CHAPTER-6

(SECTION-A)

STABILITY ANALYSIS OF MODIFIED RAYLEIGH-BENARD CONVECTION IN POROUS MEDIUM IN PRESENCE OF VARIABLE VISCOSITY AND NON-UNIFORM TEMPERATURE GRADIENT
6.1 INTRODUCTION

Thermal convection in a porous medium is of great importance to Geophysicists and fluid dynamicists, as the fluid in the core of the Earth is at very high temperature and the earth’s crust serves as a porous medium. Treating viscosity as constant, earlier workers like Lapwood [1948] and Wooding [1960] have determined the critical temperature gradient which sets up thermal convection in a porous medium. They have used Darcy model and the analysis is based on normal mode technique of Chandrasekhar. Prabhamani and Rudraiah [1973] treated the Brinkman model by using both normal mode and energy method with constant viscosity.

Rogers and Marison [1950] and Rogers et al. [1951] considered the onset of convection induced by buoyancy in a saturated porous medium with non-linear basic temperature distributions. The effect of non-uniform temperature profile for Rayleigh-Bénard convection in a saturated porous medium has also been considered by Nield [1975] and Thangaraj [2000].

Motivated by the above investigations, our aim in this section of the chapter is to examine the effect of non-uniform temperature gradient and temperature dependent viscosity on the Rayleigh-Bénard convection in a saturated porous medium in view of modified theory of Banerjee et al. [1983].

6.2 THE PHYSICAL CONFIGURATION

Consider an infinite, viscous saturated porous fluid layer in the force field of gravity heated from below. The fluid layer of density $\rho$ is confined between two horizontal boundaries $z = 0$ and $z = d$, maintained respectively at constant temperature $T_0$ and $T_1 (T_0 > T_1)$. Let the origin be taken on the lower boundary $z = 0$ with $z$-axis perpendicular to it so that $xy$-plane constitutes the horizontal plane $z = 0$.

Our objective is to examine the onset of instability on convection otherwise stable configuration, when the temperature gradient is non-uniform and the viscosity is temperature dependent. It is assumed that the saturated fluid and the porous layer are incompressible and the porosity of the medium is constant.
The basic hydrodynamic equations that govern the above physical problem in view of Darcy’s Law (cf. Wankat and Schowalter, [1970]) are given by:

(i) **Equations of Motion**

\[ \rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho X_i - \frac{\partial p}{\partial x_i} - \mu \frac{u_i}{K_i} \]

(ii) **Equation of Continuity**

\[ \frac{\partial u_j}{\partial x_j} = 0 \]

(iii) **Equation of Heat Conduction**

\[ \epsilon \left[ \frac{\partial (cT)}{\partial t} + u_j \frac{\partial (cT)}{\partial x_j} + (1 - \epsilon) \frac{\partial (p c_s T)}{\partial t} \right] = \frac{\partial}{\partial x_j} \left( K \frac{\partial T}{\partial x_j} \right) \]

The above basic hydrodynamic equations must be supplemented by equations of state and the following equations for substances with which we are concerned:

\[ \rho = \rho_0 \left[ 1 - \alpha \left( T - T_0 \right) \right]; \quad c = c_0 \left[ 1 - \alpha_2 \left( T - T_0 \right) \right]; \]

\[ \rho_s = \rho_{s0} \left[ 1 - \alpha_1 \left( T - T_0 \right) \right]; \quad c_s = c_{s0} \left[ 1 - \alpha_3 \left( T - T_0 \right) \right]; \]

\[ K = K_0 \left[ 1 - \alpha_5 \left( T - T_0 \right) \right]; \quad \alpha = \alpha_0 \left[ 1 - \alpha_6 \left( T - T_0 \right) \right]; \quad (6.2.1) \]

In the above equations; \( u_i \) is the \( i \)th component of velocity; \( X_i \) is the \( i \)th component of external force; \( p \) is the pressure; \( \rho_s \) is the solid density; \( K_1 \) is the permeability of the porous medium; \( c \) is the heat capacity of the fluid; \( c_s \) is the heat capacity of the solid; \( \epsilon \) is the porosity of the medium. \( K \) is the thermal conductivity of the fluid-solid mixture, \( \alpha \) is the thermal expansion coefficient of the fluid and \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \) are the respective coefficients of variations analogous \( \alpha \) and are in the range of \( 10^{-3} \) to \( 10^{-4} \).

