{W}{\tau} \right] &= 0, \quad (3.3.79) \\
- \frac{Q\sigma}{\sigma} \left[ (D^2 - a^2 - \frac{p\sigma}{\sigma})h_z + Dw \right] &= 0. \quad (3.3.80)
\end{align*}$$

Equation (2.2.37) reduces to the above equations with

$$A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{\overline{Ra}^2}{p} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.$$
\[
B = \begin{bmatrix}
-(2a^2 + Q + \frac{p}{\sigma}) & \frac{Ra^2}{p} & 0 & 0 \\
\frac{Ra^2}{p} & - \frac{Ra^2}{p}(2a^2 + p) & 0 & 0 \\
0 & 0 & - \tau \bar{R}_s a^2 & 0 \\
0 & 0 & 0 & - \frac{Qp^* \sigma_1}{\sigma}
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
a^4 + \frac{pa^2}{\sigma} + \frac{Ra^2}{p} & - \frac{Ra^4}{p} & - \bar{R}_s a^2 & 0 \\
- \frac{Ra^4}{p} & \frac{Ra^4}{p}(a^2 + p) & 0 & 0 \\
- \bar{R}_s a^2 & 0 & \tau \bar{R}_s a^2 (a^2 + \frac{p}{\tau}) & 0 \\
0 & 0 & 0 & \frac{Qp^* \sigma_1}{\sigma} (a^2 + \frac{p \sigma_1}{\sigma})
\end{bmatrix},
\]

\[
E = \begin{bmatrix}
0 & 0 & 0 & \frac{Qp^* \sigma_1}{\sigma} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
- \frac{Qp^* \sigma_1}{\sigma} & 0 & 0 & 0
\end{bmatrix},
\]

and \[
X(z) = \begin{bmatrix}
W(z) \\
\Theta(z) \\
\varnothing(z) \\
h_z(z)
\end{bmatrix},
\]

Further the boundary conditions on \(X\) conform to those of \(W\), \(\Theta\), \(\varnothing\) and \(h_z\). Also \(A_1\), \(B_1\) and \(C_1\) come out to be diagonal matrices with diagonal entries:
\[ \text{dg}(A_1) = p_1(0, \frac{Ra^2}{|p|^2}, 0, 0), \]
\[ \text{dg}(B_1) = -p_1\left(\frac{1}{\sigma}, \frac{2Ra^2}{|p|^2}|a^2 + pr|, 0, -\frac{Q\sigma_1}{\sigma}\right), \]
and
\[ \text{dg}(C_1) = -p_1(a^2 \left[ \frac{R}{|p|^2} - \frac{1}{\sigma} \right], -\frac{Ra^4}{|p|^2}(a^2 + 2pr), -Rs^2a^2, \frac{Qa^2\sigma_1}{\sigma}). \]

Now expressing \( B_1 \) and \( C_1 \) as
\[ B_1 = -p_1(B_3 - B_4), \]
\[ C_1 = -p_1(C_3 - C_4), \]
where \( B_3, B_4, C_3 \) and \( C_4 \) are diagonal matrices with diagonal entries
\[ \text{dg}(B_3) = \left(\frac{1}{\sigma}, \frac{2Ra^2}{|p|^2}|a^2 + pr|, 0, 0\right), \]
\[ \text{dg}(B_4) = \left(0, 0, 0, \frac{Q\sigma_1}{\sigma}\right), \]
\[ \text{dg}(C_3) = \left(0, 0, 0, \frac{Qa^2\sigma_1}{\sigma}\right), \]
and
\[ \text{dg}(C_4) = \left(-a^2 \left[ \frac{R}{|p|^2} - \frac{1}{\sigma} \right], \frac{Ra^4}{|p|^2}(a^2 + 2pr), -Rs^2a^2, 0\right). \]

Proceeding exactly as in Theorem 7 of the present chapter, we see that the conditions of Theorem 7 together with inequality (2.2.28) are satisfied and therefore from (2.2.29), we have
\[ \text{either } \frac{R}{|p|^2} - \frac{1}{\sigma} > 0 \text{ or } \frac{Q}{|p|} - \frac{1}{\sigma} > 0. \]
which gives
\[ |p|^2 < \text{greater of } (-R\sigma, Q\sigma^2). \]

This proves the theorem.

(i) **In presence of rotation:**

The governing equations and the boundary conditions for the present problem under Boussinesq approximation are given by equations (3.3.20)-(3.3.25) with \( R = -\overline{R}, \ R_s = -\overline{R}_s, \)
where \( \overline{R} > 0, \ R_s > 0. \)

**THEOREM 14:** If \((p, W, \Theta, \phi, \eta), \ p = p_r + ip_i, \ p_r \geq 0, \)
p\(i \neq 0,\) is a solution of the governing equations and boundary conditions for the problem under consideration, then
\[ |p|^2 < \text{greater of } (-R\sigma, T\sigma^2). \]  \((3.3.81)\)

**PROOF:** Since \( p_i \neq 0, \) we can write the governing equations for the present problem in the following convenient forms:
\[
(D^2 - a^2)(D^2 - a^2 - \frac{p}{\sigma})W + \overline{Ra}^2 \left[ \frac{W}{p} + \frac{1}{p}(D^2 - a^2)\Theta \right] - \overline{R}_s a^2 \phi - TDZ = 0, \]  \((3.3.82)\)
\[
\frac{\overline{Ra}^2}{p} (D^2 - a^2) \left[ (D^2 - a^2 - p)\Theta + W \right] = 0, \]  \((3.3.83)\)
\[
- \tau \overline{R}_s a^2 \left[ (D^2 - a^2 - \frac{p}{\sigma})\phi + \frac{W}{\tau} \right] = 0, \]  \((3.3.84)\)
\[
T \left[ (D^2 - a^2 - \frac{p}{\sigma})Z + DN \right] = 0. \]  \((3.3.85)\)

Equation (2.2.37) reduces to the above equations with
\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{\overline{Ra}^2}{p} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
-(2a^2 + \frac{\rho}{\sigma}) & \frac{\overline{Ra}^2}{p} & 0 & 0 \\
\frac{\overline{Ra}^2}{p} & -(2a^2 + \frac{\rho}{\sigma}) & 0 & 0 \\
0 & 0 & -\tau\overline{R}a^2 & 0 \\
0 & 0 & 0 & \tau\overline{R}a^2
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
\frac{a^4 + \frac{\rho a^2}{\sigma} + \frac{\overline{Ra}^2}{p}}{p} & -\frac{\overline{Ra}^4}{p} & -\overline{R}a^2 & 0 \\
-\frac{\overline{Ra}^4}{p} & \frac{\overline{Ra}^4}{p} & 0 & 0 \\
-\tau\overline{R}a^2 & 0 & \tau\overline{R}a^2 & 0 \\
0 & 0 & 0 & \tau\overline{R}a^2 + \frac{\rho}{\sigma}
\end{bmatrix}
\]

\[
E = \begin{bmatrix}
0 & 0 & 0 & -\tau \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\tau & 0 & 0 & 0
\end{bmatrix}
\]

and \(X(z) = \begin{bmatrix} W(z) \\ \Theta(z) \\ \varnothing(z) \\ Z(z) \end{bmatrix}\)
Further the boundary conditions on X conform to those of W, \( \emptyset \), \( \emptyset \) and Z. Also \( A_1 \), \( B_1 \) and \( C_1 \) come out to be diagonal matrices with diagonal entries

\[
dg(A_1) = p_1(0, \frac{Ra^2}{|p|^2}, 0, 0),
\]

\[
dg(B_1) = -p_1\left(\frac{1}{\sigma}, \frac{2Ra^2}{|p|^2}, [a^2 + p_r], 0, 0\right),
\]

and

\[
dg(C_1) = -p_1(a^2 - \frac{\bar{R}}{|p|^2}, -\frac{1}{\sigma}), -\frac{Ra^4}{|p|^2}(a^2 + 2p_r), -\bar{R}s a^2, \frac{T}{\sigma}).
\]

Now expressing

\[
C_1 = -p_1(C_3 - C_4),
\]

where \( C_3 \) and \( C_4 \) are diagonal matrices with diagonal entries

\[
dg(C_3) = (0, 0, 0, \frac{T}{\sigma}),
\]

\[
dg(C_4) = (a^2 - \frac{1}{\sigma}, \frac{\bar{R}}{|p|^2}, \frac{Ra^4}{|p|^2}(a^2 + 2p_r), \bar{R}s a^2, 0).\]

Proceeding exactly as in Theorem 8 of the present chapter, we see that the conditions of Theorem 7 together with inequality (2.2.28) are satisfied and therefore from (2.2.29), we have

either \( \frac{\bar{R}}{|p|^2} - \frac{1}{\sigma} > 0 \) or \( \frac{T\sigma}{|p|^2} - \frac{1}{\sigma} > 0 \)

which gives

\[ |p|^2 < \text{greater of } (-\bar{R}\sigma, T\sigma^2). \]

This proves the theorem.
(j) **For a visco-elastic fluid:**

The governing equations and the boundary conditions for the present problem when the visco-elastic fluid is described by Maxwell's constitutive relation are given by equations (3.3.32)-(3.3.35) with $R = -\overline{R}$, $R_s = -\overline{R_s}$, where $\overline{R} > 0$, $\overline{R_s} > 0$.

