PROBLEM I

EXACT SOLUTION OF THE FORCED CONVECTION ENERGY EQUATION FOR TIMELESS LINEAR VARIATION OF INLET TEMPERATURE IN A CHANNEL

An exact solution of the transient forced convection energy equation of a viscous incompressible fluid with fully developed flow between two parallel flat plates is obtained in the present problem when the inlet temperature varies linearly with time. The Laplace transform technique has been used to obtain the solution and its interpretations for the case of laminar flows are given.

NOMENCLATURE

\( T \)

temperature

\( c_p \)

specific heat at constant pressure

\( h \)

half distance between parallel plates

\( K \)

thermal conductivity

\( t \)

time

\( u \)

velocity component in \( x \)-direction

\( \bar{u} \)

average velocity

\( x, y \)

cartesian coordinates (\( x \)-flow direction, \( y \)-distance from channel centreline)

\( \rho \)

fluid density

\( \nu \)

kinematic coefficient of viscosity

* Accepted for publication in Indian J. Pure Appl. Mathematics.
R  Reynolds number \( \left( = \frac{f u}{d} \right) \)

P  Prandtl number \( \left( = \frac{\rho u c_p}{k} \right) \)

\( T_0, T_1, T_2 \)  constants

**INTRODUCTION**

The study of unsteady forced convection heat transfer in tubes and ducts has recently become of greater importance in connection with the control of modern high performance heat transfer devices. Literature on thermal transient problems is limited but increasing. In the solutions of the problem of transient forced convection in laminar flow, it has generally been assumed that the inlet temperature of the fluid is constant across the flow with a specified time-wise variation of wall temperature, wall heat flux or internal heat generation.

In the present problem a solution of the transient forced convection energy equation is obtained for laminar flow in a channel bounded by two parallel flat plates under a prescribed boundary condition with an inlet temperature varying linearly with time,
1. FORMULATION OF THE PROBLEM

We consider the steady laminar flow of a viscous incompressible fluid through a parallel plate channel whose sides are separated by a distance $2h$. The fluid entering the channel has a temperature which is spatially uniform across the entrance section but varies linearly with time. Therefore we can write the inlet condition as

$$\bar{T}(0, \bar{y}, \bar{t}) = T_0 + T_1 \left( \frac{\bar{y}}{h} \right)$$  \hspace{1cm} (3.1.1)$$

The unsteady energy equation for a fully developed flow in a parallel plate channel is given by

$$\frac{\partial \bar{T}}{\partial \bar{t}} + u \frac{\partial \bar{T}}{\partial \bar{x}} = \frac{k}{\rho c_p} \frac{\partial^2 \bar{T}}{\partial \bar{y}^2}$$  \hspace{1cm} (3.1.2)$$

The inlet and the boundary conditions of the problem are as follows:

$$\bar{T} = T_0 + T_1 \left( \frac{\bar{y}}{h} \right) \text{ when } \bar{x} = 0,$$  \hspace{1cm} (3.1.3)$$

$$\left( \frac{\partial \bar{T}}{\partial \bar{y}} \right)_{\bar{y} = 0} = 0, \quad \bar{T} = T_2 \text{ at } \bar{y} = h \quad (\bar{t} > 0).$$  \hspace{1cm} (3.1.4)$$

The equation (3.1.2) is subjected to the following restrictions:

(a) Fully developed laminar velocity profile between the parallel plates.
(b) Frictional dissipation of energy is negligible.

(c) Axial conduction is negligible as compared to bulk transport in the \( \bar{z} \)-direction. This is a reasonable assumption when Péclet number exceeds 100.

(d) Fluid property variations are also neglected.

(e) Thermal resistance of the channel wall is negligible.

Further, to simplify the method of analysis the case of constant velocity will be treated here and for this purpose we substitute \( \bar{u} \) ( = mean velocity) for the velocity profile in (3.1.2).

We now introduce the following non-dimensional quantities:

\[
\theta = \frac{T - T_0}{T_1}, \quad x = \frac{x}{h}, \quad y = \frac{y}{h},
\]

\[
t = \frac{\bar{u} \tau}{h^2}, \quad \Theta = \frac{T_1 - T_0}{T_1}.
\]

Equation (3.1.2) then becomes

\[
\frac{\partial \Theta}{\partial t} + R \frac{\partial \Theta}{\partial x} = \frac{1}{P} \frac{\partial^2 \Theta}{\partial y^2}.
\]  

(3.1.5)
The inlet and the boundary conditions reduce to

\[ \theta = t \quad \text{when} \quad x = 0, \quad (3.1.6) \]

\[ \left( \frac{\partial \theta}{\partial y} \right)_{y=0} = 0, \quad \theta = \theta_0 \quad \text{at} \quad y = 1 \quad (t > 0). \quad (3.1.7) \]

2. Solution

We now separate the foregoing problem into two as follows:

\[ \theta(x, y, t) = \theta_1(x, y) + \theta_2(x, y, t), \quad (3.2.1) \]

where \( \theta_1 \) and \( \theta_2 \) satisfy the following problems:

\[ PR \frac{\partial \theta_1}{\partial x} = \frac{\partial^2 \theta_1}{\partial y^2}, \quad (3.2.2) \]

\[ \theta_1 = 0 \quad \text{when} \quad x = 0, \quad (3.2.3) \]

\[ \left( \frac{\partial \theta_1}{\partial y} \right)_{y=0} = 0, \quad \theta_1 = \theta_0 \quad \text{at} \quad y = 1. \quad (3.2.4) \]

\[ \left[ \frac{\partial \theta_2}{\partial t} + R \frac{\partial \theta_2}{\partial x} \right] = \frac{1}{P} \frac{\partial^2 \theta_2}{\partial y^2}, \quad (3.2.5) \]

\[ \theta_2 = t \quad \text{when} \quad x = 0, \quad (3.2.6) \]

\[ \left( \frac{\partial \theta_2}{\partial y} \right)_{y=0} = 0, \quad \theta_2 = 0 \quad \text{at} \quad y = 1 \quad (t > 0). \quad (3.2.7) \]
Let \( \overline{\theta}_1 = \int_0^\infty e^{-\beta x} \theta_1 \, dx \). \hfill (3.2.8)

Multiplying (3.2.2) and (3.2.4) by the kernel \( e^{\beta x} \) of the Laplace transform and integrating w.r.t. \( x \) between 0, \( \infty \) and using (3.2.3), (3.2.8), we obtain

\[
\frac{d^2 \overline{\theta}_1}{dy^2} - \beta^2 \overline{\theta}_1 = 0 , \hfill (3.2.9)
\]

where \( \beta^2 = PR \). \hfill \( \beta \)

The solution of (3.2.9) under the boundary conditions

\[
\left( \frac{d \overline{\theta}_1}{dy} \right)_{y=0} = 0 , \quad \overline{\theta}_1 = \frac{\theta_0}{\delta} \text{ at } y = 1
\]

is given by

\[
\overline{\theta}_1 = \begin{bmatrix} \frac{\theta_0 \cosh \beta y}{\delta \cosh \beta} \end{bmatrix} . \hfill (3.2.10)
\]

Inverting (3.2.10), we get

\[
\theta_1(x, y) = \theta_0 \left[ 1 - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left( \frac{2(n+1)\pi y}{2} \right) \exp \left[ -\frac{(2n+1)^2 \pi^2}{4PR} x \right] \right] .
\]

In obtaining \( \theta_2(x, y, t) \) we assume that

\[
\theta_2(x, y, t) = t \phi(x, y) + \psi(x, y) , \hfill (3.2.11)
\]
where the new temperature functions $\phi$ and $\psi$ satisfy the following problems:

$$\frac{\partial \phi}{\partial x} = \frac{\partial^2 \phi}{\partial y^2}, \quad \text{or} \quad P \frac{\partial \phi}{\partial x} + P \phi = \frac{\partial^2 \phi}{\partial y^2},$$

(3.2.12)

$$\phi = 1 \quad \text{when} \quad x = 0, \quad \text{or} \quad \psi = 0 \quad \text{when} \quad x = 0, \quad \text{or} \quad \phi = 0 \quad \text{at} \quad y = 1, \quad \psi = 0 \quad \text{at} \quad y = 1.$$  

(3.2.13)  

(3.2.14)  

(3.2.15)  

(3.2.16)  

(3.2.17)

Let

$$\Phi = \int_0^\infty e^{\rho x} \phi \, dx,$$

(3.2.18)

and

$$\Psi = \int_0^\infty e^{\rho x} \psi \, dx.$$

(3.2.19)

Multiplying (3.2.12) and (3.2.14) by $e^{-\rho x}$ and integrating w.r.t. $x$ between 0, $\infty$ and using (3.2.13), (3.2.18), we get

$$\frac{\partial^2 \Phi}{\partial y^2} - \rho^2 \Phi = -PR.$$  

(3.2.20)
The solution of (3.2.20) under the boundary conditions

\[ \left( \frac{d\varphi}{dy} \right)_{y=0} = 0, \quad \varphi = 0 \quad \text{at} \quad y = 1 \]

is given by

\[ \overline{\varphi} = \frac{1}{\beta} \left[ 1 - \frac{\cosh \beta y}{\cosh \beta} \right]. \tag{3.2.21} \]

On inversion, we get

\[ \varphi(x, y) = \frac{L}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left[ \frac{2n+1}{2} \pi y \right] \cdot \exp \left[ - \frac{(2n+1)^2 \pi^2}{4PR} x \right]. \tag{3.2.22} \]

Substituting (3.2.22) in (3.2.15) and then multiplying by \( e^{\beta x} \) and integrating \( \text{w.r.t.} \ x \) between 0, \( \infty \) and using (3.2.16), (3.2.19), we obtain

\[ \frac{d^2 \bar{\psi}}{dy^2} - \frac{\rho^2}{\kappa} \bar{\psi} = \frac{4PR}{\kappa} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left[ \frac{2n+1}{2} \pi y \right] - \frac{(2n+1)^2 \pi^2}{4PR} \cdot \left[ \rho + \frac{(2n+1)^2 \pi^2}{4PR} \right]. \]

The solution of the above equation under the boundary conditions

\[ \left( \frac{d\bar{\psi}}{dy} \right)_{y=0} = 0, \quad \bar{\psi} = 0 \quad \text{at} \quad y = 1 \]
is given by

\[
\bar{\Psi} = -\frac{4}{\pi R} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot \cos \left[ \frac{(2n+1)^2 \pi^2}{4PR} \right]
\]

(3.2.23)

Inverting (3.2.23), we get

\[
\psi(x,y) = -\frac{4x}{\pi R} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot \cos \left[ \frac{(2n+1)^2 \pi^2}{4PR} \right] \cdot \exp \left[ -\frac{(2n+1)^2 \pi^2}{4PR} \right].
\]

(3.2.24)

Thus

\[
\theta(x,y,t) = \theta_0 + \frac{4}{\pi} \left( t - \frac{x}{R} - \theta_0 \right) \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot \cos \left[ \frac{(2n+1)^2 \pi^2}{4PR} \right] \cdot X_n.
\]

(3.2.25)

where \( X_n = \exp \left[ -\frac{(2n+1)^2 \pi^2}{4PR} \right]. \)

**Discussion**

Equation (3.2.25) gives the temperature distribution, \( \theta(x,y,t) \), as a function of time and space in the form of an infinite series each term of which contains an exponential term in \( x \) except the first term, \( \theta_0 \), on the R.H.S. of (3.2.25), which is a constant.
This means that each mode of the temperature field
decays exponentially along the channel and this decay
is inversely proportional to $P$ and $R$. Therefore as the
Prandtl number and the Reynolds number increase in a given
flow, the decay decreases.
PROBLEM II

EXACT SOLUTION OF THE FORCED CONVECTION ENERGY EQUATION FOR TIME-WISE LINEAR VARIATION OF INLET TEMPERATURE IN A CIRCULAR PIPE

An exact solution of the transient forced convection energy equation of a viscous incompressible fluid with fully developed flow in a circular pipe is obtained in the present problem when the inlet temperature varies linearly with time. The Laplace transform technique has been used to obtain the solution and its interpretations for the case of laminar flows are given.

NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>temperature</td>
</tr>
<tr>
<td>$c_p$</td>
<td>specific heat at constant pressure</td>
</tr>
<tr>
<td>$a$</td>
<td>pipe radius</td>
</tr>
<tr>
<td>$k$</td>
<td>thermal conductivity</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$u$</td>
<td>velocity component in $\bar{y}$-direction</td>
</tr>
<tr>
<td>$\overline{u}$</td>
<td>average velocity</td>
</tr>
<tr>
<td>$\bar{v}, \phi, \bar{y}$</td>
<td>cylindrical polar coordinates ($\bar{y}$-flow direction)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>fluid density</td>
</tr>
<tr>
<td>$\nu$</td>
<td>kinematic coefficient of viscosity</td>
</tr>
</tbody>
</table>

* Submitted for publication in Indian J. Pure and Appl. Mathematics in revised form.
The study of unsteady forced convection heat transfer in tubes and ducts has recently become of greater importance in connection with the control of modern high performance heat transfer devices. Literature on thermal transient problems is limited but increasing. In the solutions of the problem of transient forced convection in laminar flow, it has generally been assumed that the inlet temperature of the fluid is constant across the flow with a specified timewise variation of wall temperature, wall heat flux or internal heat generation.

In the present problem a solution of the transient forced convection energy equation is obtained for laminar flow in a circular pipe under a prescribed boundary condition with an inlet temperature varying linearly with time.
1. FORMULATION OF THE PROBLEM

We consider the steady laminar flow of a viscous incompressible fluid through a circular pipe of radius $a$. The fluid entering the pipe has a temperature which is spatially uniform across the entrance section but varies linearly with time. Therefore we can write the inlet condition as

$$
\overline{T}(\bar{\nu}, 0, \overline{z}) = T_0 + T_1 \left( \frac{\overline{u} \overline{T}}{a^2} \right).
$$

(3.1.1)

The unsteady energy equation for a fully developed flow in a circular pipe is given by

$$
\frac{1}{\rho c_p} \frac{\partial \overline{T}}{\partial \bar{\nu}} + u \frac{\partial \overline{T}}{\partial \bar{\nu}} = \frac{k}{\rho c_p} \left[ \frac{\partial^2 \overline{T}}{\partial \bar{\nu}^2} + \frac{1}{\bar{\nu}} \frac{\partial \overline{T}}{\partial \bar{\nu}} \right].
$$

(3.1.2)

The inlet and the boundary conditions of the problem are as follows:

$$
\overline{T} = T_0 + T_1 \left( \frac{\overline{u} \overline{T}}{a^2} \right) \quad \text{when} \quad \bar{\nu} = 0,
$$

(3.1.3)

$$
\overline{T} \quad \text{is finite at} \quad \bar{\nu} = 0, \quad \overline{T} = \overline{T}_a \quad \text{at} \quad \bar{\nu} = a \left( \overline{T} > 0 \right).
$$

(3.1.4)

The equation (3.1.2) is subjected to the following restrictions:
(a) Fully developed laminar velocity profile in the circular pipe.
(b) Frictional dissipation of energy is negligible.
(c) Axial conduction is negligible as compared to bulk transport in the $\frac{y}{a}$ direction. This is a reasonable assumption when Peclet number exceeds 100.
(d) Fluid property variations are also neglected.
(e) Thermal resistance of the pipe wall is negligible.