The various other symbols have the same meanings as defined in earlier chapters.

Now, applying the modified Boussinesq approximations as suggested by Banerjee et. al. [1983] and Pathania [1986], the above equations yield the following modified simplified equations governing the present problem in porous medium are;

\[ \rho_0 \frac{\partial u_i}{\partial t} + \rho_0 u_j \frac{\partial u_i}{\partial x_j} = \rho X_i - \frac{\partial p}{\partial x_i} - \mu_0 \frac{u_i}{K_i} \quad (6.2.2) \]

\[ \frac{\partial u_j}{\partial x_j} = 0 \quad (6.2.3) \]

\[ \left[ M \left[ 1 - T (\alpha_1 + \alpha_3) + (1 - \alpha_2 T) \right] \right] \frac{\partial T}{\partial t} + (1 - \alpha_2 T) u_j \frac{\partial T}{\partial x_j} = \kappa \nabla^2 T; \quad (6.2.4) \]

Here, and \( \kappa = \frac{K_0}{\rho_0 c_0} \) is the thermal diffusivity defined as the thermal conductivity of the fluid- solid mixture divided by the heat capacity and density of the fluid and \( M = \frac{\rho_{s0} c_{s0}}{\rho_0 c_0} \frac{1 - \epsilon}{\epsilon} \) is a constant.
6.3. INITIAL STATIONARY STATE AND SOLUTION

Since the equilibrium state under consideration is a static one, so it is clearly characterized by the following equations;
\[ u_i = (u, v, w) = (0,0,0); \quad T = T_b(z); \quad p = p_b(z) \quad \mu = \mu_0 f(z); \]
and \[ \rho = \rho_b(z); \quad F(z) = -\frac{d}{dx} f(z); \]  \hspace{1cm} (6.3.1)
where, the subscript 'b' indicates basic state value and the monotonic non-dimensional basic temperature gradient \( F(z) \) which is non-negative satisfies the condition \( \int_0^1 F(z)dz = 1 \) (see Nield [1975]).

The non-uniformity of \( T_b \) as defined in (6.3.1) finds its origin in transient heating or cooling at the boundaries.

When no motions are present, the hydrodynamical equations (6.2.2) require that the pressure distribution is governed by
\[ \frac{\partial \rho}{\partial z} = \rho_b X_i = -\rho_b g \]  \hspace{1cm} (6.3.2)
where, \( X_i = (0,0,-g); \ g \) is the acceleration due to gravity.

Also, equation (6.2.4) implies that the basic temperature distribution is governed by
\[ \frac{d^2 T_b}{dz^2} = 0 \]  \hspace{1cm} (6.3.3)
and the density distribution is given by;
\[ \rho = \rho_0[1 - \alpha(T_b - T_0)]. \]  \hspace{1cm} (6.3.4)
where, \( \rho_0 \) represents the density at the lower boundary \( z = 0 \).

6.4. THE PERTURBATION EQUATIONS AND BOUNDARY CONDITIONS

Let the initial state described by equations (6.3.1) be slightly perturbed, so that the perturbed state is given by;
\[ (u_1', u_2', u_3') = (0,0,0) + (u, v, w); \quad p' = p_b + \delta p; \quad T' = T_b + \theta; \]
\[ \rho' = \rho_0[1 - \alpha(T_b + \theta - T_0)] \]  \hspace{1cm} (6.4.1)
where, \((u, v, w), \delta p \) and \( \theta \) are the perturbations in the initial velocities, pressure \( p \) and temperature \( T \), respectively.

Substituting the perturbed quantities given in (6.4.1) in equations (6.2.2)-(6.2.4), using the initial state solutions given in (6.3.1)-(6.3.4), linearizing the resulting equations by neglecting the products of perturbations and higher order terms containing perturbations, we get the following linearized perturbation equations;
\[ \rho_0 \frac{\partial u}{\partial t} = -\frac{\partial (\delta p)}{\partial x} - \mu_0 f(z) \frac{\epsilon}{k_1} u \]  \hspace{1cm} (6.4.2)
\[ \rho_0 \frac{\partial v}{\partial t} = - \frac{\partial (\delta p)}{\partial y} - \mu_0 f(z) \frac{e}{k_1} v \quad (6.4.3) \]
\[ \rho_0 \frac{\partial w}{\partial t} = - \frac{\partial (\delta p)}{\partial x} - \mu_0 f(z) \frac{e}{k_1} w + \rho_0 (\alpha \theta) g \quad (6.4.4) \]
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (6.4.5) \]
\[ A \frac{\partial \theta}{\partial t} = (1 - \alpha_2 T_0) F(z) \frac{\Delta T}{d} w + \kappa \nabla^2 \theta \quad (6.4.6) \]

where, \( A = M[1 - \alpha_2 T_0 (\alpha_1 + \alpha_3) + (1 - \alpha_2 T_0)] \).