**THEOREM 15:** If $(p, W, \theta, \varnothing), p = p_r + ip_i, p_r \geq 0, p_i \neq 0,$

is a solution of the governing equations and boundary conditions for the problem under consideration, then

$$│p│^2 < \frac{-\overline{R_s} \gamma \sigma + \sqrt{\overline{R_s}^2 \gamma^2 \sigma^2 - 4R \sigma^2}}{2}$$

(3.3.86)

**PROOF:** Since $p_i \neq 0$, we can write the governing equations for the present problem in the following convenient forms:

$$\begin{align*}
(D^2 - a^2) [D^2 - a^2 - \frac{\rho(1 + \gamma p)}{\sigma}] W + \overline{R} a^2 (1 + \gamma p) \left[ \frac{W}{p} \right. \\
+ \frac{1}{p} (D^2 - a^2) \theta] - \overline{R_s} a^2 (1 + \gamma p) \varnothing = 0 ,
\end{align*}$$

(3.3.87)

$$\overline{R} a^2 (1 + \gamma p^* \frac{p}{p}) (D^2 - a^2) [(D^2 - a^2 - p) \theta + W] = 0 ,$$

(3.3.88)

$$- \tau \overline{R_s} a^2 (1 + \gamma p^*) [(D^2 - a^2 - \frac{D}{\tau}) \varnothing + \frac{W}{\tau}] = 0 ,$$

(3.3.89)

Equation (2.1.1) reduces to the above equations with

$$A = \begin{bmatrix}
1 & 0 & 0 \\
0 & \overline{R} a^2 (1 + \gamma p^*) \frac{p}{p} & 0 \\
0 & 0 & 0
\end{bmatrix} ,$$
\[
\begin{bmatrix}
-\frac{2a^2 + \frac{p(1+\gamma p)}{\sigma}}{\sigma} & \frac{-Ra^2(1+\gamma p)}{p} & 0 \\
\frac{-Ra^2(1+\gamma p)}{p} & -\frac{Ra^2(1+\gamma p)}{p} & -\frac{2a^2 + p}{p} \\
0 & 0 & -\frac{\tau Ra^2(1+\gamma p)}{p} \\
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
\frac{a^4 + \frac{pa^2(1+\gamma p)}{\sigma}}{\sigma} & -\frac{Ra^4(1+\gamma p)}{p} & -\frac{Ra^2(1+\gamma p)}{p} \\
\frac{-Ra^4(1+\gamma p)}{p} & -\frac{Ra^4(1+\gamma p)}{p} & 0 \\
\frac{-Ra^2(1+\gamma p)}{p} & \frac{-Ra^2(1+\gamma p)}{p} & \frac{(a^2 + p)}{\tau} \\
\end{bmatrix}
\]

and \( X(z) = \begin{bmatrix} W(z) \\ \Theta(z) \\ \Phi(z) \end{bmatrix} \).

Further the boundary conditions on \( X \) conform to those of \( W, \Theta \) and \( \Phi \). Also \( A_1, B_1 \) and \( C_1 \) come out to be diagonal matrices with diagonal entries

\[
dg(A_1) = p_1(0, \frac{-Ra^2}{|p|}, 0),
\]

\[
dg(B_1) = -p_1\left(\frac{1}{\sigma} (1 + 2\gamma p), \frac{-Ra^2}{|p|} + \gamma + \frac{2p}{|p|^2}, 0 \right),
\]

\[-\tau Ra^2 \gamma \).
Now expressing $B_1$ and $C_1$ as

$$B_1 = - p_1 (B_3 - B_4),$$

and

$$C_1 = - p_1 (C_3 - C_4),$$

where $B_3$, $B_4$, $C_3$ and $C_4$ are diagonal matrices with diagonal entries

$$dG(B_3) = \left( \frac{1}{\sigma} [1 + 2 \tau \rho R], \frac{2a^2}{|p|^2} + \tau R + \frac{2pR}{|p|^2}, 0 \right),$$

$$dG(B_4) = (0, 0, \tau \rho R a^2),$$

$$dG(C_3) = (0, 0, \tau \rho R a^2),$$

and

$$dG(C_4) = \left( a^2 \frac{1}{\sigma} + \frac{2\tau \rho R}{|p|^2} - \frac{R}{|p|^2}, \frac{2a^2}{|p|^2} + \tau R + \frac{2pR}{|p|^2}, \frac{2a^2}{|p|^2} \right).$$

we show that the inequality (2.2.22) is satisfied if $H$ is a diagonal matrix with diagonal entries

$$dG(H) = \left( \frac{\tau \rho R a^2}{|p|^2}, 0, 0 \right).$$

Now

$$\int \left[ (\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 (X) X \right] dV = \tau \rho R a^2 \int_0^1 [\| \partial \|^2 + a^2 \| \partial \|^2] dz$$

(3.3.90)
Integrating by parts the right hand side of equation (3.3.90) and using the boundary conditions (3.3.35), we have

\[ \int [(\text{grad } X)^\dagger \text{ B}_4(\text{grad } X) + X^\dagger C_3 X] \, dv = -\tau R_3 a^2 \int_0^1 \phi \left( D^2 - a^2 \right) \phi \, dz \]  

(3.3.91)

Using Schwartz inequality, we get

\[ \int [(\text{grad } X)^\dagger \text{ B}_4(\text{grad } X) + X^\dagger C_3 X] \, dv = \leq \tau R_3 a^2 \left[ \int_0^1 |\phi|^2 \, dz \right]^{1/2} \left[ \int_0^1 \left| (D^2 - a^2)\phi \right|^2 \, dz \right]^{1/2} \]  

(3.3.92)

Now from equation (3.3.34), we have

\[ \frac{1}{\tau^2} \int_0^1 |W|^2 \, dz = \int_0^1 \left| (D^2 - a^2)\phi \right|^2 \, dz + \frac{2p}{\tau} \int_0^1 \left[ |D\phi|^2 + a^2 |\phi|^2 \right] \, dz + \frac{1}{\tau^2} \int_0^1 |\phi|^2 \, dz . \]  

(3.3.93)

Since \( p \gg 0 \), \( p_1 \neq 0 \), equation (3.3.93) implies that

\[ \int_0^1 \left| (D^2 - a^2)\phi \right|^2 \, dz < \frac{1}{\tau^2} \int_0^1 |W|^2 \, dz , \]  

(3.3.94)

and

\[ \int_0^1 |\phi|^2 \, dz < \frac{1}{|p|^2} \int_0^1 |W|^2 \, dz . \]  

(3.3.95)

Therefore from inequalities (3.3.92), (3.3.94) and (3.3.95), we have

\[ \int [(\text{grad } X)^\dagger \text{ B}_4(\text{grad } X) + X^\dagger C_3 X] \, dv < \frac{\tau R_3 a^2}{|p|} \int_0^1 |W|^2 \, dz = \int X^\dagger H X \, dv . \]
Thus with $l = p_1$, the conditions of Theorem 4 are satisfied, therefore

$$\frac{\bar{R}_s \gamma a^2}{|p|} > a^2 \left[ \frac{1}{\sigma} + \frac{2 \gamma \rho_R}{\sigma} - \frac{\bar{R}}{|p|^2} \right].$$

Since $p_r \geq 0$, we have

$$\frac{\bar{R}}{|p|^2} + \frac{\bar{R}_s \gamma'}{|p|} - \frac{1}{\sigma} > 0,$$

which implies that

$$|p| < \frac{-\bar{R}_s \gamma \sigma + \sqrt{\bar{R}^2 \gamma^2 \sigma^2 - 4R\sigma}}{2}.$$

This proves the theorem.

(k) For a visco-elastic fluid in presence of magnetic field:

The governing equations and the boundary conditions for the present problem when the visco-elastic fluid is described by Maxwell's constitutive relation are given by equations (3.3.47)-(3.3.51) with $R = -\bar{R}$, $R_s = -\bar{R}_s$, where $\bar{R} > 0, \bar{R}_s > 0$.