Further, to simplify the method of analysis, the case of constant velocity will be treated here and for this purpose we substitute $\bar{u}$ (mean velocity) for the velocity profile in (3.1.2)

We now introduce the following non-dimensional quantities:

$$\theta = \frac{T - T_0}{T_1}, \quad \frac{y}{a} = \frac{\bar{y}}{a}, \quad r = \frac{\bar{r}}{a},$$

$$t = \frac{\omega \bar{t}}{a^2}, \quad \theta_0 = \frac{T_2 - T_0}{T_1}$$

Equation (3.1.2) then becomes

$$\frac{\partial \theta}{\partial t} + R \frac{\partial \theta}{\partial \frac{y}{a}} = \frac{1}{P} \left[ \frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} \right].$$

(3.1.5)
The inlet and the boundary conditions reduce to

\[ \theta = t \quad \text{when} \quad \gamma = 0, \quad (3.1.6) \]

\[ \theta \quad \text{is finite at} \quad \gamma = 0, \quad \theta = \theta_0 \quad \text{at} \quad \gamma = 1 \quad (t > 0). \quad (3.1.7) \]

2. SOLUTION

We now separate the foregoing problem into two as follows:

\[ \theta(\gamma, \gamma, t) = \theta_1(\gamma, \gamma) + \theta_2(\gamma, \gamma, t), \quad (3.2.1) \]

where \( \theta_1 \) and \( \theta_2 \) satisfy the following problems:

\[ PR \frac{\partial \theta_1}{\partial \gamma} = \left[ \frac{\partial^2 \theta_1}{\partial \gamma^2} + \frac{1}{\gamma} \frac{\partial \theta_1}{\partial \gamma} \right], \quad (3.2.2) \]

\[ \theta_1 = 0 \quad \text{when} \quad \gamma = 0, \quad (3.2.3) \]

\[ \theta_1 \quad \text{is finite at} \quad \gamma = 0, \quad \theta_1 = \theta_0 \quad \text{at} \quad \gamma = 1. \quad (3.2.4) \]

\[ \frac{\partial \theta_2}{\partial t} + R \frac{\partial \theta_2}{\partial \gamma} = \frac{1}{P} \left[ \frac{\partial^2 \theta_2}{\partial \gamma^2} + \frac{1}{\gamma} \frac{\partial \theta_2}{\partial \gamma} \right], \quad (3.2.5) \]

\[ \theta_2 = t \quad \text{when} \quad \gamma = 0, \quad (3.2.6) \]

\[ \theta_2 \quad \text{is finite at} \quad \gamma = 0, \quad \theta_2 = 0 \quad \text{at} \quad \gamma = 1 \quad (t > 0). \quad (3.2.7) \]

Let \[ \overline{\theta}_1 = \int_0^\infty e^{-\frac{\gamma}{R}} \theta_1 \, d\gamma. \quad (3.2.8) \]
Multiplying (3.2.2) and (3.2.4) by the kernel
\( e^{-qy} \)
of the Laplace transform and integrating w.r.t. \( y \) between 0, \( \infty \) and using (3.2.3), (3.2.8), we get

\[
\frac{d^2 \tilde{\theta}_1}{d\gamma^2} + \frac{1}{\gamma} \frac{d\tilde{\theta}_1}{d\gamma} - \frac{1}{\gamma} \phi \tilde{\theta}_1 = 0 ,
\]

(3.2.9)

where
\( \phi = PR \).

The solution of (3.2.9) under the boundary conditions

\( \tilde{\theta}_1 \) is finite at \( \gamma = 0 \), \( \tilde{\theta}_1 = \theta_0 \) at \( \gamma = 1 \)

is given by

\[
\tilde{\theta}_1 = \left[ \frac{\theta_0 I_0(\phi \gamma)}{\phi I_0(\phi)} \right] ,
\]

(3.2.10)

where \( I_0 \) is the modified Bessel function of the first kind of order zero.

Inverting (3.2.10), we get

\[
\theta_1(\gamma, \xi) = \theta_0 \left[ 1 - 2 \sum_{n=1}^{\infty} \frac{\frac{\frac{\theta_0}{\xi_n}}{I_0(\xi_n) \xi_n}}{J_0(\xi_n)} \exp(-\frac{\xi_n^2}{PR} \gamma) \right] ,
\]

where the \( \xi_n \) are the positive roots of \( J_0(\xi) = 0 \).

In obtaining \( \theta_2(\gamma, \xi, t) \) we assume that

\[
\theta_2(\gamma, \xi, t) = t \phi(\gamma, \xi) + \psi(\gamma, \xi) ,
\]

(3.2.11)
where the new temperature functions \( \phi \) and \( \psi \) satisfy the following problems:

\[
PR \frac{\delta \phi}{\delta \gamma} = \left[ \frac{\delta^2 \phi}{\delta \gamma^2} + \frac{1}{\gamma} \frac{\delta \phi}{\delta \gamma} \right], \quad (3.2.12)
\]

\[
\phi = 1 \quad \text{when} \quad \gamma = 0, \quad (3.2.13)
\]

\[
\phi \quad \text{is finite at} \quad \gamma = 0, \quad \phi = 0 \quad \text{at} \quad \gamma = 1. \quad (3.2.14)
\]

\[
PR \frac{\delta \psi}{\delta \gamma} + \gamma \phi = \left[ \frac{\delta^2 \psi}{\delta \gamma^2} + \frac{1}{\gamma} \frac{\delta \psi}{\delta \gamma} \right], \quad (3.2.15)
\]

\[
\psi = 0 \quad \text{when} \quad \gamma = 0, \quad (3.2.16)
\]

\[
\psi \quad \text{is finite at} \quad \gamma = 0, \quad \psi = 0 \quad \text{at} \quad \gamma = 1. \quad (3.2.17)
\]

Let

\[
\bar{\phi} = \int_0^\infty e^{-\phi \gamma} \phi \, d\gamma, \quad (3.2.18)
\]

\[
\bar{\psi} = \int_0^\infty e^{-\psi \gamma} \psi \, d\gamma. \quad (3.2.19)
\]

Multiplying (3.2.12) and (3.2.14) by \( e^{-\phi \gamma} \) and integrating w.r.t. \( \gamma \) between 0, \( \infty \) and using (3.2.13), (3.2.18), we get

\[
\frac{\delta^2 \bar{\phi}}{\delta \gamma^2} + \frac{1}{\gamma} \frac{\delta \bar{\phi}}{\delta \gamma} - \bar{\phi}^2 = -PR. \quad (3.2.20)
\]

The solution of (3.2.20) under the boundary conditions

\[
\bar{\phi} \quad \text{is finite at} \quad \gamma = 0, \quad \bar{\phi} = 0 \quad \text{at} \quad \gamma = 1.
\]
is given by

$$\overline{\phi} = \frac{1}{\beta} \left[ 1 - \frac{I_0(\beta \gamma)}{I_0(\beta)} \right]$$  \hspace{1cm} (3.2.21)$$

On inversion, we get

$$\phi(\gamma, \hat{\gamma}) = 2 \sum_{n=1}^{\infty} \frac{J_0(\lambda_n \gamma)}{\lambda_n J_1(\lambda_n)} \exp\left(-\frac{\lambda_n^2}{PR} \hat{\gamma}\right)$$  \hspace{1cm} (3.2.22)$$

Substituting (3.2.22) in (3.2.15) and then multiplying by $e^{\beta \gamma}$ and integrating w.r.t. $\hat{\gamma}$ between 0, $\infty$ and using (3.2.16), (3.2.19), we obtain

$$\frac{d^2 \overline{\psi}}{d\gamma^2} + \frac{1}{\gamma} \frac{d \overline{\psi}}{d\gamma} - \beta^2 \overline{\psi} = 2 \sum_{n=1}^{\infty} \frac{J_0(\gamma \lambda_n)}{\lambda_n J_1(\lambda_n)} \cdot \frac{1}{\left(\beta + \frac{\lambda_n^2}{PR}\right)}$$

The solution of the above equation under the boundary conditions

$$\overline{\psi} \text{ is finite at } \gamma = 0, \quad \overline{\psi} = 0 \text{ at } \gamma = 1$$

is given by

$$\overline{\psi} = -\frac{2}{R} \sum_{n=1}^{\infty} \frac{J_0(\lambda_n \gamma)}{\lambda_n J_1(\lambda_n)} \cdot \frac{1}{\left(\beta + \frac{\lambda_n^2}{PR}\right)^2}$$  \hspace{1cm} (3.2.23)$$
Inverting (3.2.23), we get

\[
\psi (\gamma, \tilde{z}) = - \frac{2}{R} \gamma \sum_{n=1}^{\infty} \frac{J_0 (d_n \gamma)}{\alpha_n J_1 (d_n)} \exp \left[ - \frac{\alpha_n^2}{PR} \tilde{z} \right].
\]  

(3.2.24)

Thus

\[
\Theta (\gamma, \tilde{z}, t) = \theta_0 + 2 (t - \frac{\gamma}{R} - \theta_0) \sum_{n=1}^{\infty} \frac{J_0 (d_n \gamma)}{\alpha_n J_1 (d_n)} X_n ,
\]  

(3.2.25)

where \(X_n = \exp \left[ - \frac{\alpha_n^2}{PR} \tilde{z} \right]\).

**DISCUSSION**

Equation (3.2.25) gives the temperature distribution, \(\Theta (\gamma, \tilde{z}, t)\), as a function of time and space in the form of an infinite series each term of which contains an exponential term in \(\tilde{z}\) except the first term, \(\theta_0\), on the R.H.S. of (3.2.25), which is a constant. This means that each mode of the temperature field decays exponentially along the circular pipe and this decay is inversely proportional to \(P\) and \(R\). Therefore as the Prandtl number and the Reynolds number increase in a given flow, the decay decreases.
PROBLEM XIX

EXACT SOLUTION OF THE TRANSIENT FORCED CONVECTION ENERGY EQUATION FOR TIMewise VARIATION OF INLET TEMPERATURE IN A CHANNEL.

An exact solution of the equation of transient forced convection for time varying inlet temperature of a viscous incompressible laminar forced convection heat transfer with fully developed flow in a channel bounded by two parallel flat plates has been obtained in the present problem. The Laplace transform technique has been used as the method of analysis.

NOMENCLATURE

\( T \) temperature
\( \kappa \) thermal conductivity
\( c_p \) specific heat at constant pressure
\( h \) half distance between parallel plates
\( \bar{v} \) time
\( u \) velocity component in \( \bar{x} \)-direction
\( \bar{u} \) average velocity
\( x, \bar{y} \) cartesian coordinates (\( \bar{x} \)-flow direction, \( \bar{y} \)-distance from channel centerline)

* Communicated for publication in Bulletin Mathematique, Romania
\( \rho \)  
**fluid density**

\( \mu \)  
**coefficient of viscosity**

\[ \omega = \frac{\mu}{\rho} \]

\( P \)  
**Prandtl number**  
\[ \text{Pr} = \frac{\mu C_p}{K} \]

\( R \)  
**Reynolds number**  
\[ \text{Re} = \frac{hU}{\nu} \]

\( \frac{\nu}{\text{Re}} = 0 \)  
**corresponds to channel centerline**

**INTRODUCTION**

The study of unsteady forced convection heat transfer in tubes and ducts has recently become of greater importance in connection with the control of modern high performance heat transfer devices. Literature on thermal transients is limited but increasing. In the solutions of the problem of transient forced convection in laminar flow, it has generally been assumed that the inlet temperature of the fluid is constant across the flow with a given timewise variation of wall temperature, wall heat flux or internal heat generation per unit volume. There are also some works on the thermal transient problems in heat exchangers.

The problem of transient forced convection heat transfer may be stated as follows: the temperature
distribution is to be obtained in the system at an arbitrary instant of time, given:

(i) The inlet temperature distribution as an arbitrary function of time and space.

(ii) Initial temperature distribution for \( x > 0 \) as an arbitrary function of time and space.

(iii) A prescribed boundary condition which may take many forms.

In the present problem the solution has been obtained for laminar flow in a channel bounded by two parallel flat plates under a given boundary condition with an inlet temperature varying sinusoidally with time.

1. FORMULATION OF THE PROBLEM

We consider a steady laminar flow in a parallel plate channel whose sides are separated by a distance \( 2h \). The fluid entering the channel has a temperature which is spatially uniform across the entrance section but varies sinusoidally with time. Therefore we can write the inlet condition as

\[
T(0, y, t) = T_0 + (\Delta T)_0 \sin \omega t, \tag{3.1.1}
\]
where $T_0$ is the cycle mean temperature, $(\Delta T)_o$ is the amplitude and $\omega$ is the inlet frequency.

The unsteady energy equation for a fully developed hydrodynamic flow in a parallel plate channel is given by

$$\left[ \frac{\partial T}{\partial \tau} + u \frac{\partial T}{\partial \xi} \right] = \frac{K}{\rho c_p} \frac{\partial^2 T}{\partial y^2} \tag{3.1.2}$$

The inlet and the boundary conditions of the problem are as follows:

$$\bar{T} = T_o + (\Delta T)_o \sin \omega \bar{\tau} \quad \text{when} \quad \bar{\tau} = 0 ;$$

$$\left( \frac{\partial \bar{T}}{\partial \bar{y}} \right)_{\bar{y}=0} = 0 , \quad \bar{T} = T_1 \quad \text{at} \quad \bar{y} = h \quad \bar{T} > 0 .$$

The system satisfying equation (3.1.2) is subjected to the following restrictions:

(a) Fully developed laminar velocity profile between the parallel plates.