Since the fluid under consideration is confined between two horizontal planes \( z = 0 \) and \( z = d \), the fluid parameters must satisfy certain boundary conditions on them.

In view of the identical nature of the physical problem and the bounding surfaces as discussed in Chapter 5, we have the following boundary conditions on \( w \) and \( \theta \):

\[ w = 0 = \frac{\partial \theta}{\partial x} = \frac{\partial w}{\partial x} \text{ on a rigid surface} \quad (6.4.7) \]
\[ w = 0 = \frac{\partial \theta}{\partial x} = \frac{\partial^2 w}{\partial x^2} \text{ on a free surface.} \quad (6.4.8) \]

**6.5 THE ANALYSIS IN TERMS OF NORMAL MODES**

We shall now investigate the stability of the system by analyzing an arbitrary perturbation into a complete set of normal modes individually.

Proceeding exactly as in Chapter 2, we take the following dependence of the perturbations \( u, v, w, \delta p \) and \( \theta \):

\[ g(x, y, z, t) = g(z) e^{i(k_x x + k_y y + n t)} \quad (6.5.1) \]

For the perturbations with dependence on \( x, y \) and \( t \), we have

\[ \frac{\partial}{\partial t} \equiv n; \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = -k^2; \quad \nabla^2 \equiv \frac{\partial^2}{\partial z^2} - k^2. \quad (6.5.2) \]

The linearized perturbation equations (6.4.2)-(6.4.6), in view of (6.5.1) and (6.5.2) become

\[ \rho_0 n u = -ik_x \delta p - \mu_0 f(z) \frac{e}{k_1} u \quad (6.5.3) \]
\[ \rho_0 n v = -ik_y \delta p - \mu_0 f(z) \frac{e}{k_1} v \quad (6.5.4) \]
\[ \rho_0 n w = -\frac{d}{dz} \delta p - \mu_0 f(z) \frac{e}{k_1} w + \rho_0 (\alpha \theta) g \quad (6.5.5) \]
\[ \frac{dw}{dz} = -i(k_x u + k_y v) \quad (6.5.6) \]
\[ An \theta = (1 - \alpha_2 T_0) F(z) \frac{\Delta T}{d} w F(z) + \kappa \left( \frac{d^2}{dz^2} - k^2 \right) \theta \quad (6.5.7) \]

In the above equations, \( u, v, w, \delta p \) and \( \theta \) are now functions of \( z \) only.

Multiplying equation (6.5.3) by \( ik_x \) and equation (6.5.4) by \( ik_y \), adding the resulting equations and making use of (6.5.6), we have

\[ \rho_0 \frac{dw}{dz} = k^2 \delta p - \mu_0 f(z) \frac{e}{k_1} Dw \quad (6.5.8) \]
Eliminating $\delta \rho$ between equations (6.5.5) and (6.5.8), we have

$$[\rho_0 n + \mu_0 f(z) \frac{e^{k_z}}{k_z}] \left( \frac{d^2}{dz^2} - k^2 \right) w + \frac{\mu_0}{k_z} Df Dw = -\rho_0 (\alpha \theta) k^2 g$$

(6.5.9)

where $\delta_0 = \frac{\mu_0}{\rho_0}$.