THEOREM 16: If $(p, W, \Theta, \Phi, h_z), p = p_r + ip_i, p_r \geq 0, p_i \neq 0$, is a solution of the governing equations and boundary conditions for the problem under consideration, then

$$|p|^2 < \text{greater of } \left[ \left( \frac{-\bar{R}_s \gamma \sigma + \sqrt{\bar{R}^2 \gamma^2 \sigma^2 - 4R\sigma}}{2} \right), \Omega^2 \sigma^2 \right].$$

(3.3.96)
PROOF: Since \( p_1 \neq 0 \), we can write the governing equations for the present problem in the following convenient forms:

\[
(D^2 - a^2) \left[ (D^2 - a^2 - \frac{p(1 + \gamma p)}{\sigma}) \right] W + \frac{Ra}{p}(D^2 - a^2) \left[ \frac{W}{p} \right] \\
+ \frac{1}{p}(D^2 - a^2) \left[ \frac{\sigma}{\sigma} - \frac{p(1 + \gamma p)}{\sigma} \right] - \frac{Ra}{p} \left[ \frac{W}{p} \right] = 0,
\]

\[
(D^2 - a^2) \left[ (D^2 - a^2 - \frac{p(1 + \gamma p)}{\sigma}) \right] \left( \frac{W}{p} \right) = 0,
\]

\[
\left[ \frac{Ra}{p}(1 + \gamma p^*) \right] \left( D^2 - a^2 - \frac{p(1 + \gamma p)}{\sigma} \right) W + \left( \frac{W}{p} \right) = 0
\]

\[
\left[ \frac{-\sigma}{\sigma} \right] \left( 1 + \gamma p^* \right) \left[ (D^2 - a^2 - \frac{p(1 + \gamma p)}{\sigma}) \right] h_z + D \left( \frac{W}{p} \right) = 0.
\]

Equation (2.2.37) reduces to the above equations with

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{-Ra}{p}(1 + \gamma p^*) & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
-\frac{-Ra}{p}(1 + \gamma p^*) & \frac{-Ra}{p}(1 + \gamma p^*) & 0 & 0 \\
\frac{-Ra}{p}(1 + \gamma p^*) & \frac{-Ra}{p}(1 + \gamma p^*) & 0 & 0 \\
0 & 0 & \frac{-\sigma}{\sigma} \left( 1 + \gamma p^* \right) & 0 \\
0 & 0 & 0 & \frac{-\sigma}{\sigma} \left( 1 + \gamma p^* \right)
\end{bmatrix}
\]
\[ C = \begin{pmatrix}
 a^4 + \frac{p a^2}{(1 + \gamma p)} & -\frac{Ra^4}{p} & -Ra^2(1 + \gamma p) & 0 \\
 + \frac{Ra^2}{p} & -\frac{Ra^4}{p} & -Ra^2(1 + \gamma p) & 0 \\
 -\frac{Ra^4}{p} & -\frac{Ra^4}{p} & 0 & 0 \\
 (a^2 + p) & (a^2 + p) & 0 & \frac{Qp * \sigma_1}{\sigma} & 0 \\
 -Ra^2(1 + \gamma p) & 0 & \tau Ra^2(1 + \gamma p) & 0 & (a^2 + p) \\
 0 & 0 & 0 & 0 & \frac{Qp * \sigma_1}{\sigma} (1 + \gamma p)
\end{pmatrix} \]

Further the boundary conditions on \( X \) conform to those of \( W, \Phi, \Theta \) and \( h_2 \). Also \( A_1, B_1 \) and \( C_1 \) come out to be diagonal matrices with diagonal entries:
\[ \text{dg}(A) = p_1(0, \frac{\bar{R}_a^2}{|p|}, 0, 0), \]
\[ \text{dg}(B) = -p_1\left(\frac{1}{\sigma}\gamma + \psi \eta\right), \quad \bar{R}_a^2 \left[ \frac{2a^2}{|p|^2} + \gamma + \frac{2p_r}{|p|^2} \right], \]
\[ \quad - \tau \bar{R}_s a^2, \quad \frac{\gamma \sigma_1}{\sigma} \right), \]

where \( \gamma = (1 + 2 \sigma \eta \rho) \); and

\[ \text{dg}(C) = -p_1\left(\frac{\bar{R}}{|p|^2} - \frac{1}{\sigma} - \frac{2\eta p_r a^2}{\bar{s}^2}, -\bar{R} a^4 \left[ \gamma + \frac{a^2 + 2p_r}{|p|^2} \right]\right), \]
\[ \tau \bar{R}_s a^2 \left[ \gamma \sigma_1 \left( \gamma + \frac{\gamma |p|^2 \sigma_1}{a^2 \sigma} \right) \right). \]

Now expressing \( B \) and \( C \) as

\[ B = -p_1(B_3 - B_4), \]
and \[ C = -p_1(C_3 - C_4), \]

where \( B_3, B_4, C_3 \) and \( C_4 \) are diagonal matrices with diagonal entries

\[ \text{dg}(B_3) = \left( \frac{1}{\sigma} \left[ 1 + 2 \sigma \eta \rho \right], \bar{R}_a^2 \left[ \frac{2a^2}{|p|^2} + \gamma + \frac{2p_r}{|p|^2} \right], 0, 0 \right), \]
\[ \text{dg}(B_4) = \left( 0, 0, \tau \bar{R}_s a^2, \frac{\gamma \sigma_1}{\sigma} \right), \]
\[ \text{dg}(C_3) = \left( 0, 0, \tau \bar{R}_s a^4, \frac{\gamma \sigma_1}{\sigma} \left[ \gamma + \frac{\gamma |p|^2 \sigma_1}{a^2 \sigma} \right] \right), \]

and

\[ \text{dg}(C_4) = \left( a^2 \frac{1}{\sigma} + \frac{2\eta p_r a^2}{\bar{s}^2}, -\bar{R}_a^4 \left[ \gamma + \frac{a^2 + 2p_r}{|p|^2} \right], \bar{R}_s a^2, 0 \right), \]
We show that the inequality (2.2.28) is satisfied if G and H are diagonal matrices with diagonal entries:

$$dg(G) = \left( \frac{QY}{|P|} + Qr, 0, 0, 0 \right),$$

and $$dg(H) = \left( \frac{R_s a^2}{|P|}, 0, 0, 0 \right).$$

Now

$$\int \left[ (\text{grad } X)^\top B_4(\text{grad } X) + X^\top C_3 X \right] dV = \frac{Q \pi |P|^2 \sigma_1^2}{\sigma^2} \int |h_2| dz$$

$$+ \tau \frac{R_s a^2}{\sigma} \left[ \int \|D\|^2 + a^2 \|\varphi\|^2 \right] dz + \frac{Q \sigma_1^2}{\sigma} \int \left[ \|Dh_2\|^2 + a^2 \|h_2\|^2 \right] dz.$$

Integrating by parts and using the boundary conditions (3.3.51), we have

$$\int \left[ (\text{grad } X)^\top B_4(\text{grad } X) + X^\top C_3 X \right] dV = \frac{Q \pi |P|^2 \sigma_1^2}{\sigma^2} \int |h_2| dz$$

$$- \tau \frac{R_s a^2}{\sigma} \int \varphi (D^2 - a^2) \varphi dz - \frac{Q \sigma_1^2}{\sigma} \int h_2 (D^2 - a^2) h_2^* dz.$$

Using Schwartz inequality, we have

$$\int \left[ (\text{grad } X)^\top B_4(\text{grad } X) + X^\top C_3 X \right] dV \leq \frac{Q \pi |P|^2 \sigma_1^2}{\sigma^2} \int |h_2| dz$$

$$+ \tau \frac{R_s a^2}{\sigma} \left[ \int \|\varphi\|^2 dz \right]^{1/2} \left[ \int \|D^2 - a^2\| \varphi \|^2 dz \right]^{1/2}$$

$$+ \frac{Q \sigma_1^2}{\sigma} \left[ \int \|h_2\|^2 dz \right]^{1/2} \left[ \int \|(D^2 - a^2)h_2\|^2 dz \right]^{1/2}.$$ 

(3.3.101)

Since $$p_\tau \gg 0, p_1 \neq 0$$, inequalities (3.3.94), (3.3.95), (3.2.19) and (3.2.20) hold here also and therefore we have
\[ \int_V \left[ (\nabla X)^\dagger B_4 (\nabla X) + X^\dagger C_3 X \right] dV < \frac{P \overline{R}_s a^2}{|P|} \int_0^1 |W|^2 dz + \frac{Q}{|P|} \int_0^1 |D|W|^2 dz + Q \int_0^1 |D|W|^2 dz \]
\[ = \int_V \left[ (\nabla X)^\dagger G(\nabla X) + X^\dagger HX \right] dV \]

Thus, with \( l = p_i \) the conditions of Theorem 7 together with inequality (2.2.28) are satisfied and therefore from (2.2.29), we have

either \( \frac{\overline{R}}{|P|} + \frac{\overline{R}_s l^2}{|P|} - \frac{1}{\sigma} > 0 \) or \( \frac{Q}{|P|} - \frac{1}{\sigma} > 0 \),

which implies that

\[ |P|^2 < \text{greater of } \left[ \frac{-R_s l^2 \sigma + \sqrt{R_s^2 l^2 \sigma^2 - 4R \sigma \sigma_2}}{2} \right], Q^2 \sigma^2 \].

This proves the theorem.

(1) **For a visco-elastic fluid in presence of rotation:**

The governing equations and the boundary conditions for the present problem when the visco-elastic fluid is described by Maxwell's constitutive relation are given by equations (3.3.58)-(3.3.62) with \( R = -\overline{R}, R_s = -\overline{R}_s \), where \( \overline{R} > 0, \overline{R}_s > 0 \).

**THEOREM 17:** If \( (p, \psi, \theta, \varphi, z), p = p_r + ip_i, p_r \geq 0, p_i \neq 0 \) is a solution of the governing equations and boundary conditions for the present problem, then

\[ |P|^2 < \text{greater of } \left[ \frac{-R_s l^2 \sigma + \sqrt{R_s^2 l^2 \sigma^2 - 4R \sigma \sigma_2}}{2} \right], Q^2 \sigma^2 \].

