CREEP TRANSITION
CHAPTER V

CREEP TRANSITION OF A
TRANSVERSELY ISOTROPIC DISC
HAVING VARIABLE THICKNESS UNDER
INTERNAL PRESSURE

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Applied Mathematics)
5.1 INTRODUCTION

Literature survey indicates that several workers have analyzed circular disc with constant material properties under various conditions. Solution for thin isotropic discs can be found in most of the standard elasticity, plasticity and creep text books [34, 39, 41, 82, 88, 94-95, 99-100, 109, 111]. Gramer [37] has shown that discs made of non-linear hardening material can also be treated analytically for some special case of hardening characteristics. However, in the above works, the variation of the disc thickness has been neglected. In analyzing the problem, these authors have used some simplifying assumptions. First, the deformation is small enough to make infinitesimal strain theory applicable. Second, simplifications were made regarding the constitutive equations of the material like incompressibility of the material and a yield criterion. Incompressibility of the material is one of the most important assumptions that simplify the problem. In fact, in most of the cases, it is not possible to find a solution in closed form without this assumption. Seth's transition theory does not require these assumptions and thus poses and solves a more general problem from which cases pertaining to the above assumptions can be worked out. Seth's transition theory utilizes the concept of generalized strain measure and asymptotic solution through the critical points of differential system defining the
deformed field and has successfully been applied to a large number of problems [8, 51, 53, 57, 136, 142, 148, 153].

In this chapter, we calculated the creep stresses and strain rates for a transversely isotropic disc having variable thickness subjected to internal pressure by using Seth’s transition theory. The thickness $h$ is assumed to vary in the radial direction as given in equation (1.6.24). Results obtained have been discussed numerically and depicted graphically.

### 5.2 GOVERNING EQUATIONS

We consider a thin disc of non-constant thickness of transversely isotropic material with internal and external radii ‘$a$’ and ‘$b$’ respectively subjected to internal pressure $p$. The disc is thin and it is effectively in a state of plane stress.

The governing equations for this problem are same as given in section 2.2.

### 5.3 SOLUTION THROUGH PRINCIPAL STRESS DIFFERENCE

It has been shown [58, 67-68, 70-71, 74-76, 146, 150, 152-153] that the asymptotic solution through the principal
stress-difference at the transition point \( P \to -1 \) leads to creep state.

For finding the creep stresses at the transition point \( P \to -1 \), we define the transition function \( R_4 \) as,

\[
R_4 = T_{n'} - T_{n0} = \frac{2C_66}{n} \beta^n \left[ 1 - (1 + P)^n \right]. \tag{5.3.1}
\]

Taking logarithmic differentiation of equation (5.3.1) w.r.t. \( r' \), we get,

\[
\frac{d}{dr} (\log R_4) = \frac{2C_66}{rR_4} \left[ P \beta^n - P \beta^n (1 + P)^n - P \beta^{n+1} (1 + P)^{n+1} \frac{dP}{d\beta} \right]. \tag{5.3.2}
\]

Substituting the value of \( \frac{dP}{d\beta} \) from equation (2.2.4) in equation (5.3.2), we have,

\[
\frac{d}{dr} (\log R_4) = \frac{2C_66}{rR_4} \left[ \frac{2P \beta^n - 2h'r}{nh} - \frac{2C_66 \beta^n}{nA} + \frac{2C_66 \beta^n}{nA} (1 + P)^n \right. \]
\[
\left. + \frac{2C_66 \beta^n h'r}{nAh} - \frac{2C_66 P \beta^n}{A} + \frac{h'r}{nh} \beta^n (1 + P)^n \right]. \tag{5.3.3}
\]

Asymptotic value of equation (5.3.3) as \( P \to -1 \), is

\[
\frac{d}{dr} (\log R_4) = \frac{1}{r} \left[ -2n + C_1(n-1) \right] + \frac{h'}{h} \left[ 1 - C_2 \right] + \frac{r^n h'}{hD^n} \left[ C_2 - 2 \right]. \tag{5.3.4}
\]

Asymptotic value of \( \beta \) as \( P \to -1 \) is \( \frac{D}{r} \), where D is a constant.
Integrating equation (5.3.4) w.r.t. 'r', we get,

\[ R_i = T_{r_i} - T_{r_0} = A_i r' h' \exp f, \quad (5.3.5) \]

where \( s = \left[ -2n + C_2(n-1) \right] \), \( t = (1 - C_2) \), \( C_2 = \frac{2C_{66}}{A} \),

\[ f = (2 - C_2) \frac{K}{nD^n} r'' \] and \( A_i \) is constant of integration.