(b) Frictional dissipation of energy is negligible.

(c) Axial conduction is negligible as compared to bulk transport in the $\bar{y}$-direction. This is a reasonable assumption when Péclet number is greater than 100.
(d) Fluid property variations are also neglected.
(e) Thermal resistance of the channel wall is negligible.

Further, to simplify the method of analysis the case of constant velocity will be considered here and for this purpose we substitute \( \bar{u} \) (= mean velocity) for the velocity profile in (3.1.2).

We now introduce the following dimensionless variables:

\[
\theta = \frac{T - T_0}{(\Delta T)_0}, \quad \chi = \frac{x}{h}, \quad \eta = \frac{y}{h},
\]

\[
t = \frac{\bar{u}t}{h^2}, \quad \omega = \frac{h^2 \bar{u}}{\bar{u}}, \quad \theta_0 = \frac{\bar{u} - T_0}{(\Delta T)_0}.
\]

Equation (3.1.2) then becomes

\[
\left[ \frac{\partial \theta}{\partial t} + \bar{u} \frac{\partial \theta}{\partial x} \right] = \frac{1}{P} \frac{\partial^2 \theta}{\partial y^2}.
\]

(3.1.3)

The inlet and the boundary conditions reduce to

\[
\theta = \sin \omega t \quad \text{when} \quad \chi = 0; \quad \theta = \theta_0 \quad \text{at} \quad \eta = 1 \quad (t > 0). \]

(3.1.4)

(3.1.5)
2. SOLUTION

We now separate the foregoing problem into two as follows:

\[ \theta(x, y, t) = \theta_1(x, y) + \theta_2(x, y, t), \]

(3.2.1)

where the new temperature functions satisfy the following problems:

\[ PR \frac{\partial \theta_1}{\partial x} = \frac{\partial^2 \theta_1}{\partial y^2}, \]

(3.2.2)

\[ \theta_1 = 0 \quad \text{when} \quad x = 0; \]

(3.2.3)

\[ \left( \frac{\partial \theta_1}{\partial y} \right)_{y=0} = 0, \quad \theta_1 = \theta_0 \quad \text{at} \quad y = 1. \]

(3.2.4)

\[ \left[ \frac{\partial \theta_2}{\partial t} + R \frac{\partial \theta_2}{\partial x} \right] = \frac{1}{P} \frac{\partial^2 \theta_2}{\partial y^2}, \]

(3.2.5)

\[ \theta_2 = \sin \omega t \quad \text{when} \quad x = 0; \]

(3.2.6)

\[ \left( \frac{\partial \theta_2}{\partial y} \right)_{y=0} = 0, \quad \theta_2 = 0 \quad \text{at} \quad y = 1 \quad (t > 0). \]

(3.2.7)

Now let

\[ \overline{\theta}_1 = \int_0^\infty e^{-px} \theta_1 dx. \]

(3.2.8)

Multiplying the differential equation (3.2.2) and the condition (3.2.4) by the kernel \( e^{px} \) of the Laplace transform and integrating w.r.t. \( x \) between 0, \( \infty \)
and using (3.2.3), (3.2.8), we get

\[ \frac{d^2 \theta_1}{dy^2} + p^2 \theta_1 = 0 , \]  

(3.2.9)

where

\[ p^2 = P R \frac{\theta}{\phi} . \]

The solution of (3.2.9) under the conditions

\[ \left( \frac{d \theta_1}{dy} \right)_{y=0} = 0 , \quad \theta_1 = \frac{\theta_0}{p} \quad \text{at } y=1 \]  

(3.2.10)

is given by

\[ \theta_1 = \left[ \frac{\theta_0 \cosh p y}{p \cosh p} \right] . \]  

(3.2.11)

The inversion theorem gives

\[ \theta_1(x, y) = \theta_0 \left[ 1 - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cdot \cos \left( \frac{(2n+1)\pi y}{2} \right) \cdot \exp \left[ - \frac{(2n+1)^2 \pi^2}{4 PR} x \right] \right] . \]  

(3.2.12)

In obtaining \( \theta_2(x, y, t) \) we define the following auxiliary problem:

\[ \frac{d \theta_2}{dt} + R \frac{d \theta_2}{dx} = \frac{1}{p} \frac{d^2 \theta_2}{dy^2} , \]  

(3.2.13)

\[ \theta_2' = \cos \omega t \quad \text{when} \quad x = 0 ; \]  

(3.2.14)

\[ \left( \frac{d \theta_2}{dy} \right)_{y=0} = 0 , \quad \theta_2' = 0 \quad \text{at } y=1 . \]  

(3.2.15)
Here we note that the auxiliary problem is similar to the original problem for \( \theta_2 \), except that the periodic condition is shifted by \( \frac{\pi}{2} \).

Let us define a new temperature function \( \theta_c(x,y,t) \) such that

\[
\theta_c = \theta_2' + i \theta_2 ,
\]

then the problems given by equations (3.2.5) and (3.2.13) can be combined to give the following problem:

\[
\frac{\partial \theta_c}{\partial t} + R \frac{\partial \theta_c}{\partial x} = \frac{1}{P} \frac{\partial^2 \theta_c}{\partial y^2} ,
\]

\[
\theta_c = e^{i \omega t} \quad \text{when} \quad x = 0 ;
\]

\[
(\frac{\partial \theta_c}{\partial y})_{y=0} = 0 , \quad \theta_c = 0 \quad \text{at} \quad y = 1 .
\]

We now assume a periodic solution of the following type

\[
\theta_c (x,y,t) = e^{i \omega t} \cdot \phi (x,y) .
\]

Substituting (3.2.20) into (3.2.17), we get

\[
i \omega \phi + R \frac{\partial \phi}{\partial x} = \frac{1}{P} \frac{\partial^2 \phi}{\partial y^2} .
\]

Inlet and boundary conditions for this problem become

\[
\phi = 1 \quad \text{when} \quad x = 0 ;
\]

\[
(\frac{\partial \phi}{\partial y})_{y=0} = 0 , \quad \phi = 0 \quad \text{at} \quad y = 1 .
\]
Let
\[ \overline{\phi} = \int_{0}^{\infty} e^{-\xi x} \phi(x) \, dx. \]  

Multiplying (3.2.21) by \( e^{\xi x} \) and integrating w.r.t. \( x \) between 0, \( \infty \) and using (3.2.22), (3.2.24), we get
\[ \frac{d^2 \overline{\phi}}{dy^2} - q^2 \overline{\phi} = -PR, \]  
where
\[ q^2 = PR \left( \frac{\rho}{R} + \frac{i \omega}{R} \right). \]

The solution of (3.2.25), with \( \frac{d \overline{\phi}}{dy} \bigg|_{y=0} = 0 \)
and \( \overline{\phi} = 0 \) at \( y=1 \), is
\[ \overline{\phi} = \frac{1}{(\rho + \frac{i \omega}{R})} \left[ 1 - \frac{\cosh qy}{\cosh q} \right]. \]

Inverting (3.2.26), we get
\[ \phi(x,y) = \frac{L}{\lambda} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left( \frac{2n+1}{a} \right) \pi y \cdot \exp \left\{ \left( \frac{(2n+1)^2}{4PR} - \frac{i \omega}{R} \right) x \right\}. \]

Therefore
\[ \Theta_2(x,y,t) = \frac{L}{\lambda} \sin \left( \omega t - \frac{\omega}{R} x \right) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left( \frac{2n+1}{a} \right) \pi y \cdot \exp \left\{ -\left( \frac{(2n+1)^2}{4PR} \right) x \right\}. \]
Thus

\[ \theta(x, y, t) = \theta_0 \left[ 1 - \frac{L}{K} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left( \frac{2n+1}{2} \right) \Gamma_y \cdot \exp \left\{ -\frac{(2n+1)x^2}{LPR} \right\} \right] \]

\[ + \frac{L}{K} \left[ \sin \left( \omega t - \omega \frac{PR}{X} \right) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left( \frac{2n+1}{2} \right) \Gamma_y \cdot \exp \left\{ -\frac{(2n+1)x^2}{LPR} \right\} \right] \]

(3.2.28)

\[ \Theta(x, y, t) \] gives the dimensionless temperature distribution between two parallel flat plates when the inlet temperature has been changed periodically. The solution represents the exact solution of energy equation for slug flow assumption.

CONCLUSIONS

The expression (3.2.28) gives the temperature distribution as a function of time and space in the form of infinite series each term of which includes an exponential term in \( x \). This means that each mode of temperature distribution decays exponentially along the channel and this decay is inversely proportional to \( P \) and \( R \). Therefore as Prandtl number and Reynolds number increase, decay decreases. It is also observed that phase lag along
the channel is linear and slope of this is \( \frac{\omega}{R} \). Thus, as the inlet frequency is increased, phase lag increases and as Reynolds number is increased, phase lag decreases. These are now being verified by experiment and the results will be reported very soon.
PROBLEM IV

EXACT SOLUTION OF THE TRANSIENT FORCED CONVECTION ENERGY EQUATION FOR TIMewise VARIATION OF INLET TEMPERATURE IN A CIRCULAR PIPE

NOMENCLATURE

\( T \) 

temperature

\( K \) 

thermal conductivity

\( c_p \) 

specific heat at constant pressure

\( a \) 

pipe radius

\( t \) 

time

\( u \) 

velocity component in \( \frac{2}{3} \) direction

\( \overline{u} \) 

average velocity

\( r, \phi, \frac{2}{3} \) 

cylindrical polar coordinates (\( \frac{2}{3} \) flow direction)

\( \rho \) 

fluid density

\( P \) 

Prandtl number \( \left( = \frac{\mu c_p}{K} \right) \)

\( R \) 

Reynolds number \( \left( = \frac{a \overline{u}}{\nu} \right) \)

\( \theta \) 

dimensional as temperature \( \left( = \frac{\overline{T} - T_o}{(\Delta T)_o} \right) \)

Meaning of any other symbols are given in the text as they occur.

* Accepted for publication in Indian J. Pure and Appl. Math.
INTRODUCTION

The study of unsteady forced convection heat transfer in tubes and ducts has recently become of greater importance in connection with the control of modern high performance heat transfer devices. Literature on thermal transients is limited but increasing. In the solutions of the problem of transient forced convection in laminar flow, it has generally been assumed that the inlet temperature of the fluid is constant across the flow with a specified time-wise variation of wall temperature, wall heat flux or internal heat generation per unit volume. There is also some work done on the thermal transient problems in heat exchangers.

The problem of transient forced convection heat transfer may be stated as follows: the temperature distribution is to be obtained in the system at an arbitrary instant of time, given:

(i) The inlet temperature distribution as an arbitrary function of time and space.

(ii) Initial temperature distribution for \( \frac{\partial}{\partial t} > 0 \) as an arbitrary function of time and space.

(iii) A prescribed boundary condition which may take many forms.
In the present problem a solution is obtained for laminar flow in a circular pipe under a prescribed boundary condition with an inlet temperature varying sinusoidally with time.

1. FORMULATION OF THE PROBLEM

We consider a steady laminar flow through the circular pipe. The fluid entering the pipe has a temperature which is spatially uniform across the entrance section but varies sinusoidally with time. Therefore we can write the inlet condition as

$$
T(\bar{v}, 0, \bar{z}) = T_0 + (\Delta T_0) \sin \bar{\omega} \bar{z},
$$

where $T_0$ is the cycle mean temperature, $(\Delta T_0)$ is the amplitude and $\bar{\omega}$ is the inlet frequency.

The unsteady energy equation for a fully developed hydrodynamic flow in a circular pipe is given by

$$
\frac{\partial T}{\partial \bar{z}} + u \frac{\partial T}{\partial \bar{y}} = \frac{K}{\rho c_p} \left[ \frac{3T}{\partial \bar{z}^2} + \frac{1}{\bar{y}} \frac{\partial T}{\partial \bar{y}} \right].
$$

The inlet and the boundary conditions of the problem are as follows:

$$
\bar{T} = T_0 + (\Delta T_0) \sin \bar{\omega} \bar{z}
$$

when $\bar{z} = 0$;
\[ T \text{ is finite at } \bar{y} = 0, \quad \bar{y} = a. \quad (\bar{T} > 0) \]

The system satisfying equation (3.1.2) is subjected to the following restrictions:

(a) Fully developed laminar velocity profile in the circular pipe.

(b) Frictional dissipation of energy is negligible.

(c) Axial conduction is negligible with respect to bulk transport in the \( \bar{y} \)-direction. This is a reasonable assumption when Peclet number exceeds 100.

(d) Fluid property variations are also neglected.

(e) Thermal resistance of the pipe wall is negligible.

Further, to simplify the method of analysis the case of constant velocity will be treated here and for this purpose we substitute \( \bar{u} \) (mean velocity) for the velocity profile in (3.1.2).

We now introduce the following non-dimensional quantities:
\[
\theta = \frac{T - T_0}{(\Delta T)_0}, \quad \frac{\theta}{a} = \frac{3}{a}, \quad \gamma = \frac{T}{a},
\]

\[
\omega = \frac{a\omega}{\omega}, \quad t = \frac{2t}{a^2}, \quad e_0 = \frac{T - T_0}{(\Delta T)_0}.
\]

Equation (3.1.2) then becomes

\[
\frac{d\theta}{dt} + R \frac{d\theta}{d\gamma} = \frac{1}{P} \left[ \frac{d^2\theta}{d\gamma^2} + \frac{1}{\gamma} \frac{d\theta}{d\gamma} \right].
\] (3.1.3)

The inlet and the boundary conditions reduce to

\[
\theta = \sin \omega t \quad \text{when} \quad \gamma = 0;
\] (3.1.4)

\[
\theta \text{ is finite at } \gamma = 0,
\]

\[
\theta = e_0 \text{ at } \gamma = 1. \quad \{ t > 0 \}
\] (3.1.5)

2. **SOLUTION**

We now separate the foregoing problem into

two as follows:

\[
\theta(\gamma, \gamma, t) = \theta_1(\gamma, \gamma) + \theta_2(\gamma, \gamma, t),
\] (3.2.1)

where the new temperature functions satisfy the following problems:
\[ PR \frac{\partial \theta_1}{\partial \gamma} = \left[ \frac{\partial^2 \theta_1}{\partial \gamma^2} + \frac{1}{\gamma} \frac{\partial \theta_1}{\partial \gamma} \right], \]  
(3.2.2)

\[ \theta_1 = 0 \text{ when } \gamma = 0; \]  
(3.2.3)

\[ \theta_1 \text{ is finite at } \gamma = 0, \]  
(3.2.4)

\[ \theta_1 = \theta_0 \text{ at } \gamma = 1. \]  
(3.2.5)

\[ \frac{\partial \theta_2}{\partial t} + R \frac{\partial \theta_2}{\partial \gamma} = \frac{1}{p} \left[ \frac{\partial^2 \theta_2}{\partial \gamma^2} + \frac{1}{\gamma} \frac{\partial \theta_2}{\partial \gamma} \right], \]  
(3.2.6)

\[ \theta_2 = \sin \omega t \text{ when } \gamma = 0; \]  
(3.2.7)

\[ \theta_2 \text{ is finite at } \gamma = 0, \]  
(3.2.8)

\[ \theta_2 = 0 \text{ at } \gamma = 1. \]  
(3.2.9)

Let

\[ \bar{\theta}_1 = \int_0^\infty e^{-\gamma \gamma} \theta_1 \cdot d\gamma. \]  
(3.2.10)

Multiplying the differential equation (3.2.2) and the condition (3.2.4) by the kernel \( e^{-\gamma \gamma} \) of the Laplace transform and integrating w.r.t. \( \gamma \) between 0, \( \infty \) and using (3.2.3), (3.2.8), we get

\[ \frac{\partial^2 \bar{\theta}_1}{\partial \gamma^2} + \frac{1}{\gamma} \frac{\partial \bar{\theta}_1}{\partial \gamma} - \beta^2 \bar{\theta}_1 = 0, \]  
(3.2.9)

where \( \beta^2 = PRb \).