(3.3.102)
PROOF: Since \( p \neq 0 \), we can write the governing equations for the present problem in the following convenient forms:

\[
(D^2 - a^2) \left[ D^2 - a^2 - \frac{p(1 + \Gamma^* p)}{\sigma} \right] W + \frac{Ra^2}{p} (1 + \Gamma^* p) \left[ \frac{W}{p} \right] + \frac{1}{p} (D^2 - a^2) \Phi - \frac{\tau R_s a^2}{(1 + \Gamma^* p)} \Phi - T(1 + \Gamma^* p) DZ = 0 ,
\]

\[
(3.3.103)
\]

\[
\frac{Ra^2}{p} \left( D^2 - a^2 \right) \left[ (D^2 - a^2 - p) \Phi + W \right] = 0 ,
\]

\[
(3.3.104)
\]

\[
-\frac{\tau R_s a^2}{p} \left( 1 + \Gamma^* p \right) \left[ (D^2 - a^2 - \frac{p}{T}) \Phi + \frac{W}{T} \right] = 0 ,
\]

\[
(3.3.105)
\]

\[
\frac{T}{(1 + \Gamma^* p)} \left[ (D^2 - a^2 - \frac{p(1 + \Gamma^* p)}{\sigma}) Z + (1 + \Gamma^* p) D W \right] = 0 .
\]

\[
(3.3.106)
\]

Equation (2.2.37) reduces to the above equations with

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{Ra^2(1 + \Gamma^* p)^*}{p} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
-\left[ 2a^2 + \frac{p(1 + \Gamma^* p)}{\sigma} \right] \frac{Ra^2(1 + \Gamma^* p)^*}{p} & 0 & 0 \\
\frac{Ra^2(1 + \Gamma^* p)^*}{p} & -\frac{Ra^2(1 + \Gamma^* p)^*}{p} & 0 & 0 \\
0 & 0 & \frac{T}{(1 + \Gamma^* p)} & 0 \\
0 & 0 & 0 & \frac{T}{(1 + \Gamma^* p)}
\end{bmatrix}
\]
Further the boundary conditions on $X$ conform to those of $W$, $\Theta$, $\emptyset$ and $Z$. Also $A_1$, $B_1$ and $C_1$ come out to be diagonal matrices with diagonal entries
\[
\text{dg}(A_1) = p_1(0, \frac{Ra^2}{|p|^2}, 0, 0),
\]
\[
\text{dg}(B_1) = -p_1\left(\frac{1}{\sigma} [1 + 2\eta p_x], \frac{Ra^2}{|p|^2} + \eta' + \frac{2p_x}{|p|^2}\right),
\]
\[
- tRs a^2 \eta', \frac{2\eta'}{|(1 + \eta p)|^2} \left(1 + |\eta p_x|\right),
\]
\[
\text{dg}(C_1) = -p_1\left(\frac{R}{|p|} - \frac{1}{\sigma} - \frac{2\eta p_x}{\sigma}, - Ra^4\eta' + \frac{(a^2 + 2p_x)}{|p|^2}\right),
\]
\[
-tRs a^2 [\eta a^2 - \frac{1}{\eta'}], - T\left[\frac{2\eta^2 a^2 (1 + \eta p_x)}{|(1 + \eta p)|^2} - \frac{1}{\sigma}\right].
\]

Now expressing \(B_1\) and \(C_1\) as

\[
B_1 = - p_1(B_3 - B_4),
\]

and \(C_1 = - p_1(C_3 - C_4),\)

where \(B_3, B_4, C_3\) and \(C_4\) are diagonal matrices with diagonal entries:

\[
\text{dg}(B_3) = \left(\frac{1}{\sigma} [1 + 2\eta p_x], \frac{Ra^2}{|p|^2} + \eta' + \frac{2p_x}{|p|^2}\right), 0,
\]
\[
\frac{2\eta^2 p_x}{|(1 + \eta p)|^2},
\]
\[
\text{dg}(B_4) = (0, 0, \eta tRs a^2, \frac{-2\eta'}{|(1 + \eta p)|^2}),
\]
\[
\text{dg}(C_3) = (0, 0, \eta tRs a^4, \left[\frac{T}{\sigma} - \frac{2\eta tRs a^2}{|(1 + \eta p)|^2}\right]),
\]

and
\[
\begin{align*}
dg(C_4) &= \left(a^2 \left[ \frac{1}{\sigma} + \frac{2\Gamma \rho \Gamma}{\sigma} - \frac{\bar{R}}{|p|^2} \right], \quad \bar{R} a^4 \left[ \Gamma + \frac{(a^2 + 2p \Gamma)}{|p|^2} \right], \quad \bar{R} a^2 \frac{2T \Gamma^2 a^2 \rho \Gamma}{(1 + \Gamma p)^2} \right),
\end{align*}
\]

We show that the inequality (2.2.28) is satisfied if \( G \) and \( H \) are diagonal matrices with diagonal entries:

\[
\begin{align*}
dg(G) &= \left( \frac{T \sigma}{|p|^2}, 0, 0, 0 \right),
\end{align*}
\]
and

\[
\begin{align*}
dg(H) &= \left( \bar{R} a, 0, 0, 0 \right).
\end{align*}
\]

Now,

\[
\begin{align*}
\int \left[ [(\text{grad} \times) + B_4(\text{grad} X) + X^t C_3 X] \right] dv = \tau \Gamma \bar{R} a^2 \frac{1}{|p|^2} \int |\text{grad} \times|^2 + a^2 |\text{grad} \times|^2 dz + \frac{T}{\sigma} \int |Z|^2 dz = \frac{2T \Gamma}{(1 + \Gamma p)^2} \int |DZ|^2 + a^2 |Z|^2 dz.
\end{align*}
\]

Integrating by parts the first term on the right hand side and using the boundary conditions (3.3.62), we get

\[
\begin{align*}
\int \left[ [(\text{grad} \times) + B_4(\text{grad} X) + X^t C_3 X] \right] dv &=
\end{align*}
\]

Using Schwartz inequality we get

\[
\begin{align*}
\int \left[ [(\text{grad} \times) + B_4(\text{grad} X) + X^t C_3 X] \right] dv \leq \tau \Gamma \bar{R} a^2 \frac{1}{|p|^2} \int |\text{grad} \times|^2 + a^2 |Z|^2 dz. \quad (3.3.107)
\end{align*}
\]
Since $p_r \geq 0$, $p_1 \neq 0$, inequalities (3.3.94), (3.3.95) and (3.3.71) hold here also and therefore we have

$$
\int [(\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 X] \, dV
$$

$$
< \frac{R}{|p|} \int_0^1 |W|^2 \, dz + \frac{T_0}{|p|^2} \int_0^1 |D^2 W|^2 \, dz
$$

$$
= \int [(\text{grad } X)^\dagger G (\text{grad } X) + X^\dagger HX] \, dV .
$$

Thus with $l = p_1$ the conditions of Theorem 7 together with inequality (2.2.28) are satisfied and therefore from (2.2.29), we have

$$
\text{either } \frac{R}{|p|^2} + \frac{R_0}{|p|} - \frac{1}{c} > 0 \text{ or } \frac{T_0}{|p|^2} - \frac{1}{c} > 0 ,
$$

which implies that

$$
|p|^2 < \text{greater of } \left[\frac{-R_0 c + \sqrt{R_0^2 c^2 - 4R_0 c}}{2}, T_0^2\right] .
$$

This proves the theorem.

.4 STABILITY OF RAYLEIGH-TAYLOR CONFIGURATION

(a) Rayleigh-Taylor problem:

The governing equations and boundary conditions of this problem under Boussinesq approximation for a fluid of constant viscosity are

$$
n(D^2 - k^2)(D^2 - k^2 - \frac{n}{p})W = \frac{gk^2}{\mu} (DP) W , \quad (3.4.1)
$$

and
either \( W = 0 = D W \) at \( z = 0 \) and \( z = d \), 
\begin{align*}
\text{rigid boundaries)} \\
\text{or } W = 0 = D^2 W \text{ at } z = 0 \text{ and } z = d , 
\end{align*}

(3.4.2)

where \( W(z) \) is the single Fourier component of the vertical perturbation velocity; \( z \) is the vertical coordinate; \( D \) stands for \( \frac{d}{dz} \); \( n = n_x + in_i \) is the complex growth rate of the wave number \( k \); \( \rho_0 \) is a constant, \( \rho(z) \) and \( \mu \) respectively represent the density and viscosity of the fluid; \( g \) is the acceleration due to gravity; \( d \) is the depth of the fluid layer and \( \nu = \mu/\rho_0 \).

THEOREM 18: If \((n, W), n = n_x + in_i \) and \( n_i \neq 0 \) is a solution of equations (3.4.1)-(3.4.2), then

\[ |n|^2 < \sup_{\text{flow domain}} \left[ - \frac{gD\rho}{\rho_0} \right]. \]

(3.4.3)

PROOF: Since \( n_i \neq 0 \), equation (3.4.1) can be put in the following convenient form:

\[ [(D^2 - k^2)(D^2 - k^2 - \frac{n}{\nu}) - \frac{gk^2(D\rho)}{n\mu}]W = 0. \]

(3.4.4)

Equation (2.1.1) reduces to the above equation with

\[ A = 1, \quad B = - (2k^2 + \frac{n}{\nu}), \quad C = \left[ k^2(k^2 + \frac{n}{\nu}) - \frac{gk^2(D\rho)}{\mu} \right], \]

and \( X(z) = W(z) \).