Using the following non-dimensional quantities

$$z^* = \frac{z}{a}, \quad D = \frac{d}{dz}, \quad a^* = kd, \quad P = \frac{k_z}{ed^2}$$

$$p^* = \frac{\eta a^2}{k_z}, \quad w^* = w, \quad \theta^* = \frac{\eta \theta}{\Delta T d}, \quad \sigma^* = \frac{\delta_0}{\kappa}$$

(6.5.10)

in equations (6.5.9) and (6.5.7), dropping the star for convenience in writing, we obtain the following linearized perturbation equations governing the present problem;

$$[\frac{p}{\sigma} + \frac{f}{p}] (D^2 - a^2) w + \frac{Df Dw}{p} = -Ra^2 \theta$$

(6.5.11)

$$(D^2 - a^2 - Ap) \theta = -(1 - \alpha^2 \theta) F(z) w$$

(6.5.12)

In the above equations; $a^2$ represents the square of the wave number, $p(= p_r + ip_i)$ is the complex growth rate; $f(T)$ is the temperature dependent viscosity factor, $R = \frac{g a \Delta T d^3}{\kappa \theta_0}$ is the thermal Rayleigh number, $\sigma = \frac{\delta_0}{\kappa}$ is the Prandtl number, $\theta_0 = \frac{\mu_0}{\rho_0}$ is the kinematic viscosity.

The boundary conditions (6.4.7)-(6.4.8) in view of (6.5.1) and (6.5.10) assume the following forms;

$$w = 0 = D\theta = Dw \text{ on a free surface}$$

$$w = 0 = D\theta = Dw \text{ on a rigid surface}.$$

The solution of the above equations must be sought subject to certain boundary conditions. In the present analysis, we have considered the following cases of combinations of boundary conditions;

**Case 1:** Both boundaries dynamically free and thermally insulating

$$w = 0 = D\theta = D^2 w \text{ at } z = 0 \text{ and } z = 1$$

(6.5.13)

**Case 2:** Both boundaries rigid and thermally insulating

$$w = 0 = D\theta = Dw \text{ at } z = 0 \text{ and } z = 1$$

(6.5.14)

**Case 3:** Lower rigid and upper boundary free

$$w = 0 = D\theta = Dw \text{ at } z = 0 \text{ and } w = 0 = D\theta = D^2 w \text{ at } z = 1$$

(6.5.15)

Thus, the system of equations (6.5.11)-(6.5.12) together with either of the boundary conditions (6.5.13)-(6.5.15) constitutes an eigenvalue for Rayleigh number $R$, for given values of other parameters; namely $\sigma, p$ and $a$ and govern the problem of
onset of convection in modified Rayleigh-Bénard convection in saturated porous medium with non uniform temperature gradient and variable viscosity. Here viscosity and temperature gradient are both arbitrary functions of \( z \).

**Remarks 3:** The system of equations (6.5.11)-(6.5.12) together with either of the boundary conditions (6.5.13)-(6.5.15) yields the eigen value problems in a saturated porous medium governing:

a. Modified Rayleigh-Bénard Convection in a porous medium (MRBCPM) with variable viscosity, if we take \( F(z) = 1 \) and \( T_0 \alpha_2 < 1 \).

b. Rayleigh-Bénard Convection (RBCPM) in a porous medium with variable viscosity, if we take \( F(z) = 1 \) and \( \alpha_2 = 0 \).

c. Rayleigh-Bénard Convection in a porous medium (RBCPMNUTG) with non uniform temperature gradient and variable viscosity, if we take \( F(z) \) is any arbitrary function of \( z \) and \( \alpha_2 = 0 \).

Further, when the temperature dependent viscosity factor \( f = 1 \), the above problems refers to the respective configurations with constant viscosity in a saturated porous medium.

As we know, a given state of the system is stable, neutral or unstable according as \( \text{Pr} \) (real part of \( p \)) is negative, zero or positive respectively. Further, if \( \text{Pr} = 0 \) implies \( \text{Pl} = 0 \) for every wave number \( \alpha \), then the principle of exchange of stability (PES) is valid, which means that instability sets in as stationary convection, otherwise we shall have overstability at least when instability sets in as certain modes.

### 6.6 MATHEMATICAL ANALYSIS

We shall mathematically investigate the system of equations (6.5.11)-(6.5.12) subject to either of the boundary conditions (6.5.13)-(6.5.15).

First of all, we shall investigate the validity of principle of exchange of stabilities for the Rayleigh-Bénard problem when the temperature gradient is non uniform together with the temperature dependent viscosity.

We shall first prove the validity of PES for the problem.

**Theorem 6.1.** If \((p, w, \theta)\) is a solution of equations (6.5.11)-(6.5.12) together with either of the boundary conditions (6.5.13)-(6.5.15), \( T_0 \alpha_2 < 1 \) and \( A > 0 \), then \( \text{Pr} = 0 \).