The solution of (3.2.9) under the boundary conditions
\[ \overline{\theta}_1 \text{ is finite at } \gamma = 0, \]
\[ \overline{\theta}_1 = \frac{\theta_0}{P} \text{ at } \gamma = 1 \]  
\[ (3.2.10) \]

is
\[ \overline{\theta}_1 = \left[ \frac{\theta_0 \, I_0(\beta \gamma)}{\beta \, I_0(\beta)} \right]. \]
\[ (3.2.11) \]

The inversion theorem gives
\[ \theta_1(\gamma, \chi) = \theta_0 \left[ 1 - 2 \sum_{n=1}^{\infty} \frac{J_n(\chi \gamma)}{\lambda_n \, J_0(\lambda_n)} \cdot \exp \left( - \frac{\lambda_n^2}{PR \gamma} \right) \right], \]
\[ (3.2.12) \]
where the \( \lambda_n \, (n = 1, 2, 3, \ldots) \) are the positive roots of \( J_0(\lambda) = 0 \).

In obtaining \( \theta_2(\gamma, \chi, t) \) we define the following auxiliary problems:
\[ \frac{d\theta'_2}{dt} + R \frac{d\theta'_2}{d\chi} = \frac{1}{P} \left[ \frac{d^2\theta'_2}{d\chi^2} + \frac{1}{\chi} \frac{d\theta'_2}{d\chi} \right], \]
\[ (3.2.13) \]
\[ \theta'_2 = \cos \omega t \quad \text{when } \chi = 0; \]
\[ (3.2.14) \]
\[ \theta'_2 \text{ is finite at } \gamma = 0, \]
\[ \theta'_2 = 0 \text{ at } \gamma = 1, \]  
\[ (3.2.15) \]

Here we note that the auxiliary problem is similar to the original problem for \( \theta_2 \) except that the periodic condition is shifted by \( \frac{\pi}{2} \).
Let us define a new temperature function 
\[ \theta_c(\gamma, \jmath, t) \] such that
\[ \theta_c = \theta'_c + i \theta_2 \]

(3.2.16)

then the problems given by equations (3.2.5) and (3.2.13) can be combined to give the following problem:

\[ \frac{\partial \theta_c}{\partial t} + R \frac{\partial \theta_c}{\partial \jmath} = \frac{1}{P} \left[ \frac{\partial^2 \theta_c}{\partial \gamma^2} + \frac{1}{\jmath} \frac{\partial \theta_c}{\partial \gamma} \right] \]

(3.2.17)

\[ \theta_c = e^{i \omega t} \quad \text{when} \quad \gamma = 0 \]

(3.2.18)

\[ \theta_c \quad \text{is finite at} \quad \gamma = 0, \quad \gamma = 1 \]

(3.2.19)

We now assume a periodic solution of the following type

\[ \theta_c(\gamma, \jmath, t) = e^{i \omega t} \psi(\gamma, \jmath) \]

(3.2.20)

Introducing the definition given by equation (3.2.20) into equation (3.2.17), we get

\[ i \omega \psi + R \frac{\partial \psi}{\partial \jmath} = \frac{1}{P} \left[ \frac{\partial^2 \psi}{\partial \gamma^2} + \frac{1}{\jmath} \frac{\partial \psi}{\partial \gamma} \right] \]

(3.2.21)

Boundary and inlet conditions for this problem become

\[ \psi = 1 \quad \text{when} \quad \jmath = 0 \]

(3.2.22)
\[ \psi \text{ is finite at } \gamma = 0, \]
\[ \psi = 0 \text{ at } \gamma = 1. \]  \hfill (3.2.23)

Let
\[ \overline{\psi} = \int_0^\infty e^{-\gamma y} \psi(\gamma, y) \, dy . \]  \hfill (3.2.24)

Multiplying (3.2.21) by \( e^{\gamma y} \) and integrating v.r.t. \( y \) between 0, \( \infty \) and using (3.2.22), (3.2.24), we get
\[ \frac{d^2 \overline{\psi}}{dy^2} + \frac{1}{\gamma} \frac{d \overline{\psi}}{dy} - PR(\beta + i\omega R) \overline{\psi} = -PR . \]  \hfill (3.2.25)

The solution of (3.2.25), with \( \overline{\psi} = 0 \)
at \( \gamma = 1 \) and \( \overline{\psi} \) finite at \( \gamma = 0 \), is
\[ \overline{\psi} = \frac{1}{(\beta + i\omega R)} \left[ 1 - \frac{I_0(\gamma \omega)}{I_0(\gamma)} \right] , \]  \hfill (3.2.26)

where \( \gamma^2 = PR(\beta + i\omega R) \).

Inverting (3.2.26), we get
\[ \psi(\gamma, y) = 2 \sum_{n=1}^\infty \frac{J_0(\alpha_n \gamma)}{\alpha_n J_1(\alpha_n)} \exp \left[ -\left( \frac{\alpha_n^2}{PR} + \frac{i\omega}{R} \right) y \right] \]

Therefore
\[ \Theta_s(\gamma, \eta, t) = 2 \sin \left( \omega t - \frac{\omega \gamma}{R \eta} \right) \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \eta)}{\alpha_n J_1(\alpha_n)} \cdot \exp \left( -\frac{\alpha_n^2}{PR \eta} \right) \]  

(3.2.27)

Thus

\[ \Theta(\gamma, \eta, t) = \Theta_0 \left[ 1 - 2 \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \eta)}{\alpha_n J_1(\alpha_n)} \cdot \exp \left( -\frac{\alpha_n^2}{PR \eta} \right) \right] \]

\[ + 2 \sin \left( \omega t - \frac{\omega \gamma}{R \eta} \right) \sum_{n=1}^{\infty} \frac{J_0(\gamma \alpha_n)}{\alpha_n J_1(\alpha_n)} \cdot \exp \left( -\frac{\alpha_n^2}{PR \eta} \right) \]  

(3.2.28)

\[ \Theta(\gamma, \eta, t) \] gives the dimensionless temperature distribution in a circular pipe when the inlet temperature has been changed periodically. The solution represents the exact solution of energy equation for slug flow assumption.

**CONCLUSIONS**

The expression (3.2.28) gives the temperature distribution as a function of time and space in the form of infinite series, each term of which includes an exponential term in \( \eta \). This means that each mode of temperature distribution decays exponentially along the pipe and this decay is inversely proportional to \( P \) and \( R \). Therefore for a given flow regime as Prandtl number and Reynolds number increase, decay decreases. It is also
observed that phase lag along the pipe is linear and slope of this is \( \frac{\omega}{R} \). Thus, as the inlet frequency is increased, phase lag increases and as Reynolds number is increased, \( \frac{\omega}{R} \) decreases. These are now verified by experiment which is still under further investigations.
PROBLEM V

AN EXACT SOLUTION OF THE TRANSIENT FORCED CONVECTION ENERGY EQUATION FOR TIMELIKE LINEAR VARIATION OF INLET TEMPERATURE IN A CHANNEL

NOMENCLATURE

\( T \)  
\( c_p \)  
\( h \)  
\( k \)  
\( \bar{u} \)  
\( x, y \)  
\( u \)  
\( \rho \)  
\( \nu \)  
\( R \)  
\( P \)  
\( T_0, T_1, T_2 \)

temperature

specific heat at constant pressure

half distance between parallel plates

thermal conductivity

time

velocity component in \( \bar{x} \)-direction

mean velocity or average velocity

cartesian coordinates (\( \bar{x} \)-flow direction, \( \bar{y} \)-distance from channel centerline)

fluid density

kinematic coefficient of viscosity

Reynolds number \( \left( = \frac{h \bar{u}}{\nu} \right) \)

Prandtl number \( \left( = \frac{\rho \bar{u} c_p}{k} \right) \)

known constant temperatures

* Communicated for publication in Indian J. Pure and Appl. Math.
INTRODUCTION

The study of unsteady forced convection heat transfer in tubes and ducts has recently become of greater importance in connection with the control of modern high performance heat transfer devices. Literature on thermal transient problems is limited but increasing. In solutions of the problems of transient forced convection in laminar flow, it has usually been assumed that the inlet temperature of the fluid is constant across the flow with a specified time-wise variation of wall temperature, wall heat flux or internal heat generation.

In the present problem an exact solution of the transient forced convection energy equation of a viscous incompressible fluid with fully developed flow in a parallel plate channel is obtained under prescribed boundary conditions when the inlet temperature varies linearly with time and its interpretations for the case of laminar flows are given.

1. FORMULATION OF THE PROBLEM

We consider the steady laminar flow of a viscous incompressible fluid in a channel bounded by two parallel flat plates. The fluid entering the channel has a temperature which is spatially uniform across the entrance
section but varies linearly with time. Therefore we can write the inlet condition as

\[ \bar{T}(0, \bar{y}, \bar{t}) = T_0 + T_1 \left( \frac{\bar{u} \bar{t}}{h^2} \right) \]  \hspace{1cm} (3.1.1) \]

The unsteady energy equation for a fully developed flow in the present case is given by

\[ \frac{\partial \bar{T}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{T}}{\partial \bar{x}} = K \frac{\bar{c}_p}{\bar{c}_p} \left[ \frac{\partial^2 \bar{T}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} \right] \] \hspace{1cm} (3.1.2) \]

The inlet and the boundary conditions of the problem are as follows:

\[ \bar{T} = T_0 + T_1 \left( \frac{\bar{u} \bar{t}}{h^2} \right) \quad \text{when} \quad \bar{x} = 0 \]  \hspace{1cm} (3.1.3) \]

\[ \frac{\partial \bar{T}}{\partial \bar{y}} = 0 \quad \text{at} \quad \bar{y} = h \quad (\bar{t} > 0) \]  \hspace{1cm} (3.1.4) \]

The equation (3.1.2) is subjected to the following restrictions:

(a) Fully developed laminar velocity profile between the parallel plates.

(b) Frictional dissipation of energy is negligible.

(c) Fluid property variations are neglected.
(d) Thermal resistance of the channel wall is negligible.

Further, to simplify the method of analysis the case of constant velocity will be considered here and for this purpose we substitute \( \overline{u} \) for the velocity profile in (3.1.2).

We now introduce the following non-dimensional quantities:

\[
\theta = \frac{T - T_0}{T_1}, \quad \chi = \frac{x}{h}, \quad \gamma = \frac{y}{h},
\]

\[
\tau = \frac{\sqrt{T}}{h^2}, \quad \Theta_0 = \frac{T_0 - T_0}{T_1}.
\]

Equation (3.1.2) then becomes

\[
\frac{\partial \theta}{\partial \tau} + R \frac{\partial \theta}{\partial \chi} = \frac{1}{P} \left[ \frac{\partial^2 \theta}{\partial \chi^2} + \frac{\partial^2 \theta}{\partial \gamma^2} \right].
\]  

(3.1.5)

The inlet and the boundary conditions reduce to

\[
\theta = \tau \quad \text{when} \quad \chi = 0,
\]

(3.1.6)

\[
\left( \frac{\partial \theta}{\partial \gamma} \right)_{\gamma=0} = 0, \quad \Theta = \Theta_0 \quad \text{at} \quad \gamma = 1 \quad (\tau > 0).
\]

(3.1.7)
2. Solution

We separate the above problem into two as follows:

\[ \Theta(x, y, t) = \Theta_1(x, y) + \Theta_2(x, y, t), \quad (3.2.1) \]

where \( \Theta_1 \) and \( \Theta_2 \) satisfy the following problems:

\[ P \frac{\partial \Theta_1}{\partial x} = \left[ \frac{\partial^2 \Theta_1}{\partial x^2} + \frac{\partial^2 \Theta_1}{\partial y^2} \right] \]

\[ \Theta_1 = 0 \quad \text{when} \quad x = 0, \]

\[ \left( \frac{\partial \Theta_1}{\partial y} \right)_{y=0} = 0, \quad \Theta_1 = \Theta_0 \quad \text{at} \quad y = 1. \quad (3.2.2) \]

\[ \frac{\partial \Theta_2}{\partial t} + R \frac{\partial \Theta_2}{\partial x} = \frac{1}{P} \left[ \frac{\partial^2 \Theta_2}{\partial x^2} + \frac{\partial^2 \Theta_2}{\partial y^2} \right], \quad (3.2.3) \]

\[ \Theta_2 = t \quad \text{when} \quad x = 0, \quad (3.2.4) \]

\[ \left( \frac{\partial \Theta_2}{\partial y} \right)_{y=0} = 0, \quad \Theta_2 = 0 \quad \text{at} \quad y = 1 \quad (t > 0). \quad (3.2.5) \]

Solving (3.2.2), we get
\[
\Theta_1(x, y) = \Theta_0 \left[ 1 - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left( \frac{2n+1}{2} \pi y \right) \exp(-\lambda_n x) \right],
\]

where

\[
2 \lambda_n = \left[ \sqrt{P^2 R^2 + (2n+1)^2 \pi^2} - PR \right].
\]