Further the boundary conditions on \( X \) conform to those of \( W \).

Also \( A_1, B_1 \) and \( C_1 \) come out to be

\[ A_1 = 0, \quad B_1 = - n_i \left( \frac{1}{\nu} \right); \quad \text{and } C_1 = - n_i \left[ - \frac{k^2}{\nu} - \frac{gk^2(D\rho)}{\mu n^2} \right]. \]
Thus, with \( l = n_1 \), conditions of Theorem 1 are satisfied and therefore

\[
|n|^2 < \sup_{\text{flow domain}} \left[ -\frac{g\rho^2}{\rho_0} \right].
\]

This proves the theorem.

THEOREM 19: If \((n, W)\), \(n = n_r + in_i\) is a solution of equations (3.4.1)-(3.4.2) and \(DP < 0\), then

\[
n_r < 0.
\] (3.4.5)

PROOF: As in Theorem 18 of this chapter, we take

\[
A = 1, \quad B = -(2k^2 + \frac{n_2}{\nu}), \quad C = [k^2(k^2 + \frac{n_2}{\nu}) - \frac{gk^2}{\mu n^r} (DP)],
\]

and \(X(z) = W(z)\).

Here \(A_1', B_1', C_1'\) come out to be

\[
A_1' = 1, \quad B_1' = -(2k^2 + \frac{n_2}{\nu}), \quad C_1' = [k^2(k^2 + \frac{n_2}{\nu}) - \frac{gk^2(DP)n_r}{\mu |n|^2}].
\]

Thus, with \( l = 1 \) conditions of Theorem 2 are satisfied, therefore

\[
n_r < 0.
\]

This proves the theorem.

(b) In presence of magnetic field:

The governing equations and boundary conditions for the present problem under Boussinesq approximation are

\[
n(D^2 - k^2)(D^2 - k^2 - \frac{n_2}{\nu})W = \frac{gk^2(DP)}{\mu} W - \frac{\mu e^{Hn}}{4\pi \rho^2} \frac{\rho_0}{\nu^2} D(D^2 - k^2)h_z
\] (3.4.6)
\[(D^2 - k^2 - \frac{n}{\eta})h_z = -\left(\frac{H}{\eta}\right)DW\]  \tag{3.4.7}

and

either \( W = DW = 0 = h_z \) at \( z = 0 \) and \( z = d \)  
\{ (rigid perfectly conducting boundaries) \}

or \( W = D^2W = 0 = h_z \) at \( z = 0 \) and \( z = d \)  
\{ (free perfectly conducting boundaries) \}

where \( h_z \) is the \( z \)-component of perturbed magnetic field, \( H \) is the uniform magnetic field in the \( z \)-direction, \( \mu_e \) is the magnetic permeability, \( \eta \) is the electrical resistivity.

**THEOREM 20:** If \( (n, W, h_z) \), \( n = n_r + in_i \), \( n_r \gg 0, n_i \neq 0 \) is a solution of equations (3.4.6)-(3.4.8), then

\[|n|^2 < \text{greater of} \left( \frac{\int D \rho}{\rho_o} \right), \left( \frac{\mu_e H^2}{4\pi \rho_o \eta} \right) \].  \tag{3.4.9}

**PROOF:** Since \( n_i \neq 0 \), equations (3.4.6)-(3.4.7) can be put in the following convenient forms:

\[(D^2 - k^2)(D^2 - k^2 - \frac{n}{\eta})W - \frac{gk^2(D\rho)}{\mu n} W + \frac{\mu_e H}{4\pi \rho_o \eta} (\frac{n}{\eta} Dn_z - \frac{H}{\eta} D^2W) = 0,\]

\[-\frac{\mu_e n^*}{4\pi \rho_o \eta} \left[(D^2 - k^2 - \frac{n}{\eta})h_z + \frac{H}{\eta} DW \right] = 0.\]  \tag{3.4.10, 3.4.11}

Equation (2.2.37) reduces to the above equations with

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -(2k^2 + \frac{n}{\eta} + \frac{\mu_e H^2}{4\pi \rho_o \eta}) & 0 \\ 0 & \frac{\mu_e n^*}{4\pi \rho_o \eta} \end{bmatrix},
\]
\[ C = \begin{bmatrix} k^2(k^2 + \frac{n^2}{\mu^2}) - \frac{q \kappa^2 (D \rho) n^*}{\mu |n|^2} & 0 \\ 0 & \frac{\mu e^r}{4\pi r^2} (k^2 + \frac{n^2}{\eta}) \end{bmatrix} \]

\[ E = \begin{bmatrix} 0 & \frac{\mu e^r}{4\pi r^2} \eta \\ -\frac{\mu e^r}{4\pi r^2} \eta & 0 \end{bmatrix} \], and \[ X(z) = \begin{bmatrix} W(z) \\ h_z(z) \end{bmatrix} \]

Further the boundary conditions on \( X \) conform to those of \( W \) and \( h_z \). Also \( A_1, B_1 \) and \( C_1 \) come out to be diagonal matrices with diagonal entries:

\[ \text{dg}(A_1) = (0, 0) \],

\[ \text{dg}(B_1) = -n_i\left(\frac{1}{\eta}, -\frac{\mu e^r}{4\pi r^2}\right) \],

and

\[ \text{dg}(C_1) = -n_i\left(-k^2\left[\frac{1}{\eta} + \frac{q \kappa^2 (D \rho)}{\mu |n|^2}\right], \frac{\mu e^r k^2}{4\pi r^2}\right) \].

Now expressing \( B_1 \) and \( C_1 \) as

\[ B_1 = -n_i(B_3 - B_4) \],

and \[ C_1 = -n_i(C_3 - C_4) \],

where \( B_3, B_4, C_3 \) and \( C_4 \) are diagonal matrices with diagonal entries

\[ \text{dg}(B_3) = \left(\frac{1}{\rho}, 0\right) \].
\[ dg(B_4) = \left(0, \frac{\mu e}{4\pi \rho_0 v}\right), \]

\[ dg(C_3) = \left(0, \frac{\mu e \kappa^2}{4\pi \rho_0 v}\right), \]

and \[ dg(C_4) = \left(\kappa^2\left\{\frac{1}{v} + \frac{qD}{\mu |n|^2}\right\}, 0\right). \]

We show that the inequality (2.2.28) is satisfied if \( H \) is a null matrix and \( G \) is a diagonal matrix with diagonal entries:

\[ dg(G) = \left(\frac{\mu e H^2}{4\pi \rho_0^2 |\mu| |n|}, 0\right). \]

Now

\[
\int_V \left[(\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 X\right] dv
= \frac{\mu e}{4\pi \rho_0 v} \int_0^d \left(\left|Dh_z\right|^2 + \kappa^2 \left|h_z\right|^2\right) dz .
\]

Integrating by parts and using the boundary conditions (2.4.8), we get

\[
\int_V \left[(\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 X\right] dv
= - \frac{\mu e}{4\pi \rho_0 v} \int_0^d h_z^* (D^2 - \kappa^2) h_z dz .
\]

Using Schwartz inequality, we have

\[
\int_V \left[(\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 X\right] dv
\leq \frac{\mu e}{4\pi \rho_0 v} \left[\int_0^d |h_z|^2 dz\right]^{1/2} \left[\int_0^d |(D^2 - \kappa^2) h_z|^2 dz\right]^{1/2} .
\]

(3.4.12)
Multiplying equation (3.4.7) by its complex conjugate and integrating over the range of \( z \), we have

\[
\frac{H^2}{\eta^2} \int_0^d |DW|^2 \, dz = \int_0^d (D^2 - k^2 - \frac{\eta}{n}) h_z (D^2 - k^2 - \frac{\eta^*}{n}) h_z^* \, dz .
\]

Integrating by parts and using the boundary conditions, we have

\[
\frac{H^2}{\eta^2} \int_0^d |DW|^2 \, dz = \int_0^d |(D^2 - k^2) h_z|^2 \, dz + \frac{2n \eta}{\eta} \int_0^d |h_z^2| dz + k^2 \int_0^d |h_z|^2 \, dz + \frac{\eta}{\eta^2} \int_0^d |h_z|^2 \, dz . \tag{3.4.13}
\]

Since \( n_r \gg 0 \), \( n_i \neq 0 \), equation (3.4.13) gives

\[
\int_0^d |h_z|^2 \, dz < \frac{H^2}{|n|^2} \int_0^d |DW|^2 \, dz , \tag{3.4.14}
\]

and

\[
\int_0^d |(D^2 - k^2) h_z|^2 \, dz < \frac{H^2}{\eta} \int_0^d |DW|^2 \, dz . \tag{3.4.15}
\]

Using (3.4.14) and (3.4.15) in (3.4.12), we get

\[
\int_V [(\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 X] dV < \frac{\mu e H^2}{4\pi \rho_o^2 \eta |n|} \int_0^d |DW|^2 \, dz \leq \int_V (\text{grad } X)^\dagger G(\text{grad } X) dV .
\]

Thus, with \( l = n_i \) the conditions of Theorem 7 together with inequality (2.2.28) are satisfied, therefore from (2.2.29), we have

\[
\text{either } \left[ \frac{1}{\rho} + \frac{gD\rho}{\mu |n|^2} \right] < 0 \quad \text{or} \quad \frac{\mu e H^2}{4\pi \rho_o^2 \eta |n|} - \frac{1}{\rho} > 0 ,
\]

which implies that

\[
|n|^2 \leq \text{greater of } \left[ \sup_{\text{flow domain}} \left( -\frac{gD\rho}{\rho_o} \right) , \left( \frac{\mu e H^2}{4\pi \rho_o \eta} \right)^2 \right] .
\]

This proves the theorem.
(c) **In presence of rotation**

The governing equations and boundary conditions for the present problem under Boussinesq approximation are

\[
n(D^2 - k^2)(D^2 - k^2 - \frac{n}{\nu})W = \frac{gk^2(D \rho)}{\mu} W + \frac{2\Omega n}{\nu} DZ ,
\]

\[
(D^2 - k^2 - \frac{n}{\nu})Z = - (\frac{2\Omega}{\nu}) DW ,
\]

and

\[W = 0 = DW = Z\text{ at } z = 0 \text{ and } z = d ,\]

where \(\Omega\) is the uniform rotation in the z-direction and Z is the z-component of perturbed vorticity.