In particular, \( \text{Pr} = 0 \) implies \( \text{Pl} = 0 \) i.e. PES is valid.

**Proof.** Equation (6.5.11) can be written as

\[
\frac{1}{P} D[f Dw] - \frac{L}{P} \alpha_2 w + \frac{P}{\sigma} (D^2 - \alpha^2) w = -Ra^2 \theta
\]  

(6.6.1)
Multiplying equation (6.6.1) by \( w^* \) and integrating by parts the resulting equation over the vertical range of \( z \), and using the relevant boundary conditions (6.5.13)-(6.5.15), we have

\[
\frac{1}{P} \int_0^1 w^* D[f Dw] dz - \frac{1}{P} a^2 \int_0^1 f w^* wdz + \frac{P}{\sigma} \int_0^1 (|Dw|^2 + a^2 |w|^2) dz = -Ra^2 \int_0^1 w^* \theta dz
\]

Now, using equation (6.5.12) in the right hand side of the above equation, and performing the integrating by parts a suitable number of times, we have

\[
-\frac{1}{P} \int_0^1 f |Dw|^2 dz - \frac{1}{P} a^2 \int_0^1 f |w|^2 dz - \frac{P}{\sigma} \int_0^1 (|Dw|^2 + a^2 |w|^2) dz
\]

\[
= -Ra^2 \int_0^1 \frac{(|D\theta|^2 + a^2 |\theta|^2 - Ap^* |\theta|^2)}{BF(z)} dz
\]

where, \( B = 1 - T_0 a_2 \).

Comparing the imaginary parts of equation (6.6.2), we have

\[
\frac{p_t}{\sigma} \left[ \int_0^1 (|Dw|^2 + a^2 |w|^2) dz + Ra^2 A \int_0^1 \frac{|\theta|^2 dz}{F(z)} \right] = 0
\]

(6.6.3)

Since \( R \) is positive for the present configuration (for fluid layer heated from below) and \( T_0 a_2 < 1 \) for most of the cases of the fluid and for a moderate temperature difference \( \Delta T \) (see Banerjee et.al. [1983]). Also, \( F(z) \) is non-negative in the interval \( z \in [0,1] \) and \( A > 0 \), we have, \( p_t = 0 \).

Since \( p_r \geq 0 \), in particular, \( p_r = 0 \) implies \( p_t = 0 \) i.e. PES is valid.

From the above result we get that, under the given conditions, the stationary convection is the only mode of onset of convection whether the viscosity is variable or constant (classical Bénard Problem).

**NUMERICAL ANALYSIS**

Now, we shall use the Galerkin’s method to find the value of Rayleigh numbers for each case of boundary combinations.

Since, the instability sets in as stationary convection, therefore by putting \( p = 0 \), equations (6.5.11) and (6.5.12) after little simplification reduce to the following forms;

\[
\frac{1}{P} D[f Dw] w + a^2 w = -Ra^2 \theta
\]

(6.6.4)

\[
(D^2 - a^2) \theta = BF(z) w
\]

(6.6.5)

The system of equations (6.6.4)-(6.6.5) together with either of the boundary conditions (6.5.13)-(6.5.15) now constitutes an eigenvalue problem for \( R \) for the onset of stationary convection.

We shall use the Galerkin’s method by taking a single term in the expansions for \( w \) and \( \theta \). Writing the variables \( w \) and \( \theta \) in equations (6.6.4)-(6.6.5) in the terms of the following trial functions;
\[ w = A w_1(z) \quad \text{and} \quad \theta = B \theta_1(z), \]

where, \( w_1 \) and \( \theta_1 \) are the suitably chosen trial functions which satisfy the respective boundary conditions given in (6.6.13)-(6.6.15), and \( A \) and \( B \) are arbitrary constants.

Now, multiplying the resulting equations (obtained after substituting the above trial functions in (6.6.4) and (6.6.5)) by \( w_i \) and \( \theta_i \) respectively, integrating each of the resulting equations by parts, using the relevant boundary conditions (6.6.13)-(6.6.15), and eliminating constants \( A \) and \( B \) from the resulting equations, we obtain following expression for Rayleigh number as;

\[
R = \frac{\int_0^1 f \left[ (Dw)^2 + a^2(w^2) \right] \, dz \int_0^1 (D\theta)^2 + a^2(\theta^2) \, dz}{P(1 - \alpha z)^2 a^2 \left( \int_0^1 F(z)w \theta \, dz \right) \left( \int_0^1 w \theta \, dz \right)} \quad (6.6.6)
\]

It is remarkable to note here that the expression (6.6.6) is valid for all cases of boundary conditions and for arbitrary function of viscosity variation and non-uniform temperature gradient.