Using equation (5.3.5) in equation (2.2.3), we get,

\[ hT_{r_i} = A_3 - A_2 \int F \, dr, \quad (5.3.6) \]

where \( A_2 = A_i h_0^{1+i}, F = r^{s-1} \left( \frac{r}{b} \right)^{-K(1+i)} \exp f, \quad s = \left[ -2n + C_2(n-1) \right] \),

and \( A_i \) is another integrating constant.

Using boundary conditions (2.2.5) in equation (5.3.6), we get,

\[ A_2 = \frac{-p h_0 \left( \frac{b}{a} \right)^K}{\int_c^b F \, dr}, \]

and \( A_3 = \frac{-p h_0 \left( \frac{b}{a} \right)^K}{\int_c^b F \, dr} \).
Substituting values of $A_2$ and $A_3$ in equations (5.3.6) and (5.3.5), we get,

$$T_{rr} = - \frac{p h(a)}{b} \int_a^b Fdr \quad (5.3.7)$$

$$T_{\theta\theta} = T_{rr} + \frac{p h(a)}{b} F . \quad (5.3.8)$$

Equations (5.3.7) and (5.3.8) gives creep stresses for a transversely isotropic disc with variable thickness under internal pressure.

We introduce the following non-dimensional quantities,

$$R = \frac{r}{b} , R_o = \frac{a}{b} , \sigma_r = \frac{T_{rr}}{p} , \sigma_\theta = \frac{T_{\theta\theta}}{p} , P_i = \frac{p}{C_{66}} .$$

Stresses (5.3.7) and (5.3.8) in non-dimensional form are given by,

$$\sigma_r = \frac{1}{\frac{1}{R_o^K} R^K} \int_{R_o}^R R^{s-1-K(1+t)} \exp f_1 dR ,$$

$$\sigma_\theta = \sigma_r + \frac{1}{\frac{1}{R_o^K} R^{1+K}} \int_{R_o}^R R^{s-1-K(1+t)} \exp f_1 dR , \quad (5.3.9)$$

where $f_1 = \left[2 - \frac{2C_{66}/C_{11}}{1 - C_{13}/C_{33}C_{11}} \right] n \left[ \frac{bR}{D} \right]^t , t = (1 - C_2)$ and $s = [-2n + C_2(n-1)]$. 

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FLAT DISC

For a disc having uniform thickness \(K = 0\), equation (5.3.9) becomes,

\[
\sigma_r = -\frac{1 - R^{(-2n + C_2(n-1))}}{1 - R^{(-2n + C_2(n-1))}},
\]

\[
\sigma_\theta = \frac{(1 - 2n + C_2(n-1)) R^{(-2n + C_2(n-1))} - 1}{1 - R_{o}^{(-2n + C_2(n-1))}}. \tag{5.3.10}
\]

5.4 ISOTROPIC MATERIAL

Using equation (1.6.3) in equation (5.3.9), the creep stresses for isotropic materials are,

\[
\sigma_r = \frac{-R_o^{-K} R^{K}}{\int_{R_o}^{R_{t+1}} R^{x-K(t+1)} \exp f_3 \, dR} \int_{R_o}^{R^{x-K(t+1)}} \exp f_3 \, dR,
\]

\[
\sigma_\theta = \sigma_r + \frac{R_o^{-K} R^{i+K}}{\int_{R_o}^{R^{x-K(t+1)}} \exp f_3 \, dR} \int_{R_o}^{R^{x-K(t+1)}} \exp f_3 \, dR. \tag{5.4.1}
\]

where \( f_3 = \left[ \frac{3 - 2C}{2 - C} \right] + \frac{K b R}{n D} \), \( s_1 = \left[ -2n + \frac{1}{2 - C} (n-1) \right] \),

\[
t_1 = \left[ \frac{1-C}{2-C} \right] \quad \text{and} \quad C = \frac{2\mu}{\lambda + 2\mu} = \frac{1 - 2\sigma}{1 - \sigma}.
\]
**FLAT DISC (Isotropic case)**

For a disc having uniform thickness \(K = 0\), equation (5.4.1) becomes,

\[
s_{r} = -\frac{R^{2n+\frac{1}{2}-C(n-1)}}{1 - R^{2n+\frac{1}{2}-C(n-1)}}.
\]

\[
s_{\theta} = -\frac{R^{2n+\frac{1}{2}-C(n-1)}}{1 - R^{2n+\frac{1}{2}-C(n-1)}} - 1.
\]

These equations are same as obtained by Gupta and Pankaj [57].