In obtaining \( \Theta_2(x, y, t) \) we assume that

\[
\Theta_2(x, y, t) = t \phi(x, y) + \psi(x, y), \quad (3.2.6)
\]

where the new temperature functions \( \phi \) and \( \psi \) satisfy the following problems:

\[
P R \frac{\partial \phi}{\partial x} = \left[ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right],
\]

\[
\phi = 1 \quad \text{when} \quad x = 0,
\]

\[
(\frac{\partial \phi}{\partial y})_{y=0} = 0, \quad \phi = 0 \quad \text{at} \quad \psi = 1. \quad (3.2.7)
\]

\[
P R \frac{\partial \psi}{\partial x} + P \phi = \left[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right],
\]

\[
\psi = 0 \quad \text{when} \quad x = 0, \quad (3.2.9)
\]
\[
\left( \frac{d\psi}{dy} \right)_{y=0} = 0, \quad \psi = 0 \text{ at } y = 1.
\] (3.2.10)

The solution of (3.2.7) is given by

\[
\phi(x, y) = \frac{L_1}{\Lambda} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left( \frac{2n+1}{2} \right) \cdot \exp\left( -\lambda_n x \right).
\] (3.2.11)

Then the solution of (3.2.8) under the boundary conditions (3.2.9) and (3.2.10) is given by

\[
\psi(x, y) = -\frac{L_1}{\Lambda} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \frac{\cos \left( \frac{2n+1}{2} \right) \cdot \exp\left( -\lambda_n x \right)}{\sqrt{(2n+1)^2 \Lambda^2 + P^2 R^2}}.
\]

Thus

\[
\Theta(x, y, t) = \Theta_0 \left[ 1 - \frac{L_1}{\Lambda} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left( \frac{2n+1}{2} \right) \cdot \exp\left( -\lambda_n x \right) \right]
\]

\[
+ \frac{4t}{\Lambda} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left( \frac{2n+1}{2} \right) \cdot \exp\left( -\lambda_n x \right)
\]

\[
- \frac{L_1}{\Lambda} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \frac{\cos \left( \frac{2n+1}{2} \right) \cdot \exp\left( -\lambda_n x \right)}{\sqrt{(2n+1)^2 \Lambda^2 + P^2 R^2}} \cdot \cos \left( \frac{2n+1}{2} \right) \cdot \exp\left( -\lambda_n x \right).
\] (3.2.12)
\[ \Theta(x, y, t) \] gives the dimensionless temperature distribution between two parallel flat plates when the inlet temperature varies linearly with time. The solution represents the exact solution of energy equation for slug flow assumption.

3. DISCUSSION

When the boundary condition on the wall for \[ \Theta(x, y, t) \] is homogeneous, that is, when \( \Theta_0 \) is zero, then \( \Theta_1(x, y) \) is identically zero and in that case we have

\[ \Theta(x, y, t) = \Theta_2(x, y, t) \]  \quad (3.3.1)

\( \Theta_2(x, y, t) \) shows that each mode of the temperature field decays exponentially along the channel.

In many applications heat transfer in regions away from the inlet is of interest; for such situations only the first term in the series need to be considered and from the equation (3.3.1), we then get

\[ \Theta(x, y, t) = \frac{4}{\pi} \left[ t - \frac{P x}{\sqrt{P^2 R^2 + \pi^2}} \right] \cos\left(\frac{\pi y}{2}\right) \exp\left(-\lambda_0 x\right) \]  \quad (3.3.2)
where \[ 2\lambda_0 = \left[ \sqrt{P^2 R^2 + \pi^2} - PR \right]. \]

Then temperature at any \(y\), say \(y = 0\), is given by

\[ \Theta(x, 0, t) = \frac{4}{\pi} \left[ t - \frac{P x}{\sqrt{P^2 R^2 + \pi^2}} \right] \exp(-\lambda_0 x). \]  \hspace{1cm} (3.3.3)

\(\Theta(x, 0, t)\) at various points along the channel has been presented in graphical form (Fig. given below) for various values of Reynolds number when \(P = 0.73\), \(t = 2\). From the fig. we observe that \(\Theta(x, 0, t)\) increases as Reynolds number increases.
PROBLEM VI

FORCED CONVECTION HEAT TRANSFER FOR TIME-WISE LINEAR VARIATION OF INLET TEMPERATURE IN A CIRCULAR PIPE

In the present problem an exact solution of the transient forced convection energy equation of a viscous incompressible fluid with fully developed flow in a circular pipe is obtained under a prescribed boundary condition when the inlet temperature varies linearly with time, and its interpretations for the case of laminar flows are given.

NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{T}$</td>
<td>temperature</td>
</tr>
<tr>
<td>$c_p$</td>
<td>specific heat at constant pressure</td>
</tr>
<tr>
<td>$a$</td>
<td>pipe radius</td>
</tr>
<tr>
<td>$k$</td>
<td>thermal conductivity</td>
</tr>
<tr>
<td>$\tau$</td>
<td>time</td>
</tr>
<tr>
<td>$u$</td>
<td>velocity component in $\tilde{y}$ -direction</td>
</tr>
<tr>
<td>$\bar{u}$</td>
<td>mean velocity or average velocity</td>
</tr>
<tr>
<td>$\tilde{r}, \tilde{\phi}, \tilde{y}$</td>
<td>cylindrical polar coordinates ($\tilde{y}$ -flow direction)</td>
</tr>
</tbody>
</table>

* Communicated for publication in Appl. Sci. Research
1. FORMULATION OF THE PROBLEM

We consider the steady laminar flow of a viscous incompressible fluid through a circular pipe of radius $a$. The fluid entering the pipe has a temperature which is spatially uniform across the entrance section but varies linearly with time. Therefore we can write the inlet condition as

$$\bar{T}(\nabla, \sigma, \frac{t}{a}) = T_0 + T_1 \left( \frac{u \frac{t}{a^2}} \right). \quad (3.1.1)$$

The unsteady energy equation for a fully developed flow in the present case is given by

$$\frac{d\bar{T}}{dt} + u \frac{d\bar{T}}{\partial \gamma} = \frac{K}{\rho c_p} \left[ \frac{d^2 \bar{T}}{\partial \gamma^2} + \frac{1}{\gamma} \frac{d\bar{T}}{\partial \gamma} + \frac{d^2 \bar{T}}{\partial \gamma^2} \right]. \quad (3.1.2)$$

The inlet and the boundary conditions of the problem are as follows:
\[ T = T_0 + T_1 \left( \frac{u_1}{a^2} \right) \quad \text{when } \overline{y} = 0 \, , \quad (3.1.3) \]

\[ T = \text{finite at } \overline{v} = 0, \quad T = T_2 \text{ at } \overline{v} = a \, (T > 0). \quad (3.1.4) \]

The equation (3.1.2) is subjected to the following restrictions:

(a) Fully developed laminar velocity profile in the pipe.

(b) Frictional dissipation of energy, fluid property variations and thermal resistance of the pipe wall are negligible.

Further, to simplify the method of analysis the case of constant velocity will be considered here and for this purpose we substitute \( \overline{u} \) for the velocity profile in (3.1.2)

We now introduce the following non-dimensional quantities:

\[ \Theta = \frac{T - T_0}{T_1} \, , \quad \gamma = \frac{\overline{v}}{a} \, , \quad \overline{z} = \frac{\overline{y}}{a} \, , \]

\[ t = \frac{\overline{u} \overline{t}}{a^2} \, , \quad \Theta_0 = \frac{T_2 - T_0}{T_1} \, . \]
Equation (3.1.2) then becomes

\[
\frac{d\theta}{dt} + R \frac{d\theta}{d\gamma} = \frac{1}{\rho} \left[ \frac{d^2\theta}{d\gamma^2} + \frac{1}{\gamma} \frac{d\theta}{d\gamma} + \frac{d\theta}{d\gamma^2} \right].
\] (3.1.5)

The inlet and the boundary conditions reduce to

\[
\theta = t \quad \text{when} \quad \gamma = 0,
\] (3.1.6)

\[
\theta = \text{finite at} \quad \gamma = 0, \quad \theta = \theta_0 \quad \text{at} \quad \gamma = 1 \quad (t > 0).
\] (3.1.7)

2. Solution

We separate the above problem into two as follows:

\[
\theta(\gamma, \gamma, t) = \theta_1(\gamma, \gamma) + \theta_2(\gamma, \gamma, t),
\] (3.2.1)

where \( \theta_1 \) and \( \theta_2 \) satisfy the following problems:

\[
\rho R \frac{d\theta_1}{d\gamma} = \left[ \frac{1}{\gamma^2} \frac{d^2\theta_1}{d\gamma^2} + \frac{1}{\gamma} \frac{d\theta_1}{d\gamma} + \frac{d\theta_1}{d\gamma^2} \right],
\] (3.2.2)

\[
\theta_1 = 0 \quad \text{when} \quad \gamma = 0,
\]

\[
\theta_1 = \text{finite at} \quad \gamma = 0, \quad \theta_1 = \theta_0 \quad \text{at} \quad \gamma = 1.
\]

\[
\frac{d\theta_2}{dt} + R \frac{d\theta_2}{d\gamma} = \frac{1}{\rho} \left[ \frac{d^2\theta_2}{d\gamma^2} + \frac{1}{\gamma} \frac{d\theta_2}{d\gamma} + \frac{d\theta_2}{d\gamma^2} \right],
\] (3.2.3)
\( \theta_2 = t \quad \text{when} \quad \bar{y} = 0 \), \hspace{1cm} (3.2.4)

\( \theta_2 = \text{finite at} \ \bar{y} = 0, \ \theta_2 = 0 \ \text{at} \ \bar{r} = 1 \quad (t > 0) \). \hspace{1cm} (3.2.5)

Solving (3.2.2), we get

\[
\theta_1(\bar{r}, \bar{y}) = \theta_0 \left[ 1 - 2 \sum_{n=1}^{\infty} \frac{J_0(\lambda_n \bar{r})}{\lambda_n J_1(\lambda_n)} \cdot e^{-\lambda_n \bar{y}} \right],
\]

where \( 2 \lambda_n = \left[ \frac{R^2}{4 \lambda_n^2 + PR - PR} \right] \) and the \( \lambda_n \) \( (n = 1, 2, 3, \ldots) \)

are the positive roots of \( J_1(\lambda) = 0 \).

In obtaining \( \theta_2(\bar{r}, \bar{y}, t) \) we assume that

\[
\theta_2(\bar{r}, \bar{y}, t) = t \phi(\bar{r}, \bar{y}) + \psi(\bar{r}, \bar{y}) \], \hspace{1cm} (3.2.6)

where the new temperature functions \( \phi \) and \( \psi \) satisfy the following problems:

\[
P R \frac{\partial \phi}{\partial \bar{y}} = \left[ \frac{\partial^2 \phi}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial \phi}{\partial \bar{r}} + \frac{\partial^2 \phi}{\partial \bar{y}^2} \right],
\]

\( \phi = 1 \quad \text{when} \quad \bar{y} = 0, \)

\( \phi = \text{finite at} \ \bar{r} = 0, \ \phi = 0 \ \text{at} \ \bar{r} = 1. \) \hspace{1cm} (3.2.7)
\[ P R \frac{\partial \psi}{\partial \varphi} + P \phi = \left[ \frac{\delta^2 \psi}{\delta r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\delta^2 \psi}{\delta \varphi^2} \right], \quad (3.2.8) \]

\[ \psi = 0 \quad \text{when} \quad \varphi = 0, \quad (3.2.9) \]

\[ \psi = \text{finite at} \quad r = 0, \quad \psi = 0 \quad \text{at} \quad r = 1. \quad (3.2.10) \]

The solution of (3.2.7) is given by

\[ \phi(y, \varphi) = 2 \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)}{\alpha_n J_1(\alpha_n)} \cdot e^{-\lambda n \varphi} \quad (3.2.11) \]

Then the solution of (3.2.6) under the conditions (3.2.9) and (3.2.10) is

\[ \psi(y, \varphi) = -2 P \gamma \sum_{n=1}^{\infty} \frac{1}{\sqrt{p^2 + \frac{4 \alpha_n^2}{\varphi}}} \cdot \frac{J_0(\alpha_n r)}{\alpha_n J_1(\alpha_n)} \cdot e^{-\lambda n \varphi} \]

Thus

\[ \Theta(y, \varphi, t) = \Theta_0 \left[ 1 - 2 \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)}{\alpha_n J_1(\alpha_n)} \cdot e^{-\lambda n \varphi} \right] \]

\[ + 2t \sum_{n=1}^{\infty} \frac{J_0(\alpha_n r)}{\alpha_n J_1(\alpha_n)} \cdot e^{-\lambda n \varphi} \]

\[ - 2 P \gamma \sum_{n=1}^{\infty} \frac{1}{\sqrt{p^2 + \frac{4 \alpha_n^2}{\varphi}}} \cdot \frac{J_0(\alpha_n r)}{\alpha_n J_1(\alpha_n)} \cdot e^{-\lambda n \varphi} \quad (3.2.12) \]
\( \theta(y, z, t) \) gives the dimensionless temperature distribution in the pipe when the inlet temperature varies linearly with time. The solution represents the exact solution of energy equation for slug flow assumption.

3. DISCUSSION

When the boundary condition on the wall for \( \theta(y, z, t) \) is homogeneous, that is, when \( \theta_0 \) is zero, then \( \theta_1(y, z) \) is identically zero and in that case we have

\[
\theta(y, z, t) = \theta_2(y, z, t). \tag{3.3.1}
\]

\( \theta_2(y, z, t) \) shows that each mode of the temperature field decays exponentially along the pipe.