We shall now compute the values of Rayleigh numbers for four different profiles of temperature gradients, namely;

(i). Constant Temperature Gradient: \( F(z) = 1 \) (uniform temperature gradient: the case of Bénard convection).

(ii) Inverted Parabolic Temperature Gradient: \( F(z) = 2(1 - z) \) (cf. Nield [1975]).

(iii) Cubic Temperature Gradient: \( F(z) = 3(z - 1)^2 \) (cf. Dupont et al. [1992]).

(iv) Cubic Temperature Gradient: \( F(z) = 0.6 + 1.02(z - 1)^2 \) (cf. Idris et al. [2009]).

Further, the Rayleigh numbers and consequently the values of the critical Rayleigh numbers \( R_c \) for different cases of boundaries conditions and each of the temperature gradients profiles (i)-(iv) shall be obtained for the exponential and linear non-dimensional viscosity variations given by;

\[
f_1 = e^{\delta z} \quad \text{and} \quad f_2 = (1 + \delta z),
\]

where, \( \delta \) is the temperature dependent viscosity factor.

The expressions for Rayleigh numbers for exponential and linear types of viscosity variation can be easily obtained from (6.6.6.) and are respectively given as;

\[
R = \frac{\int_0^1 e^{\delta z} \left[ (Dw)^2 + a^2(w^2) \right] \, dz \int_0^1 (D\theta)^2 + a^2(\theta^2) \, dz}{P(1 - \alpha z)^2 a^2 \left( \int_0^1 F(z)w \theta \, dz \right) \left( \int_0^1 w \theta \, dz \right)} \quad (6.6.8)
\]

and

\[
R = \frac{\int_0^1 (1 + \delta z) \left[ (Dw)^2 + a^2(w^2) \right] \, dz \int_0^1 (D\theta)^2 + a^2(\theta^2) \, dz}{P(1 - \alpha z)^2 a^2 \left( \int_0^1 F(z)w \theta \, dz \right) \left( \int_0^1 w \theta \, dz \right)} \quad (6.6.9)
\]

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Let us consider linear variation of viscosity defined in (6.6.7). The corresponding expression for the Rayleigh number for this type of viscosity variations is given by the expression (6.6.9), which is valid for all cases of boundary conditions (6.5.13)-(6.5.15).

In the following analysis, we shall now treat each of the cases of boundary conditions separately and derive the values of Rayleigh numbers for each of the temperature gradients profiles (i)-(iv).

**Case 1. Both boundaries dynamically free and thermally Insulating**

Let us consider the trial functions

\[ w = z^4 - 2z^3 + z, \quad \theta = 1, \]

which satisfies the boundary conditions

\[ w = 0 = D^2 w = D\theta \text{ at } z = 0 \text{ and } z = 1. \]

Now, using the above defined trial functions in expression (6.6.9) and obtain the values for Rayleigh number for each of the above defined temperature profiles.

(i). When \( F(z) = 1 \), the expression (6.6.9) for the given trial functions yields

\[
R = \frac{25}{(1-7_6a_2)^2} \left[ (I_8 - 4I_7)a^2 + 4I_6(a^2 + 4) - 2I_5(a^2 - 24) - 4I_4(a^2 - 9) + 8I_3 + I_2(a^2 - 12) + I_0 \right],
\]

where, \( I_n = \int_0^1 (1 + \delta z)z^n dz \); \( n = 0,1,2, ..., 8 \).

It is important to note here that the above expressions obtained for each case of temperature profiles has a common factor put in square brackets. Thus, the minimum value for each of the above expressions for \( R \) shall depend upon this common factor, as the other terms in each of these expressions are constant.