### 5.5 STRAIN RATES

The stress- strain relation (2.2.1) can also be written as,

\[
e_{y} = \frac{(A - 2C_{66})}{H} \Theta + \frac{2(A - C_{66})}{H} T_{y}.
\]

where \(\Theta = T_{11} + T_{22} + T_{33}\), \(H = 4C_{66}(C_{66} - A)\) and \(A = C_{11} - \frac{C_{13}^{2}}{C_{33}}\).

When the creep sets in, the strain should be replaced by strain rates. The stress- strain relations (5.5.1) become,

\[
\dot{e}_{y} = \frac{(A - 2C_{66})}{H} \dot{\Theta} + \frac{2(A - C_{66})}{H} \dot{T}_{y},
\]

\[
\dot{\Theta} = T_{11} + T_{22} + T_{33},
\]

\[
\dot{T}_{y} = \frac{2(A - C_{66})}{H} T_{y}.
\]
where \( \varepsilon_{ij} \) is the strain rate tensor with respect to flow parameter \( t \).

Differentiating second equation of equation (1.5.3) with respect to \( t \), we get,

\[
\dot{\varepsilon}_{t0} = -\beta^{-1} \dot{\beta}.
\]  

(5.5.3)

For SWAINGER measure \( (n=1) \) we have from equation (5.5.3),

\[
\dot{\varepsilon}_{t0} = -\beta.
\]  

(5.5.4)

The transition value of equation (5.3.1) as \( P \to -1 \) is,

\[
\beta = \left[ \frac{n}{2C_{66}} (T_{r} - T_{t0}) \right]^{\frac{1}{n}}.
\]  

(5.5.5)

Using equations (5.5.3), (5.5.4) and (5.5.5) in equation (5.5.2), we get,

\[
\dot{\varepsilon}_{rr} = -\frac{P_1 \chi}{4 \left( \frac{C_{66}}{C_{11}} - 1 + \frac{C_{13}^2}{C_{33}C_{11}} \right)} \left[ \left( 1 - \frac{C_{13}^2}{C_{33}C_{11}} - \frac{2C_{66}}{C_{11}} \right) \sigma_{\theta} - \left( 1 - \frac{C_{13}^2}{C_{33}C_{11}} \right) \sigma_{r} \right],
\]

\[
\dot{\varepsilon}_{\theta\theta} = -\frac{P_1 \chi}{4 \left( \frac{C_{66}}{C_{11}} - 1 + \frac{C_{13}^2}{C_{33}C_{11}} \right)} \left[ \left( 1 - \frac{C_{13}^2}{C_{33}C_{11}} - \frac{2C_{66}}{C_{11}} \right) \sigma_{r} - \left( 1 - \frac{C_{13}^2}{C_{33}C_{11}} \right) \sigma_{\theta} \right],
\]

\[
\dot{\varepsilon}_{zz} = -\frac{P_1 \chi}{4 \left( \frac{C_{66}}{C_{11}} - 1 + \frac{C_{13}^2}{C_{33}C_{11}} \right)} \left[ \left( 1 - \frac{C_{13}^2}{C_{33}C_{11}} - \frac{2C_{66}}{C_{11}} \right) \left( \sigma_{r} + \sigma_{\theta} \right) \right].
\]  

(5.5.6)
where \( \chi = \left[ \frac{n \rho_i}{2} (\sigma_r - \sigma_\theta) \right]^{1\,\frac{1}{n-1}} \) and \( P_i = \frac{P}{C_{66}} \).

For isotropic materials, strain rates (5.5.6) becomes,

\[
\varepsilon_{rr} = \frac{P_i \chi (2 - c)}{2(3 - 2c)} \left[ \sigma_r - \left( \frac{1-c}{2-c} \right) \sigma_\theta \right],
\]

\[
\varepsilon_{\theta \theta} = \frac{P_i \chi (2 - c)}{2(3 - 2c)} \left[ \sigma_\theta - \left( \frac{1-c}{2-c} \right) \sigma_r \right],
\]

\[
\varepsilon_{zz} = \frac{P_i \chi (2 - c)}{2(3 - 2c)} \left[ (\sigma_r + \sigma_\theta) \left( \frac{1-c}{2-c} \right) \right],
\]

(5.5.7)

where \( \chi = \left[ \frac{n \rho_i}{2} (\sigma_r - \sigma_\theta) \right]^{1\,\frac{1}{n-1}} \) and \( P_i = \frac{P}{\mu} \).