In many applications heat transfer in regions away from the inlet is of interest; for such situations only the first term in the series need to be considered and from the equation (3.3.1), we then get

\[
\theta(y, z, t) = 2 \left[ t - \frac{P^2 y}{\sqrt{P^2 R^2 + 4 \omega^2}} \right] \cdot \frac{J_0(\lambda_1 y)}{\lambda_1 J_1(\lambda_1)} \cdot \frac{e^{-\lambda_1 y}}{J_1(\lambda_1)}, \tag{3.3.2}
\]

where

\[
2\lambda_1 = \left[ \frac{P^2 R^2 + 4 \omega^2}{\sqrt{P^2 R^2 + 4 \omega^2}} - PR \right].
\]
Then temperature at any $\gamma$, say $\gamma = 0$, is given by

$$
\theta(0, \gamma, t) = 2 \left[ t - \frac{P \gamma}{\sqrt{P^2 R^2 + 4 \omega_1^2}} \right] \frac{1}{\alpha_1 J_1(\alpha_1)} e^{-\lambda_1 \gamma} \tag{3.3.3}
$$

$\theta(0, \gamma, t)$ at various points along the pipe has been presented in graphical form (Fig. given below) for $R = 100$, $1000$, $10000$ when $P = 0.73$ and $t = 2$. From the fig. we observe that $\theta(0, \gamma, t)$ increases as Reynolds number increases.
PROBLEM VII

EXACT SOLUTION OF THE FORCED CONVECTION ENERGY EQUATION FOR TRANSIENT VARIATION OF INLET TEMPERATURE IN A CIRCULAR PIPE

In the present problem an exact solution of the transient forced convection energy equation of a viscous incompressible fluid with fully developed flow in a circular pipe is obtained under a prescribed boundary condition when the inlet temperature varies sinusoidally with time, and its interpretations for the case of laminar flows are given.

NOMENCLATURE

$\bar{T}$ temperature
$K$ thermal conductivity
$c_p$ specific heat at constant pressure
$a$ pipe radius
$t$ time
$u$ velocity component in $\bar{z}$ direction
$\bar{u}$ average velocity
$\bar{r}, \phi, \bar{z}$ cylindrical polar coordinates ($\bar{z}$ - flow direction)

* Communicated for publication in Proc. Indian Acad. Sciences, Bangalore.
P  fluid density

P  Prandtl number \( \left( \frac{\mu c_p}{k} \right) \)

R  Reynolds number \( \left( \frac{a u}{\nu} \right) \)

1. FORMULATION OF THE PROBLEM

We consider a steady laminar flow through the circular pipe. The fluid entering the pipe has a temperature which is spatially uniform across the entrance section but varies sinusoidally with time. Therefore we can write the inlet condition as

\[
T(\nabla, 0, t) = T_0 + (\Delta T)_0 \sin \omega t,
\]

where \( T_0 \) is the cycle mean temperature, \( (\Delta T)_0 \) is the amplitude and \( \omega \) is the inlet frequency.

The unsteady energy equation for a fully developed hydrodynamic flow in a circular pipe is given by

\[
\frac{dT}{dt} + u \frac{dT}{dy} = \frac{K}{\rho c_p} \left[ \frac{\nabla T}{\nabla^2} + \frac{T}{\nabla^2 T} + \frac{\partial T}{\partial y} \right].
\]

The inlet and the boundary conditions of the problem are as follows:

\[
T = T_0 + (\Delta T)_0 \sin \omega t \quad \text{when} \quad y = 0,
\]
\( \tau \) is finite at \( \tau = 0 \), \( \tau = T \), at \( \tau = a \) \((3.1.4)\)

The equation \((3.1.2)\) is subjected to the following restrictions:

(a) Fully developed laminar velocity profile in the circular pipe.

(b) Frictional dissipation of energy, liquid property variations and thermal resistance of the pipe wall are negligible.

Further, to simplify the method of analysis the case of constant velocity will be considered here and for this purpose we substitute \( \bar{u} \) for the velocity profile in \((3.1.2)\).

We now introduce the following non-dimensional quantities:

\[
\theta = \frac{T - T_0}{(\Delta T)_0}, \quad \bar{y} = \frac{\bar{y}}{a}, \quad \gamma = \frac{\bar{y}}{a},
\]

\[
omega = \frac{\bar{a} \bar{\omega}}{u}, \quad \bar{t} = \frac{\bar{u} \bar{t}}{a^2}, \quad \theta_0 = \frac{T_1 - T_0}{(\Delta T)_0}.
\]

Equation \((3.1.2)\) then becomes

\[
\frac{1}{\bar{t}} \frac{d\theta}{dt} + R \frac{d\theta}{d\bar{y}} = \frac{1}{P} \left[ \frac{\bar{t}^2 \theta}{\bar{y}^2} + \frac{\bar{t} \theta}{\bar{y} \gamma} + \frac{d^2 \theta}{d\gamma^2} \right]. \quad (3.1.5)
\]
The inlet and the boundary conditions reduce to

\[ \theta = \sin \omega t \quad \text{when} \quad \gamma = 0, \quad (3.1.6) \]

\( \theta \) is finite at \( r = 0, \theta = \theta_0 \) at \( r = 1 \) \( (t > 0) \). \( (3.1.7) \)

2. SOLUTION

We separate the above problem into two as follows:

\[ \theta(r, \gamma, t) = \theta_1(r, \gamma) + \theta_2(r, \gamma, t), \quad (3.2.1) \]

where \( \theta_1 \) and \( \theta_2 \) satisfy the following problems:

\[ PR \frac{d\theta_1}{d\gamma} = \begin{bmatrix} \frac{d^2 \theta_1}{dr^2} + \frac{1}{r} \frac{d\theta_1}{dr} + \frac{d^2 \theta_1}{d\gamma^2} \end{bmatrix}, \quad (3.2.2) \]

\( \theta_1 = 0 \) when \( \gamma = 0, \theta_1 = \theta_0 \) at \( r = 1 \).

\[ \frac{d\theta_2}{dt} + R \frac{d^2 \theta_2}{d\gamma^2} = \frac{1}{P} \begin{bmatrix} \frac{d^2 \theta_2}{dr^2} + \frac{1}{r} \frac{d\theta_2}{dr} + \frac{d^2 \theta_2}{d\gamma^2} \end{bmatrix}, \quad (3.2.3) \]

\( \theta_2 = \sin \omega t \) when \( \gamma = 0, \quad (3.2.4) \)

\( \theta_2 \) is finite at \( r = 0, \theta_2 = 0 \) at \( r = 1 \) \( (t > 0) \). \( (3.2.5) \)
Solving (3.2.2), we get

$$\Theta_1(\gamma, \tilde{\gamma}) = \Theta_0 \left[ 1 - 2 \sum_{n=1}^{\infty} \frac{J_0(\lambda_n \gamma)}{\lambda_n J_1(\lambda_n)} \exp(-\lambda_n \tilde{\gamma}) \right].$$

where

$$2 \lambda_n = \sqrt{p^2 + 4 \lambda_n^2} - \frac{\lambda_n R}{p}$$

and $$\lambda_n$$ are the positive roots of $$J_0(\lambda) = 0$$.

In obtaining $$\Theta_2(\gamma, \tilde{\gamma}, t)$$ we define the following auxiliary problem:

$$\frac{d \Theta_2}{dt} + R \frac{d \Theta_2}{d \gamma} = \frac{1}{P} \left[ \frac{d^2 \Theta_2}{d \gamma^2} + 1 \frac{d \Theta_2}{d \gamma} + \frac{d^2 \Theta_2}{d \tilde{\gamma}^2} \right], \quad (3.2.6)$$

$$\Theta_2' = \cos \omega t \quad \text{when} \quad \tilde{\gamma} = 0, \quad (3.2.7)$$

$$\Theta_2' \text{ is finite at } \gamma = 0, \quad \Theta_2 = 0 \text{ at } \gamma = 1. \quad (3.2.8)$$

Let us define a new temperature function

$$\Theta_c(\gamma, \tilde{\gamma}, t) \quad \text{such that}$$

$$\Theta_c = \Theta_2' + i \Theta_2, \quad (3.2.9)$$

then the problems given by (3.2.3) and (3.2.6) can be combined to give the following problem:

$$\frac{d \Theta_c}{dt} + R \frac{d \Theta_c}{d \gamma} = \frac{1}{P} \left[ \frac{d^2 \Theta_c}{d \gamma^2} + 1 \frac{d \Theta_c}{d \gamma} + \frac{d^2 \Theta_c}{d \tilde{\gamma}^2} \right], \quad (3.2.10)$$
\[ \theta_c = e^{i\omega t} \quad \text{when} \quad \gamma = 0 \quad , \quad (3.2.11) \]

\[ \theta_c \quad \text{is finite at} \quad \gamma = 0 \quad , \quad \theta_c = 0 \quad \text{at} \quad \gamma = 1 \quad . \quad (3.2.12) \]

We now assume a periodic solution of the following type:

\[ \theta_c(\gamma, \gamma, t) = e^{i\omega t} \phi(\gamma, \gamma) \quad . \quad (3.2.13) \]

Introducing (3.2.13) into (3.2.10), we get

\[ i\omega \phi + R \frac{d\phi}{d\gamma} = \frac{1}{p} \left[ \frac{d^2\phi}{d\gamma^2} + \frac{1}{\gamma} \frac{d\phi}{d\gamma} + \frac{d^2\phi}{d\gamma^2} \right] \quad , \quad (3.2.14) \]

\[ \phi = 1 \quad \text{when} \quad \gamma = 0 \quad , \quad (3.2.15) \]

\[ \phi \quad \text{is finite at} \quad \gamma = 0 \quad , \quad \phi = 0 \quad \text{at} \quad \gamma = 1 \quad . \quad (3.2.16) \]

The solution of (3.2.14) under the conditions (3.2.15) and (3.2.16) is given by

\[ \phi(\gamma, \gamma) = 2 \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \gamma)}{\alpha_n J_1(\alpha_n)} \exp \left[ -\left( \frac{\alpha_n - \beta \alpha_n^2 + i \alpha_n^2}{2} \right)^2 \right] \quad , \]
where
\[
\sqrt{2} X_n = \left[ \frac{P^2 R^2 + 8 P^2 R \alpha_n + 16 \alpha_n^2 + 16 \omega^2 P^2}{\sqrt{P^4 R^2 + 8 P^2 R \alpha_n + 16 \alpha_n^2 + 16 \omega^2 P^2 + (P^2 R^2 + 4 \alpha_n^2)}} \right]^{1/2}
\]
\[
\sqrt{2} Y_n = \left[ \frac{P^4 R^2 + 8 P^2 R \alpha_n + 16 \alpha_n^2 + 4 \omega^2 P^2}{\sqrt{P^4 R^2 + 8 P^2 R \alpha_n + 16 \alpha_n^2 + 16 \omega^2 P^2 - (P^2 R^2 + 4 \alpha_n^2)}} \right]^{1/2}
\]

Therefore
\[
\theta_2(\gamma, \gamma, t) = 2 \sum_{n=1}^{\infty} \frac{J_6(\alpha_n \gamma)}{\alpha_n J_1(\alpha_n)} \cdot \exp \left[ -\frac{(X_n - PR \gamma)}{2} \right] \cdot \sin(\omega t - \frac{Y_n \gamma}{2})
\]

Thus
\[
\theta(\gamma, \gamma, t) = \theta_0 \left[ 1 - 2 \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \gamma)}{\alpha_n J_1(\alpha_n)} \cdot \exp(-\lambda_n \gamma) \right]
\]

\[
+ 2 \sum_{n=1}^{\infty} \frac{J_0(\alpha_n \gamma)}{\alpha_n J_1(\alpha_n)} \cdot \exp \left[ -\frac{(X_n - PR \gamma)}{2} \right] \cdot \sin(\omega t - \frac{Y_n \gamma}{2})
\]

2. DISCUSSION

When the boundary condition on the wall for
\[
\theta(\gamma, \gamma, t)
\]

is homogeneous, that is, when \( \theta_0 \) is zero,
then \( \theta_1(\gamma, \gamma) \) is identically zero and in that case we have

\[
\theta(\gamma, \gamma, t) = \theta_2(\gamma, \gamma, t)
\]

(3.3.1)
\( \Theta_2(\gamma, \gamma, t) \) shows that each mode of temperature decays exponentially along the pipe.

In many applications heat transfer in regions away from the inlet is of interest; for such situations only the first term in the series need to be considered and from the equation (3.3.1), we then get

\[
\Theta(\gamma, \gamma, t) = 2 \frac{J_0(\alpha_1 \gamma)}{\alpha_1 J_1(\alpha_1)} \sin(\omega t - \frac{\gamma_1}{2} \gamma) \exp \left[ -\left( \frac{X_1 - PR}{2} \right)^2 \right],
\]

where

\[
\sqrt{2} X_1 = \sqrt{P^4 R^2 R^2 \alpha_4^2 + 16 \alpha_4^2 + 16 \alpha_4^2 P^2 + (P^2 R^2 + 4 \alpha_4^2)}\]

\[
\sqrt{2} Y_1 = \sqrt{P^4 R^2 R^2 \alpha_4^2 + 16 \alpha_4^2 + 16 \alpha_4^2 P^2 - (P^2 R^2 + 4 \alpha_4^2)}\]

Then the temperature at any \( \gamma \), say \( \gamma = 0 \) (axis of the pipe), is given by

\[
\Theta(0, \gamma, t) = 2 \frac{2}{\alpha_1 J_1(\alpha_1)} \exp \left[ -\left( \frac{X_1 - PR}{2} \right)^2 \right] \sin(\omega t - \frac{\gamma_1}{2} \gamma).
\]

(3.3.3)
\[ \Theta (0, y, t) = A \sin \left( \omega t - \frac{y_1}{2} \right), \tag{3.3.4} \]

where

\[ A = \frac{2}{a_1 L_1} \exp \left[ \frac{-\left( \frac{x_1 - P R}{2} \frac{y}{j} \right)}{2} \right]. \]

The amplitudes at various points along the axis of the pipe have been presented in Table 1 for various values of Reynolds number. This table shows that the amplitude increases as \( R \) increases. From (3.3.3) it is obvious that phase lag along the axis of the pipe is linear and slope of this is \( \frac{y_1}{2} \) \((= \delta)\).

Phase lag along the axis of the pipe has been presented in Tables 2 and 3 for various values of Reynolds number and inlet frequencies. Table 2 shows that phase lag decreases as \( R \) increases. From Table 3 we note that phase lag increases with the increase of \( \omega \).