**THEOREM 21:** If \((n, W, Z), n = n_r + i n_i, n_r \geq 0, n_i \neq 0\) is a solution of equations (3.4.16)-(3.4.18), then

\[|n|^2 \leq \text{greater of } \left[ \sup_{\text{flow domain}} \left( -\frac{gD \rho}{\mu n} \right), 4\Omega^2 \right] ,
\]

**PROOF:** Since \(n_i \neq 0\), equations (3.4.16)-(3.4.17) can be put in the following convenient forms:

\[
(D^2 - k^2)(D^2 - k^2 - \frac{n}{\nu})W - \frac{gk^2(D \rho)}{\mu n} W - \frac{2\Omega}{\nu} DZ = 0 ,
\]

\[
[(D^2 - k^2 - \frac{n}{\nu})Z + \frac{2\Omega}{\nu} DW] = 0 .
\]

Equation (2.2.37) reduces to the above equations with

\[
A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -(2k^2 + \frac{n}{\nu}) & 0 \\ 0 & 1 \end{bmatrix},
\]
\[
C = \begin{bmatrix}
k^2(k^2 + \frac{n}{y}) - \frac{gk^2(D \cdot \rho)}{\mu n} & 0 \\
0 & -(k^2 + \frac{n}{y})
\end{bmatrix}, \quad E = \begin{bmatrix}
0 & -\frac{2\Omega}{y} \\
\frac{2\Omega}{y} & 0
\end{bmatrix},
\]

and \(X(z) = \begin{bmatrix}
W(z) \\
Z(z)
\end{bmatrix} \).

Further the boundary conditions on \(X\) conform to those of \(W\) and \(Z\). Also \(A_1\), \(B_1\) and \(C_1\) come out to be diagonal matrices with diagonal entries:

\[
dg(A_1) = (0, 0),
\]

\[
dg(B_1) = -n_1\left(\frac{1}{y}, 0\right),
\]

and \(dg(C_1) = -n_1(-k^2 \left[\frac{1}{y} + \frac{gD\rho}{\mu |n|^2}\right], \frac{1}{y})\).

Now expressing \(C_1\) as

\[
C_1 = -n_1(C_3 - C_4),
\]

where \(C_3\) and \(C_4\) are diagonal matrices with diagonal entries:

\[
dg(C_3) = (0, \frac{1}{y}),
\]

\[
dg(C_4) = (k^2 \left[\frac{1}{y} + \frac{gD\rho}{\mu |n|^2}\right], 0),
\]

we show that the inequality (2.2.28) is satisfied if \(H\) is the null matrix and \(G\) is the diagonal matrix with diagonal entries:

\[
dg(G) = \left(\frac{4\Omega^2}{|n|^2 y^2}, 0\right).
\]

Since in this case \(B_4\) is the null matrix,
\[ \int_V [(\nabla X)^\top B_4(\nabla X) + X^\top C_3X] \, dv = \int_V x^\top C_3X \, dv \]

\[ = \frac{1}{v} \int_0^d |z|^2 \, dz. \]  

(3.4.22)

Now from equation (3.4.17), we have

\[ \frac{4 \Omega^2}{v^2} \int_0^d |DN|^2 \, dz = \frac{d}{d} \left[ (D^2 - k^2 - \frac{n}{v})Z (D^2 - k^2 - \frac{n^*}{v})Z^* \right] \, dz. \]

Integrating by parts and using the boundary conditions, we have

\[ \frac{4 \Omega^2}{v^2} \int_0^d |DN|^2 \, dz = \frac{d}{d} \left| (D^2 - k^2)Z \right|^2 \, dz + \frac{2n}{v} \int_0^d \left| (DZ)^2 \right| \, dz \]

\[ + k^2 |Z|^2 \, dz + \frac{l \cdot n}{v^2} \int_0^d |Z|^2 \, dz. \]  

(3.4.23)

Since \( n_x \geq 0, n_1 \neq 0, \) equation (3.4.23) gives

\[ \int_0^d |Z|^2 \, dz < \frac{4 \Omega^2}{|n|^2} \int_0^d |DN|^2 \, dz. \]  

(3.4.24)

Equation (3.4.22) and the inequality (3.4.24) give

\[ \int_V [(\nabla X)^\top B_4(\nabla X) + X^\top C_3X] \, dv < \frac{4 \Omega^2}{v \cdot |n|^2} \int_0^d |DN|^2 \, dz \]

\[ = \int_V (\nabla X)^\top G(\nabla X) \, dv. \]

Thus, with \( 1 = n_1 \) conditions of Theorem 7 together with inequality (2.2.28) are satisfied, therefore from (2.2.29), we have

either \( \frac{1}{v} + \frac{gDf}{\mu |n|^2} < 0 \) or \( \frac{4 \Omega^2}{v \cdot |n|^2} - \frac{1}{v} > 0 \)

which implies that

\[ |n|^2 < \text{greater of } \left[ \sup_{\text{flow domain}} \left( -\frac{gDf}{\mu^2} \right), 4 \Omega^2 \right]. \]

This proves the theorem.
3.5 STABILITY OF GENERALIZED BÉNARD PROBLEM

(a) Generalized Bénard problem:

The governing equations and boundary conditions for the present problem under Boussinesq approximation are

\[(D^2 - a^2)(D^2 - a^2 - p)W = \frac{g\alpha d^2 a^2}{\nu} \theta + \frac{g\alpha d^2 D(\rho)}{p\nu^2} W, \quad (3.5.1)\]
\[(D^2 - a^2 - \sigma p)\theta = -\left(\frac{\beta d^2}{\kappa}\right) W, \quad (3.5.2)\]

and

\[W = 0 = \theta \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1, \]

and

either \(DW = 0\) at \(z = 0 \quad \text{and} \quad z = 1\) (rigid boundaries),

or \(D^2 W = 0\) at \(z = 0 \quad \text{and} \quad z = 1\) (free boundaries).

(3.5.3)

where \(W(z)\) and \(\theta(z)\) are respectively the single Fourier components of the vertical perturbation velocity and perturbation temperature; \(z\) is the vertical coordinate; \(D\) stands for \(\frac{d}{dz}\); \(p = p_r + ip_i\) is the complex growth rate of the wave mode of wave number \('a'\); \(g\) is the acceleration due to gravity; \(\alpha\) is the coefficient of volume expansion of the fluid; \(d\) is the depth of the fluid layer; \(\nu\) is the coefficient of kinematic viscosity of the fluid; \(\sigma\) is the thermal Prandtl number; \(\beta\) is the uniform adverse temperature gradient maintained between the two bounding surfaces; \(\kappa\) is the coefficient of thermal diffusivity of the fluid; \(\rho\) is the density.

THEOREM 22: If \((p, W, \theta), p = p_r + ip_i, p_i \neq 0\), is a solution of equations (3.5.1)-(3.5.3), then
\[ |p|^2 < \sup_{\text{flow domain}} \left[ -\frac{g(D\rho)a^3}{\nu^2} \right]. \]  

(3.5.4)

PROOF: Since \( p \neq 0 \), equations (3.5.1)-(3.5.2) can be put in the following convenient forms:

\[(D^2 - a^2)(D^2 - a^2 - p)W - \frac{ga^2D^3(\rho)}{p\nu^2} W - \frac{gad^2a^2}{\nu} \Theta = 0,\]

(3.5.5)

\[-\frac{gad^2}{\nu} \frac{K}{\beta} [(D^2 - a^2 - \sigma p)\Theta + \frac{\beta a^2}{K} W] = 0.\]

(3.5.6)

Equation (2.1.1) reduces to the above equations with

\[A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -12a^2 + p & 0 \\ 0 & -\frac{gad^2}{\beta \nu} \end{bmatrix},\]

\[C = \begin{bmatrix} a^2(a^2 + p) - \frac{ga^2D^3(\rho)}{p\nu^2} & -\frac{gad^2a^2}{\nu} \\ -\frac{gad^2a^2}{\nu} & \frac{gad^2k}{\beta \nu}(a^2 + \sigma p) \end{bmatrix},\]

and \( X(z) = \begin{bmatrix} W(z) \\ \Theta(z) \end{bmatrix} \).