These equations are same as given in equation (1.6.22) by Odquist [111] provided we put \( n = 1/N \).
5.6 NUMERICAL ILLUSTRATION AND DISCUSSION

**Figure 5.1:** Disc having variable thickness.

For calculating the stresses and strain rate distribution based on the above analysis, the following values of measure \( n \), \( D \) and pressure \( P_1 \) have been taken:

\[
  n = 1, \ 1/3, \ 1/7 \ (i.e. \ N = 1, \ 3 \ and \ 7)
\]

\[
  P_1 = \frac{P}{C_{66}} = 0.1, \ 1.0 \quad \text{and} \quad D = 1.
\]

Elastic constants \( C_{ij} \) have been given in Table 5.1 for isotropic material [87] (Brass, \( \sigma = 0.33 \)) and transversely isotropic material [98] (Cadmium).
TABLE 5.1 Elastic constants $C_\nu$ (in units of $10^{10}$ N / m²).

<table>
<thead>
<tr>
<th></th>
<th>$C_{33}$</th>
<th>$C_{44}$</th>
<th>$C_{11}$</th>
<th>$C_{13}$</th>
<th>$C_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Isotropic Material</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>($\sigma = 0.33$ or $c = 0.50$, Brass)</td>
<td>3.0</td>
<td>0.99997</td>
<td>3.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td><strong>Transversely Isotropic Material (Cadmium or $C_2=0.40$)</strong></td>
<td>4.69</td>
<td>1.56</td>
<td>11.0</td>
<td>3.83</td>
<td>4.04</td>
</tr>
</tbody>
</table>

Curves have been drawn in Figures 5.2, 5.3 and 5.4 between stresses and radius $R = r/b$ for $K = 0, 1.5$ and $2.0$ respectively. It can be seen from Figure 5.2 that the disc having constant thickness ($K=0$) subjected to internal pressure and made of transversely isotropic material (Cadmium) has maximum circumferential stress at the outer surface for measure $N=3$ (i.e. $n=1/3$) and this value further increases at the outer surface with the increase in measure $N$ (i.e. $n=1/7$) in comparison to disc made of isotropic material (Brass). The disc made of variable thickness for transversely isotropic material has maximum circumferential stress at the outer surface in comparison to disc having constant thickness ($K=0$) for measure $N=3$ (i.e. $n=1/3$) and $K=1.5$ and this value further increases with the increase in measure $N$ and $K$.

In Figures 5.5 - 5.10, curves has been drawn for creep strain rate along the radius $R (= r/b)$ for measure $N = 1, 3, 7$ (or $n=1, 1/3$ and $1/7$) and pressure $P_1 = 0.1$ and $1.0$ respectively. It
Figure 5.2: Creep Stresses of a Transversely Isotropic Disc With Constant
Thickness (K = 0) Under Internal Pressure.
Figure 5.3: Creep Stresses of a Transversely Isotropic Disc With Variable Thickness (K = 1.5) Under Internal Pressure.
Figure 5.4: Creep Stresses of a Transversely Isotropic Disc With Variable Thickness ($K = 2$) Under Internal Pressure.
Figure 5.5: Strain Rates for a Transversely Isotropic Disc With Variable Thickness Under Internal Pressure (P₁ = 0.1) for measure N = 1.
Figure 5.6: Strain Rates for a Transversely Isotropic Disc With Variable Thickness Under Internal Pressure (P1 = 0.1) for measure N = 1/3.
Figure 5.7: Strain Rates for a Transversely Isotropic Disc With Variable Thickness Under Internal Pressure (P₁ = 0.1) for measure N = 1/7.
Figure 5.8: Strain Rates for a Transversely Isotropic Disc With Variable Thickness Under Internal Pressure \( P_1 = 1.0 \) for measure \( N = 1 \).
Figure 5.9: Strain Rates for a Transversely Isotropic Disc With Variable Thickness Under Internal Pressure \((P_1 = 1.0)\) for measure \(N = 1/3\).
Figure 5.10: Strain Rates for a Transversely Isotropic Disc With Variable Thickness Under Internal Pressure ($P_1 = 1.0$) for measure $N = 1/7$. 