**Table 1**: \( \omega = 0.5, P = 0.73, R = 100, 500, 1000. \)

<table>
<thead>
<tr>
<th>Amplitude</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.1779</td>
<td>1.0745</td>
<td>0.7255</td>
<td>0.4664</td>
<td>0.3294</td>
<td>0.2206</td>
</tr>
<tr>
<td>A</td>
<td>1.5671</td>
<td>1.4799</td>
<td>1.3623</td>
<td>1.2702</td>
<td>1.1726</td>
<td>1.0825</td>
</tr>
<tr>
<td>A</td>
<td>1.5830</td>
<td>1.5361</td>
<td>1.4906</td>
<td>1.4322</td>
<td>1.3761</td>
<td>1.3395</td>
</tr>
</tbody>
</table>
### Table 2: \( \omega = 0.1, \ P = 0.73, \ R = 100, \ 1000, \ 10,000 \)

<table>
<thead>
<tr>
<th>Phase Lag</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>.0014</td>
<td>.0222</td>
<td>.0044</td>
<td>.0664</td>
<td>.0898</td>
<td>.1100</td>
</tr>
<tr>
<td>6</td>
<td>.0042</td>
<td>.0210</td>
<td>.0620</td>
<td>.0630</td>
<td>.0840</td>
<td>.1050</td>
</tr>
<tr>
<td>6</td>
<td>.0040</td>
<td>.0200</td>
<td>.0400</td>
<td>.0600</td>
<td>.0800</td>
<td>.1000</td>
</tr>
</tbody>
</table>

### Table 3: \( P = 0.73, \ R = 100, \ \omega = 0.02, \ 0.5, \ 1.0 \)

<table>
<thead>
<tr>
<th>Phase Lag</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>.000074</td>
<td>.000370</td>
<td>.000740</td>
<td>.001110</td>
<td>.001480</td>
<td>.001850</td>
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<tr>
<td>6</td>
<td>.000173</td>
<td>.000865</td>
<td>.001730</td>
<td>.002975</td>
<td>.003460</td>
<td>.004325</td>
</tr>
<tr>
<td>6</td>
<td>.000472</td>
<td>.002360</td>
<td>.004720</td>
<td>.007080</td>
<td>.009440</td>
<td>.011800</td>
</tr>
</tbody>
</table>
PROBLEM VIII

HEAT TRANSFER IN TWO-PHASE LAMINAR FLOW IN A CHANNEL

Exact solutions of the transient forced convection energy equations of dust particles and of liquid in a channel bounded by two parallel flat plates are obtained in the present problem when the inlet temperatures vary sinusoidally with time and an interpretation of the case of laminar flows is given.

NOMENCLATURE

\( \overline{t}_p \)  temperature of dust particle
\( \overline{T} \)  temperature of liquid
\( c_p \)  specific heat of dust particle
\( c \)  specific heat of liquid
\( h \)  half distance between parallel plates
\( k_p \)  thermal conductivity of dust particle
\( k \)  thermal conductivity of liquid
\( \overline{t} \)  time
\( \overline{u} \)  velocity component in \( \overline{x} \)-direction
\( \overline{U} \)  average velocity

* Submitted in revised form for publication in *International J. Heat and Mass Transfer*
\[ z, y \]
cartesian coordinates (\( z \)-flow direction, 
\( y \)-distance from channel centerline)

\( \rho \)
fluid density

\( m \)
mass of dust particle per unit volume 
\( (m = m \) \( n_0 \), constant)

\( \mu \)
coefficient of viscosity of liquid

\( \nu \)
kinematic coefficient of viscosity

\( P \)
Prandtl number 
\( (P = \frac{\mu c}{k}) \) 

\( R \)
Reynolds number 
\( (R = \frac{\rho u}{\nu}) \) 

\( h_p \)
heat transfer coefficient for flow over 
dust particle

\( A_p \)
surface area of dust particle

\( V_p \)
volume of dust particle

\( T_w \)
constant wall temperature

Meaning of any other symbols are given in the 
text as they occur.

**INTRODUCTION**

Heat transfer by gas-dust suspensions in pipe 
flow has been a subject of many studies because of the 
anticipated large heat-transfer coefficient due to the 
high volumetric specific heat of dust particles or 
liquid droplets compared to a gas and the demand for

In the present problem, exact solutions of the transient forced convection energy equations of dust particles and of liquid with fully developed flow in a parallel plate channel are obtained under prescribed boundary conditions when the inlet temperatures of dust particles and of liquid vary sinusoidally with time and an interpretation of the case of laminar flows is given.

1. FORMULATION OF THE PROBLEM

We consider the steady laminar flow of a dusty viscous liquid with uniform distribution of dust particles in a parallel plate channel whose sides are separated by
distance $2h$. The dust particles and the liquid entering the channel have temperatures which are specially uniform across the entrance section but vary sinusoidally with time. Therefore we can write the inlet conditions as

$$
\overline{T}_p (o, \overline{y}, \overline{z}) = T_0 + (\Delta T)_0 \sin \overline{\omega} \overline{t}, \quad (3.1.1)
$$

$$
\overline{T} (o, \overline{y}, \overline{z}) = T_0 + (\Delta T)_0 \sin \overline{\omega} \overline{t}, \quad (3.1.2)
$$

where $T_0$ is the cycle mean temperature, $(\Delta T)_0$ is the amplitude and $\overline{\omega}$ is the inlet frequency.

To obtain the heat-transfer performance and the temperatures of dust particles and of liquid it is necessary to set down two energy equations, one for the dust particles and one for the liquid-dust mixture. They are given as

$$
\left[ \frac{\partial \overline{T}_p}{\partial \overline{t}} + u_p \frac{\partial \overline{T}_p}{\partial \overline{x}} \right] = G_1 (\overline{T} - \overline{T}_p), \quad (3.1.3)
$$

$$
\frac{\partial \overline{T}}{\partial \overline{t}} + u \frac{\partial \overline{T}}{\partial \overline{x}} + \frac{m N_o c_p}{\rho c} \left[ \frac{\partial \overline{T}_p}{\partial \overline{t}} + u_p \frac{\partial \overline{T}_p}{\partial \overline{x}} \right] = \frac{\nu}{\rho} \frac{\partial^2 \overline{T}}{\partial \overline{y}^2} + \beta \left( \overline{T}_p - \overline{T} \right), \quad (3.1.4)
$$

where $G_1 = \frac{p_p A_p}{m N_o c_p V_p}$, $\beta = \frac{m N_o c_p G_1}{\rho c}$.
Simplifying equation (3.1.4), we get
\[
\left[ \frac{\partial \bar{T}}{\partial t} + u \frac{\partial \bar{T}}{\partial x} \right] = \frac{h}{P} \frac{\partial^2 \bar{T}}{\partial y^2} + \alpha \left( \bar{T}_p - \bar{T} \right) \tag{3.1.5}
\]

The inlet and the boundary conditions of the problem are as follows:

\[
\bar{T}_p = T_0 + (\Delta T)_o \sin \omega \bar{z} \quad \text{when } \bar{x} = 0, \tag{3.1.6}
\]

\[
\bar{T} = T_0 + (\Delta T)_o \sin \omega \bar{z} \quad \text{when } \bar{x} = 0. \tag{3.1.7}
\]

\[
\left( \frac{\partial \bar{T}}{\partial \bar{y}} \right)_{\bar{x}=0} = 0, \left( \frac{\partial \bar{T}}{\partial \bar{y}} \right)_{\bar{x}=0} = 0, \quad \bar{T}_p = T_w, \bar{T} = T_w \text{ at } \bar{y} = h, \tag{3.1.8}
\]

The system satisfying equations (3.1.3), (3.1.5) is subjected to the following restrictions [7]

\begin{itemize}
  \item[(i)] Radiation effect is neglected.
  \item[(ii)] The density of the liquid remains constant; thus the velocity distribution is independent of the temperature distribution.
  \item[(iii)] Liquid property variations are neglected.
  \item[(iv)] Each dust particle is small and maintains uniform temperature due to its high thermal conductivity.
  \item[(v)] The liquid and the dust particle cloud have similar velocity profiles. The presence of dust particles does not affect the liquid velocity profile.
\end{itemize}
(vi) The dust particles are uniformly distributed throughout the channel.

(vii) The effect of collision with the wall is neglected.

(viii) The suspension is extremely dilute such that each particle is assumed to see the wall without interference of other particles.

(ix) Fully developed laminar velocity profiles between the parallel plates.

(x) Axial conduction is negligible with respect to bulk transport in the \( \bar{z} \)-direction. This is a reasonable assumption when Péclet number exceeds 100 \[ 8 \].

(xi) Thermal resistance of the channel wall is negligible.

(xii) Eddy diffusivity of heat is negligible.

Further to simplify the method of analysis the case of constant velocity will be considered here and for this purpose we substitute \( \bar{u} (u = u_p) \) for the velocity profile in (3.1.3) and (3.1.5).

We now introduce the following non-dimensional quantities:

\[
\Theta = \frac{T - T_0}{(\Delta T)_0}, \quad \Theta_p = \frac{T_p - T_0}{(\Delta T)_0}, \quad x = \frac{x}{h}, \quad y = \frac{y}{h}.
\]
\[ t = \frac{2 \cdot 10^3}{\frac{h^2}{2}} , \quad \theta_0 = \frac{T_0 - T_0}{(\Delta T)_0} , \quad \omega = \frac{\frac{h^2}{2}}{2} \]

\[ \beta_3 = \frac{\frac{h^2}{2} \beta_0}{2} , \quad \beta_4 = \frac{\frac{h^2}{2} \beta_3}{2} \]

Equations (3.1.3) and (3.1.5) then become

\[ \left[ \frac{d\theta_p}{dt} + R \frac{d\theta_p}{dx} \right] = \beta_3 (\theta - \theta_p) , \quad (3.1.9) \]

\[ \left[ \frac{d\theta}{dt} + R \frac{d\theta}{dx} \right] = \frac{1}{p} \frac{\partial^2 \theta}{\partial y^2} + \beta_4 (\theta_p - \theta) . \quad (3.1.10) \]

The inlet and the boundary conditions reduce to

\[ \theta_p = \sin \omega t \quad \text{when} \quad x = 0 , \quad (3.1.11) \]

\[ \theta = \sin \omega t \quad \text{when} \quad x = 0 , \quad (3.1.12) \]

\[ (\frac{d\theta_p}{dy})_{y=0} = 0 , (\frac{d\theta}{dy})_{y=0} = 0 , \theta_p = \theta_0 , \theta = \theta_0 \quad \text{at} \quad y = 1 , \quad (t > 0) . \quad (3.1.13) \]

2. SOLUTION

The foregoing problem can be separated into

two as follows:

\[ \theta_p (x, y, t) = \theta_{p1} (x, y) + \theta_{p2} (x, y, t) , \quad (3.2.1) \]

\[ \theta (x, y, t) = \theta_1 (x, y) + \theta_2 (x, y, t) , \quad (3.2.2) \]
where \( \theta_1, \theta_2, \theta_{p1}, \text{ and } \theta_{p2} \) satisfy the following problems:

\[
R \frac{\partial \theta_{p1}}{\partial x} = \beta_3 (\theta_1 - \theta_{p1}), \quad (3.2.3)
\]

\[
R \frac{\partial \theta_1}{\partial x} = \frac{1}{P} \frac{\partial^2 \theta_1}{\partial t^2} + \beta_4 (\theta_{p1} - \theta_1), \quad (3.2.4)
\]

\[
\theta_{p1} = 0 \quad \text{when } x = 0, \quad (3.2.5)
\]

\[
\theta_1 = 0 \quad \text{when } x = 0, \quad (3.2.6)
\]

\[
\left( \frac{\partial \theta_{p1}}{\partial y} \right)_{y=0} = 0, \quad \left( \frac{\partial \theta_1}{\partial y} \right)_{y=0} = 0, \quad \theta_{p1} = \theta_0, \quad \theta_1 = \theta_0 \text{ at } y = 1. \quad (3.2.7)
\]

\[
\left[ \frac{\partial \theta_{p2}}{\partial t} + R \frac{\partial \theta_{p2}}{\partial x} \right] = \beta_3 (\theta_2 - \theta_{p2}), \quad (3.2.8)
\]

\[
\left[ \frac{\partial \theta_2}{\partial t} + R \frac{\partial \theta_2}{\partial x} \right] = \frac{1}{P} \frac{\partial^2 \theta_2}{\partial t^2} + \beta_4 (\theta_{p2} - \theta_2), \quad (3.2.9)
\]

\[
\theta_{p2} = \sin \omega t \quad \text{when } x = 0, \quad (3.2.10)
\]

\[
\theta_2 = \sin \omega t \quad \text{when } x = 0, \quad (3.2.11)
\]

\[
\left( \frac{\partial \theta_{p2}}{\partial y} \right)_{y=0} = 0, \quad \left( \frac{\partial \theta_2}{\partial y} \right)_{y=0} = 0, \quad \theta_{p2} = 0, \quad \theta_2 = 0 \text{ at } y = 1, \quad (t > 0). \quad (3.2.12)
\]

Solving equations (3.2.3) and (3.2.4) under the conditions (3.2.5) - (3.2.7), we get
\[
\Theta_1(x, y) = \Theta_0 \left[ 1 - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left( \frac{(2n+1)\pi y}{2} \right) \cdot \frac{\left( \frac{-\lambda_n e}{\lambda_n - \mu_n} \right)}{\left( \frac{-\lambda_n}{\lambda_n - \mu_n} \right)} \right],
\]

\[
\Theta_1(x, y) = \Theta_0 \left[ 1 - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left( \frac{(2n+1)\pi y}{2} \right) \cdot \frac{A_n(x)}{A_n(x)} \right],
\]

where

\[
A_n(x) = \frac{\lambda_n}{(\lambda_n - \mu_n)} \left[ 1 - \frac{R\mu_n}{B_3} \right] e^{\lambda_n x} - \frac{\mu_n}{(\lambda_n - \mu_n)} \left[ 1 - \frac{R\lambda_n}{B_3} \right] e^{-\lambda_n x}.
\]

\[
2\lambda_n = \sqrt{\left[ \frac{B_3 + B_4}{R} + \frac{(2n+1)^2 \pi^2}{4 PR} \right] + \left[ \frac{B_3 + B_4}{R} + \frac{(2n+1)^2 \pi^2}{4 PR} \right]^2 - \frac{(2n+1)^2 \pi^2 B_3}{PR^2 B_3}}.
\]

\[
2\mu_n = \sqrt{\left[ \frac{B_3 + B_4}{R} + \frac{(2n+1)^2 \pi^2}{4 PR} \right] - \left[ \frac{B_3 + B_4}{R} + \frac{(2n+1)^2 \pi^2}{4 PR} \right]^2 - \frac{(2n+1)^2 \pi^2 B_3}{PR^2 B_3}}.
\]

In obtaining \( \Theta_{p_2}(x, y, t) \) and \( \Theta_2(x, y, t) \) we define the following auxiliary problems:

\[
\frac{d\Theta_{p_2}}{dt} + R \frac{d\Theta_{p_2}}{dx} = \beta \left( \Theta_2' - \Theta_{p_2}' \right), \tag{3.2.13}
\]

\[
\frac{d\Theta_2}{dt} + R \frac{d\Theta_2}{dx} = \frac{1}{p} \frac{d^2\Theta_2}{dy^2} + \mu \left( \Theta_{p_2} - \Theta_2 \right), \tag{3.2.14}
\]
\[
\theta_{p_2} = \cos \omega t \quad \text{when} \quad x = 0 , \quad (3.2.15)
\]
\[
\theta_{p_2} = \cos \omega t \quad \text{when} \quad x = 0 , \quad (3.2.16)
\]
\[
\left( \frac{d\theta_{p_2}}{dy} \right)_{y=0} = 0 , \quad \left( \frac{d\theta_{p_2}}{dy} \right)_{y=0} = 0 , \quad \theta'_{p_2} = 0 , \quad \theta'_2 = 0 \quad \text{at} \quad y = 1 . \quad (3.2.17)
\]

Here we note that the auxiliary problems are similar to the original problems for \( \theta_{p_2} \) and \( \theta_2 \) except that the periodic condition is shifted by \( \frac{\Lambda}{2} \).