Further, the boundary conditions on \( X \) conform to those of \( W \) and \( \Theta \). Also \( A_1, B_1 \) and \( C_1 \) come out to be diagonal matrices with diagonal entries:

\[dg(A_1) = (0, 0),\]

\[dg(B_1) = -p_1(1, 0),\]

and
Thus, with \( l = p_1 \), conditions of Theorem 1 are satisfied, therefore

\[
|p|^2 < \sup_{\text{flow domain}} \left( - \frac{gd^3 D(P)}{\nu^2} \right).
\]

This proves the theorem.

(b) \textbf{In presence of magnetic field:}

The governing equations and boundary conditions for the present problem, under Boussinesq approximation are

\[
(D^2 - a^2)(D^2 - a^2 - p)W + \frac{\mu e H d}{4\pi \rho_0 \nu} D(D^2 - a^2)h_z
\]

\[
= \frac{gd^2 a^2}{\nu^2} \phi + \frac{gd^2 D(P)}{\nu^2 \rho} W, \quad (3.5.7)
\]

\[
(D^2 - a^2 - \sigma P)\phi = - \left( \frac{8d^2}{\mu} \right) W, \quad (3.5.8)
\]

\[
(D^2 - a^2 - \sigma i P)h_z = - \left( \frac{H d}{\mu} \right) \partial W, \quad (3.5.9)
\]

and

\[
W = 0 = \phi \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1
\]

and

either \( D W = 0 = h_z \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1 \)

(rigid perfectly conducting boundaries),

or \( D^2 W = 0 = h_z \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1 \)

(free perfectly conducting boundaries),

\[
(3.5.10)
\]
where $h_z$ is the $z$-component of perturbed magnetic field;
\( \eta \) is the electrical resistivity; \( \mu_e \) is the magnetic permeability.

THEOREM 23: If \((p, W, \Theta, h_z), p = p_r + ip_i, p_r \gg 0, p_i \neq 0\) is a solution of equations (3.5.7)-(3.5.10), then

\[
|p|^2 < \text{greater of } \left[ \sup_{\text{flow domain}} \left( - \frac{\alpha d^3 Dp}{\nu^2} \right), Q^2 \nu^2 \right], \tag{3.5.11}
\]

where \(Q\) is the Chandrasekhar number.

PROOF: Since \(p_i \neq 0\), equations (3.5.7)-(3.5.9) can be put in the following convenient forms:

\[
(D^2 - a^2)(D^2 - a^2 - p)W - \frac{\alpha d^2 Dp}{\nu^2} \frac{\eta}{\mu} \frac{Hd}{D^2} W = 0 ,
\]

\[
- \frac{\alpha a^2 h}{\nu^2} \left[(D^2 - a^2 - \sigma p)\Theta + \frac{Hd^2}{h} W \right] = 0 , \tag{3.5.13}
\]

\[
- \frac{\mu \eta \sigma^1 p}{4 \pi \rho^2} \left[(D^2 - a^2 - \sigma^1 p)h_z + \frac{Hd}{\eta} DW \right] = 0 . \tag{3.5.14}
\]

Equation (2.2.37) reduces to the above equations with

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
Further, the boundary conditions on $X$ conform to those of $W$, $\Theta$ and $h_z$. Also $A_1$, $B_1$ and $C_1$ come out to be diagonal matrices with diagonal entries:

$\text{dg}(A_1) = (0, 0, 0)$,

$\text{dg}(B_1) = -p_i(1, 0, -\frac{\mu e\eta \sigma_1 p^*}{4\pi P_0 p})$,
\[ dg(C_1) = -p_i \left(-a^2 \left\{ 1 + \frac{gd^2 D_P}{|P|^2} \right\}, \frac{a^{a^2}}{\beta^2}, \frac{\mu e \eta a^2}{4 \pi f \phi |P|} \right), \]

Now expressing \( B_1 \) and \( C_1 \) as

\[ B_1 = -p_i (B_3 - B_4), \]

and \( C_1 = -p_i (C_3 - C_4). \]

where \( B_3, B_4, C_3 \) and \( C_4 \) are diagonal matrices with diagonal entries

\[ dg(B_3) = (1, 0, 0), \]

\[ dg(B_4) = (0, 0, \frac{\mu e \eta \sigma_1}{4 \pi f \phi}), \]

\[ dg(C_3) = (0, 0, \frac{\mu e \eta a^2 \sigma_1}{4 \pi f \phi}), \]

and

\[ dg(C_4) = \left(a^2 \left\{ 1 + \frac{gd^2 D_P}{|P|^2} \right\}, \frac{a^{a^2}}{\beta^2}, 0 \right). \]

We show that the inequality (2.2.28) is satisfied with \( H \) as the null matrix and \( G \) as the diagonal matrix with diagonal entries:

\[ dg(C) = \left( \frac{H^2 d^2}{4 \pi f \phi \eta |P|}, 0, 0 \right) \]

Now

\[ \int_V \left[ (\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 X \right] dV \]

\[ = \frac{\mu e \eta \sigma_1}{4 \pi f \phi} \int_0^1 \left[ |Dh_z|^2 + a^2 |h_z|^2 \right] dz. \]  \hspace{1cm} (3.5.15)
Integrating by parts and using the boundary conditions (3.5.10), we get
\[
\int_V \left[ (\text{grad } X) \cdot B_4 (\text{grad } X) + X \cdot C_3 X \right] \, dv
= -\frac{\mu_e \eta \sigma_1}{4\pi \rho_0^2} \int_0^1 h_z^* (D^2 - a^2) h_z \, dz
\] (3.5.16)

Using Schwartz inequality, we get
\[
\int_V \left[ (\text{grad } X) \cdot B_4 (\text{grad } X) + X \cdot C_3 X \right] \, dv
\leq \frac{\mu_e \eta \sigma_1}{4\pi \rho_0^2} \left[ \int_0^1 |h_z|^2 \, dz \right]^{1/2} \left[ \int_0^1 |(D^2 - a^2) h_z|^2 \, dz \right]^{1/2}
\] (3.5.17)

From equation (2.4.9), we have
\[
\int_0^1 \frac{(H_d)}{\eta} DW \cdot \frac{(H_d)}{\eta} DW^* \, dz = \int_0^1 \left[ (D^2 - a^2 - \sigma_1 p) h_z^* (D^2 - a^2
- \sigma_1 p^*) h_z^* \right] dz
\]

Integrating by parts and using boundary conditions (3.5.10), we have
\[
\left( \frac{H_d}{\eta} \right)^2 \int_0^1 |DW|^2 \, dz = \int_0^1 \left[ |(D^2 - a^2) h_z|^2 + 2p_r \sigma_1 (|D h_z|^2 + a^2 |h_z|^2)
+ |p|^2 \sigma_1^2 |h_z|^2 \right] dz
\] (3.5.18)

Since \( p_r \gg 0, p_i \neq 0 \), equation (3.5.18) gives
\[
\int_0^1 |h_z|^2 \, dz < \left( \frac{H_d}{\eta} \right)^2 \frac{1}{\sigma_1^2 |p|^2} \int_0^1 |DW|^2 \, dz
\] (3.5.19)

and
\[
\int_0^1 |(D^2 - a^2) h_z|^2 \, dz < \left( \frac{H_d}{\eta} \right)^2 \int_0^1 |DW|^2 \, dz
\] (3.5.20)
Using inequalities (3.5.19) and (3.5.20), we have from inequality (3.5.17),

\[ \int \left[ (\nabla X)^\top B_4 (\nabla X) + X^\top C_3 X \right] \, dV < \frac{\mu e^{\frac{d_2 a^2}{2}}}{4\pi \rho_0 \eta \nu |p|} \int_0^1 |DW|^2 \, dz \]

Thus, with \( l = \rho_1 \), conditions of Theorem 7 together with inequality (2.2.28) are satisfied and therefore from (2.2.29), we have

either \( \sup_{\text{flow domain}} \left( \frac{-q d^3 D \rho}{\nu^2} - 1 \right) > 0 \),

or \( \frac{H^2 d^2 \mu e}{4\pi \rho_0 \eta \nu |p|} - 1 > 0 \).

This implies that

\[ |p|^2 < \text{greater of } \sup_{\text{flow domain}} \left( \frac{-q d^3 D \rho}{\nu^2}, \frac{H^2 d^2 \mu e}{4\pi \rho_0 \eta \nu} \right) \] .

This proves the theorem.

(c) In presence of rotation:

The governing equations and boundary conditions for the present problem under Boussinesq approximation are

\[ (D^2 - a^2)(D^2 - a^2 - p)W - \left( \frac{2q d^3}{\nu} \right) DZ = \frac{qa d^2 a^2}{\nu} \theta + \frac{qa d^3 D \rho}{\nu^2 p} W \]

\[ (D^2 - a^2 - \sigma p) \theta = -\left( \frac{8d^2}{w} \right) W, \]

and
\[(D^2 - a^2 - p)Z = -\left(\frac{2\Omega d}{\nu}\right)DW, \quad \text{(3.5.23)}\]

and

\[W = 0 = \Theta = DW = Z \text{ at } z = 0 \text{ and } z = 1 \quad \text{(3.5.24)}\]

where \(Z\) is the \(z\)-component of perturbed vorticity, \(\Omega\) is the uniform angular velocity in \(z\)-direction.