Let us define new temperature functions \( \theta_{pc}(x,y,t) \) and \( \theta_c(x,y,t) \) such that
\[
\theta_{pc} = \theta_{p_2} + i \theta_{p_2} , \quad (3.2.18)
\]
\[
\theta_c = \theta_2 + i \theta_2 , \quad (3.2.19)
\]

then the problems given by equations (3.2.8) - (3.2.12) and (3.2.13) - (3.2.17) can be combined to give the following problems:
\[
\left[ \frac{d\theta_{pc}}{dt} + R \frac{d\theta_{pc}}{dx} \right] = \frac{1}{\beta_3} \left( \theta_c - \theta_{pc} \right) , \quad (3.2.20)
\]
\[
\left[ \frac{d\theta_c}{dt} + R \frac{d\theta_c}{dx} \right] = \frac{1}{\beta_4} \left( \frac{d^2\theta_c}{dy^2} + \beta_4 (\theta_{pc} - \theta_c) \right) , \quad (3.2.21)
\]
\[ \theta_{pc} = e^{i \omega t} \quad \text{when} \quad x = 0, \quad (3.2.22) \]
\[ \theta_c = e^{i \omega t} \quad \text{when} \quad x = 0, \quad (3.2.23) \]
\[ \left( \frac{d \theta_{pc}}{dy} \right)_{y=0} = 0, \quad \left( \frac{d \theta_c}{dy} \right)_{y=0} = 0, \quad \theta_{pc} = 0, \theta_c = 0 \quad \text{at} \quad y = 1, \quad (3.2.24) \]

We now assume periodic solutions of the following types:
\[ \theta_{pc}(x, y, t) = e^{i \omega t} \psi(x, y), \quad (3.2.25) \]
\[ \theta_c(x, y, t) = e^{i \omega t} \phi(x, y), \quad (3.2.26) \]

where the new temperature functions \( \psi \) and \( \phi \) satisfy the following problems:
\[ \left[ i \omega \psi + R \frac{d \psi}{dx} \right] = \beta_3 (\phi - \psi), \quad (3.2.27) \]
\[ \left[ i \omega \phi + R \frac{d \phi}{dx} \right] = \frac{1}{P} \frac{d^2 \phi}{dy^2} + \beta_4 (\psi - \phi), \quad (3.2.28) \]
\[ \psi = 1, \quad \phi = 1 \quad \text{when} \quad x = 0, \quad (3.2.29) \]
\[ \left( \frac{d \psi}{dy} \right)_{y=0} = 0, \quad \left( \frac{d \phi}{dy} \right)_{y=0} = 0, \quad \psi = 0, \quad \phi = 0 \quad \text{at} \quad y = 1. \quad (3.2.30) \]
Solving equations (3.2.27) and (3.2.28) under the conditions (3.2.29) and (3.2.30), we get finally

\[ \Theta_{p_2}(x, y, t) = \frac{4}{\pi} \sin\left(\omega t - \frac{\omega}{R} x\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos\left(\frac{2n+1}{2}\pi y\right) \cdot B_n(x), \]

\[ \Theta_2(x, y, t) = \frac{4}{\pi} \sin\left(\omega t - \frac{\omega}{R} x\right) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos\left(\frac{2n+1}{2}\pi y\right) \cdot A_n(x), \]

where

\[ B_n(x) = \left[ \frac{(\lambda_n - \mu_n e^{-\mu_n x})}{(\lambda_n - \mu_n)} \right]. \]

3. DISCUSSION

When the boundary condition on the wall for

\[ \Theta_p(x, y, t) \quad \text{and} \quad \Theta(x, y, t) \quad \text{is homogeneous, that is}, \]

when \( \Theta_0 \) is zero, then

\[ \Theta_p(x, y, t) = \Theta_{p_2}(x, y, t), \quad (3.3.1) \]

\[ \Theta(x, y, t) = \Theta_2(x, y, t). \quad (3.3.2) \]

\( \Theta(x, y, t) \) and \( \Theta_2(x, y, t) \) show that the temperatures of dust particles and of liquid decay exponentially along the channel.
For single-phase system the number of dust particles per unit volume is zero (and so \( \beta = 0 \)). Hence

\[
\Theta_b(x, y, t) = \frac{L}{\pi} \sin \left( \omega t - \frac{\omega x}{R} \right) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos \left( \frac{(2n+1)^2 \pi^2}{4PR} x \right) \cdot C_n(x),
\]

where \( C_n(x) = \exp \left[ - \frac{(2n+1)^2 \pi^2}{2PR} x \right] \) and the boundary condition on the wall is homogeneous.

In many applications heat transfer in regions away from the inlet is of interest; for such situations only the first terms in the series (3.3.1), (3.3.2) and (3.3.3) need to be considered. Hence

\[
\Theta_b(x, y, t) = \frac{L}{\pi} \sin \left( \omega t - \frac{\omega x}{R} \right) \cdot \cos \left( \frac{\pi y}{2} \right) \cdot B_0(x),
\]

\[
\Theta(x, y, t) = \frac{L}{\pi} \sin \left( \omega t - \frac{\omega x}{R} \right) \cdot \cos \left( \frac{\pi y}{2} \right) \cdot A_0(x),
\]

\[
\Theta_2(x, y, t) = \frac{L}{\pi} \sin \left( \omega t - \frac{\omega x}{R} \right) \cdot \cos \left( \frac{\pi y}{2} \right) \cdot C_0(x),
\]

where

\[
B_0(x) = \left[ \frac{(\lambda_0 - \mu_0 x) \frac{\Lambda_0 e - \lambda_0 x}{\Lambda_0 - \mu_0}}{\mu_0} \right],
\]
\[ A_0(x) = \frac{\lambda_0}{(\lambda_0 - \mu_0)} \left[ 1 - \frac{R\mu_0}{B_3} \right] e^{-\mu_0 x} - \frac{\mu_0}{(\lambda_0 - \mu_0)} \left[ 1 - \frac{R\lambda_0}{B_3} \right] e^{-\lambda_0 x}, \]

\[ C_0(x) = \exp \left[ - \frac{\kappa^2}{4 PR} x \right], \]

\[ 2\lambda_0 = \left[ - \frac{B_3 + B_4}{R} + \frac{\kappa^2}{4 PR} \right] + \sqrt{\left[ \frac{B_3 + B_4}{R} + \frac{\kappa^2}{4 PR} \right]^2 - \frac{\kappa^2}{PR^2 B_3}}, \]

\[ 2\mu_0 = \left[ - \frac{B_3 + B_4}{R} + \frac{\kappa^2}{4 PR} \right] - \sqrt{\left[ \frac{B_3 + B_4}{R} + \frac{\kappa^2}{4 PR} \right]^2 - \frac{\kappa^2}{PR^2 B_3}}. \]

The temperatures at any \( y \), say \( y = 0 \), are given by

\[ \Theta_p(x, 0, t) = \frac{4}{\kappa} \sin \left( \omega t - \frac{\omega}{R} x \right). \]

\[ \Theta(x, 0, t) = \frac{4}{\kappa} \sin \left( \omega t - \frac{\omega}{R} x \right). \]

\[ \Theta_p(x, 0, t) = \frac{4}{\kappa} \sin \left( \omega t - \frac{\omega}{R} x \right). \]

where \( \Theta_p = B_0(x) \), \( \Theta = A_0(x) \), \( \Theta_p = C_0(x) \) denote amplitudes of dust particle, liquid-dust mixture and clean liquid (single-phase system) respectively.

We observe the following important points:

1. From tables 1, 2 and 5 it is obvious that the amplitudes \( \Theta_p, \Theta \) and \( \Theta_p \) increase with the increase of \( R \) and \( \Theta_p > \Theta > \Theta_p \). \( \Theta \)
Tables 3 and 4 show that $\alpha_p$ decreases with the increase of $\beta_3$ (and so $\beta_4$), but $\alpha$ increases. Also $\alpha_p > a > a_\theta$ holds (at least for values of $P$, $\frac{P_4}{P_3}$, $R$ and $\frac{P_4}{P_3}$ considered).

(11)

From tables 5 and 6 we infer that the amplitudes $\alpha_p$ and $a$ increase with the increase of $\frac{P_4}{P_3}$ and $\alpha_p > a > a_\theta$.

Thus, the effect of dust particle is to flatten the temperature profile and, consequently, to increase the heat transfer.

Equations (3.3.1), (3.3.2) and (3.3.3) suggest that the phase lags are the same for both two-phase and single-phase systems and is a limit due to the nature of the model. Also, as the inlet frequency is increased, phase lag increases and as the Reynolds number $R$ is increased, phase lag decreases.
Table 1. Comparison of the amplitudes for  
\( P = 0.73, \quad \frac{\beta}{\beta_3} = 10^5, \quad \frac{\beta_4}{\beta_3} = 0.5, \quad R = 13000 \)

<table>
<thead>
<tr>
<th>Amplitude</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_p )</td>
<td>0.99902</td>
<td>0.99902</td>
<td>0.99702</td>
<td>0.99602</td>
<td>0.99302</td>
</tr>
<tr>
<td>( a )</td>
<td>0.99998</td>
<td>0.99998</td>
<td>0.99998</td>
<td>0.99998</td>
<td>0.99998</td>
</tr>
<tr>
<td>( a_p )</td>
<td>0.99970</td>
<td>0.99740</td>
<td>0.99610</td>
<td>0.99480</td>
<td>0.99350</td>
</tr>
</tbody>
</table>

Table 2. Comparison of the amplitudes for  
\( P = 0.73, \quad \frac{\beta}{\beta_3} = 10^5, \quad \frac{\beta_4}{\beta_3} = 0.5, \quad R = 20000 \)

<table>
<thead>
<tr>
<th>Amplitude</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_p )</td>
<td>0.99945</td>
<td>0.99889</td>
<td>0.99832</td>
<td>0.99776</td>
<td>0.99719</td>
</tr>
<tr>
<td>( a )</td>
<td>0.99943</td>
<td>0.99886</td>
<td>0.99829</td>
<td>0.99773</td>
<td>0.99717</td>
</tr>
<tr>
<td>( a_p )</td>
<td>0.99916</td>
<td>0.99831</td>
<td>0.99747</td>
<td>0.99662</td>
<td>0.99578</td>
</tr>
</tbody>
</table>

Table 3. Comparison of the amplitudes for  
\( P = 0.73, \quad \frac{\beta_4}{\beta_3} = 0.5, \quad R = 20000, \quad \frac{\beta}{\beta_3} = 10^5 \)

<table>
<thead>
<tr>
<th>Amplitude</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_p )</td>
<td>0.999499</td>
<td>0.999849</td>
<td>0.9993199</td>
<td>0.9977949</td>
<td>0.9971899</td>
</tr>
<tr>
<td>( a )</td>
<td>0.999270</td>
<td>0.9998620</td>
<td>0.9992970</td>
<td>0.9977320</td>
<td>0.9971670</td>
</tr>
<tr>
<td>( a_p )</td>
<td>0.9991550</td>
<td>0.9983100</td>
<td>0.9974650</td>
<td>0.9966200</td>
<td>0.9957750</td>
</tr>
</tbody>
</table>
Table 4. Comparison of the amplitudes for 
\[ p = 0.73, \frac{B_4}{B_3} = 0.5, R = 20000, \frac{B_5}{B_3} = 10^9 \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>5</th>
<th>20</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_p )</td>
<td>0.9994351</td>
<td>0.9998701</td>
<td>0.9983051</td>
<td>0.997701</td>
<td>0.9971751</td>
</tr>
<tr>
<td>( a )</td>
<td>0.9994350</td>
<td>0.9998700</td>
<td>0.9983050</td>
<td>0.997700</td>
<td>0.9971700</td>
</tr>
<tr>
<td>( a_p )</td>
<td>0.9991550</td>
<td>0.9983100</td>
<td>0.9974650</td>
<td>0.9966200</td>
<td>0.9957750</td>
</tr>
</tbody>
</table>

Table 5. Comparison of the amplitudes for 
\[ p = 0.73, R = 25000, \frac{B_4}{B_3} = 10^5, \frac{B_5}{B_3} = 0.5 \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_p )</td>
<td>0.99959</td>
<td>0.99944</td>
<td>0.99869</td>
<td>0.99825</td>
<td>0.99780</td>
</tr>
<tr>
<td>( a )</td>
<td>0.99958</td>
<td>0.99913</td>
<td>0.99868</td>
<td>0.99823</td>
<td>0.99778</td>
</tr>
<tr>
<td>( a_p )</td>
<td>0.99933</td>
<td>0.99865</td>
<td>0.99798</td>
<td>0.99730</td>
<td>0.99663</td>
</tr>
</tbody>
</table>

Table 6. Comparison of the amplitudes for 
\[ p = 0.73, R = 25000, \frac{B_4}{B_3} = 10^5, \frac{B_5}{B_3} = 0.9 \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_p )</td>
<td>0.99967</td>
<td>0.99933</td>
<td>0.99899</td>
<td>0.99875</td>
<td>0.99831</td>
</tr>
<tr>
<td>( a )</td>
<td>0.99965</td>
<td>0.99931</td>
<td>0.99897</td>
<td>0.99873</td>
<td>0.99829</td>
</tr>
<tr>
<td>( a_p )</td>
<td>0.99933</td>
<td>0.99865</td>
<td>0.99798</td>
<td>0.99730</td>
<td>0.99663</td>
</tr>
</tbody>
</table>
REFERENCES