**Theorem 24:** If \((p, W, \Theta, Z), p = p_x + ip_y, p_x \geq 0, p_y \neq 0,\) is a solution of equations (3.5.21)-(3.5.24), then

\[|p|^2 < \text{greater of } \left[\sup_{\text{flow domain}} \left(-\frac{ga^2d^2p}{\nu^2}\right), \left(\frac{4\Omega^2d^2}{\nu^2}\right) d^2\right].\]

\[\text{(3.5.25)}\]

**Proof:** Since \(p_x \neq 0\), equations (3.5.21)-(3.5.23) can be put in the following convenient forms:

\[(D^2 - a^2)(D^2 - a^2 - p)W - \left(\frac{2\Omega d^2}{\nu}\right)DZ - \frac{ga^2d^2}{p\nu^2} \Theta - \frac{ga^2d^3p}{\nu^2} W = 0, \quad \text{(3.5.26)}\]

\[-\frac{ga^2}{\nu^2} \left[(D^2 - a^2 - p)\Theta + \frac{gd^2}{k} W\right] = 0, \quad \text{(3.5.27)}\]

\[d^2 \left[(D^2 - a^2 - p)Z + \left(\frac{2\Omega d}{\nu}\right) DW\right] = 0. \quad \text{(3.5.28)}\]

Equation (2.2.37) reduces to the above equations with

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
-(2a^2 + p) & 0 & 0 \\
0 & -\frac{ga^2}{\nu^2} & 0 \\
0 & 0 & d^2
\end{bmatrix},
\]
Further the boundary conditions on $X$ conform to those of $W$, $\Theta$ and $Z$. Also $A_1$, $B_1$ and $C_1$ come out to be diagonal matrices with diagonal entries

$$dg(A_1) = (0, 0, 0) ,$$

$$dg(B_1) = -p_1(1, 0, 0) ,$$

and

$$dg(C_1) = -p_1(-a^2 \left[ 1 + \frac{gd^3 Dp}{|p|^2 \nu^2} \right], -\frac{g\alpha a^2 K \sigma}{\beta \nu}, d^2) .$$

Now expressing $C_1$ as

$$C_1 = -p_1(C_3 - C_4) ,$$

where $C_3$ and $C_4$ are diagonal matrices with diagonal entries

$$dg(C_3) = (0, 0, d^2) ,$$

and $dg(C_4) = (a^2 \left[ 1 + \frac{gd^3 Dp}{|p|^2 \nu^2} \right], \frac{g\alpha a^2 K \sigma}{\beta \nu}, 0) ,$$
we show that the inequality (2.2.28) is satisfied with $H$ as the null matrix and $G$ as the diagonal matrix with diagonal entries

$$dg(G) = \left( \frac{4 \Omega^2 d^2}{p^2}, 0, 0 \right).$$

Since $B_4$ is the null matrix, we have

$$\int \left[(\text{grad } X)^\dagger B_4 (\text{grad } X) + X^\dagger C_3 X\right] dV = \int X^\dagger C_3 X dV = \alpha d^2 \int |Z|^2 dz .$$

(3.5.29)

From equation (3.5.23), we have

$$\int \left(\frac{2\Omega d}{p}\right) D W (\frac{2\Omega d}{p}) D W^* dZ = \int \left[(D^2 - a^2 - p)Z (D^2 - a^2 - p^*)Z^*\right] dZ .$$

Integrating by parts and using the boundary conditions (3.5.24), we have

$$(\frac{2\Omega d}{p})^2 \int |DW|^2 dZ =$$

$$\int \left[|D^2 - a^2|Z|^2 + 2 p_F (|DZ|^2 + a^2 |Z|^2) + |p|^2 |Z|^2\right] dZ .$$

(3.5.30)

Since $p_F \geq 0$, $p_i \neq 0$, equation (3.5.30) gives

$$\int |Z|^2 dZ < \frac{1}{|p|^2} (\frac{2\Omega d}{p})^2 \int |DW|^2 dZ .$$

(3.5.31)

From equation (3.5.29) and inequality (3.5.31), we have

$$\int X^\dagger C_3 X dV \leq (\frac{2\Omega d}{p})^2 \frac{d^2}{|p|^2} \int |DW|^2 dZ$$
Thus, with \( l = p_1 \), conditions of Theorem 7 together with inequality (2.2.28) are satisfied and therefore from (2.2.29), we have

\[
\text{either } \sup_{\text{flow domain}} \left[ - \frac{g \partial^3 (D \mathbf{P})}{|p|^2 \nu^2} \right] - 1 > 0
\]

or

\[
\left( \frac{4 \Omega \omega^2}{\nu^2} \right) \frac{d^2}{|p|^2} - 1 > 0.
\]

This implies that

\[
|p|^2 < \text{greater of } \left[ \sup_{\text{flow domain}} \left( - \frac{g \partial^3 (D \mathbf{P})}{\nu^2} \right), \left( \frac{4 \Omega \omega^2}{\nu^2} \right) d^2 \right].
\]

This proves the theorem.

### 3.6 CONCLUDING REMARKS

It is clearly seen that the various theorems proved in this chapter in the absence of rotation or a magnetic field can be thought of as special cases of Theorem 10 by an appropriate choice of the constants \( a_i \)'s, \( b_i \)'s, \( c_i \)'s, \( f(x) \), \( \Psi \), \( \Theta \) and \( \mu \). Therefore Theorem 10 provides us with another method of unifying the results obtained in these theorems. Further by suitable additions and modifications in equations of Section 2.4 and proving a theorem analogous to Theorem 10 all the results of this chapter can be derived from this new theorem.

The results for various rotatory configurations in the present chapter have been derived for the case of rigid boundaries only upon using Theorem 7. However, using Theorem 8,
they are derivable for all combinations of dynamically free or rigid boundaries. We illustrate this for the case of Theorem 8 (say) of the present chapter.

We write system of equations (3.3.20)-(3.3.23) in the following convenient forms:

\[
(D^2 - a^2)(D^2 - a^2 - \frac{p}{\sigma})W - Ra^2 \phi + R_s a^2 \left[ \frac{\tau}{p} (D^2 - a^2) \phi + \frac{W}{p} \right] - T D Z = 0 ,
\]

\[
- Ra^2 \left[ (D^2 - a^2 - p) \phi + W \right] = 0 ,
\]

\[
\frac{\tau^2 R_s a^2}{p^*} (D^2 - a^2) \left[ (D^2 - a^2 - \frac{p}{\sigma}) \phi + \frac{W}{\tau} \right] = 0 ,
\]

\[
- T \left[ (D^2 - a^2 - \frac{p}{\sigma}) Z + D W \right] = 0 .
\]

Equations (2.3.1) and (2.3.2) reduce to the above equations with

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & \frac{\tau^2 R_s a^2}{p^*} & 0
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
-(2a^2 + \frac{p}{\sigma}) & 0 & \frac{\tau R_s a^2}{p} \\
0 & -Ra^2 & 0 \\
\frac{\tau R_s a^2}{p^*} & 0 & -\frac{\tau^2 R_s a^2}{p^*}
\end{bmatrix},
\]

\[
(2a^2 + \frac{p}{\tau})
\]
\[
C = \begin{bmatrix}
\frac{4 + p a^2}{\sigma} + \frac{R_S a^2}{p} & -R a^2 & -\frac{\tau R_S a^4}{p} \\
-R a^2 & R a^2(a^2 + p) & 0 \\
-\frac{\tau R_S a^4}{p} & 0 & \frac{\tau^2 R_S a^2}{p} (a^4 + \frac{p a^2}{\tau})
\end{bmatrix},
\]

\[
E = \begin{bmatrix}
T & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad F = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\]

\[
X = \begin{bmatrix}
W(z) \\
\Theta(z) \\
\phi(z)
\end{bmatrix}, \quad \mathcal{H}(\omega) = \begin{bmatrix}
-Z(z) \\
0 \\
0
\end{bmatrix},
\]

and \( q = -T(a^2 + \frac{p}{\sigma}) \).

Further the boundary conditions on \( X \) and \( \mathcal{H} \) conform to those of \( W, \Theta, \phi \) and \( Z \). Also \( A_1, B_1 \) and \( C_1 \) come out to be diagonal matrices with diagonal entries:

\[
dg(A_1) = p_1(0, 0, \frac{\tau^2 R_S a^2}{|p|}),
\]

\[
dg(B_1) = -p_1(0, 0, \frac{\tau^2 R_S a^2}{|p|} (2a^2 + \frac{2p x}{\tau})),
\]

\[
dg(C_1) = -p_1(\frac{R_S a^2}{|p|} - \frac{a^2}{\sigma}, -R a^2, -\frac{\tau^2 R_S a^2}{|p|} [a^4 + 2pa^2]).
\]

Thus, with \( l = p_1, Q_i = \frac{T}{\sigma} \), conditions of Theorem 8 are satisfied and therefore from (2.3.12), we have
either \( \left( \frac{R}{|p|^2} - \frac{1}{3} \right) > 0 \) or \( \frac{T \sigma^2}{|(p + a^2 \sigma)|^2} - 1 > 0 \),

which implies that

\[ |p|^2 < \text{greater of } [R \sigma, T \sigma^2]. \]

This justifies the claim.