To find the minimum value of \( R \) in each of the above expressions, let us write the above said common factor namely;

\[
(I_8 - 4I_7)a^2 + 4I_6(a^2 + 4) - 2I_5(a^2 - 24) - 4I_4(a^2 - 9) + 8I_3 + I_2(a^2 - 12) + I_0
\]

in the following convenient form

\[
(I_8 - 4I_7 + 4I_6 - 2I_5 - 4I_4 + I_2)a^2 + (16I_6 + 48I_5 + 36I_4 + 8I_3 - 12I_2 + I_0).
\]

It is clear from the above expression that the definite integral \( I_0 \) to \( I_8 \) shall have real values and the coefficient of \( a^2 \) is positive, therefore the minimum of this expression shall depend upon the value of \( a^2 \) only. Hence, the minimum of the expression and consequently the minimum values of \( R \) exist at \( a^2 = 0 \).
In the following tables we have presented the values of $R_c$ for different values various parameters for $a^2 = 0$.

The minimum of $R$ i.e. $R_c$ exists at $a^2 = 0$, and is given by

\[ R_c = \frac{5(34+1018)}{14(1-7a^2)P}. \]

We have the following corollaries in view of Remarks 3;

**Corollary 6.7.1.** For MRBCPM, we have $R_c = \frac{85}{7(1-7a^2)P}$ for constant viscosity ($\delta = 0$).

**Corollary 6.7.2.** For RBCPM, we have $R_c = \frac{5(34+1018)}{14P}$ for variable viscosity ($\delta \neq 0$).

**Corollary 6.7.3.** For RBCPM, we have $R_c = 12.142857$, for constant viscosity ($\delta = 0$) and $a_2 = 0$; $P = 1$ which is the same value obtained by Rudraiah et. al. [1980] for Bénard Convection for this case of boundary conditions.

(ii). When $F(z) = 2(1 - z)$, the expression (6.6.9) for the given trial functions yields

\[ R = \frac{25}{(1-7a^2)P} [(l_8 - 4l_7) a^2 + 4l_6(a^2 + 4) - 2l_5(a^2 - 24) - 4l_4(a^2 - 9) + 8l_3 + l_2(a^2 - 12) + l_0]. \]

The minimum of $R$ i.e. $R_c$ exists at $a^2 = 0$, and is given by

\[ R_c = \frac{5(34+1018)}{14(1-7a^2)P}, \]

which is the same as obtained in case (i) when $F(z) = 1$.

Hence the Corollaries 6.7.1-6.7.3 are also true in this case of temperature profiles.

(iii). When $F(z) = 3(z - 1)^2$, the expression (6.6.9) for the given trial functions yields

\[ R = \frac{28}{(1-7a^2)P} [(l_8 - 4l_7) a^2 + 4l_6(a^2 + 4) - 2l_5(a^2 - 24) - 4l_4(a^2 - 9) + 8l_3 + l_2(a^2 - 12) + l_0]. \]

The minimum of $R$ i.e. $R_c$ exists at $a^2 = 0$, and is given by

\[ R_c = \frac{2(34+1018)}{5(1-7a^2)P}. \]

We have now the following corollaries in view of Remarks 3;

**Corollary 6.7.4.** For MRBCPM, we have $R_c = \frac{68}{5(1-7a^2)P}$ for constant viscosity ($\delta = 0$).

**Corollary 6.7.5.** For RBCPM, we have $R_c = \frac{2(34+1018)}{5P}$ for variable viscosity ($\delta \neq 0$).

**Corollary 6.7.6.** For RBCPM, we have $R_c = 13.6$, for constant viscosity ($\delta = 0$) and $a_2 = 0$; $P = 1$.

(iv). When $F(z) = 0.6 + 1.02(z - 1)^2$, the expression (6.6.9) for the given trial functions yields
\[ R = \frac{7000}{253(1-T_0a_2)^p} \left[ (I_6 - 4I_7)a^2 + 4I_6(a^2 + 4) - 2I_5(a^2 - 24) - 4I_4(a^2 - 9) + 8I_3 + 
I_2(a^2 - 12) + I_0 \right]. \]

The minimum of \( R \) i.e. \( R_c \) exists at \( a^2 = 0 \), and is given by
\[
R_c = \frac{100(34+101\delta)}{253(1-T_0a_2)^p}.
\]

We have now the following corollaries in view of Remarks 3;

**Corollary 6.7.7.** For MRBCPM, we have \( R_c = \frac{3400}{253(1-T_0a_2)^p} \) for constant viscosity \( (\delta = 0) \).

**Corollary 6.7.8.** For RBCPM, we have \( R_c = \frac{100(34+101\delta)}{253(1-T_0a_2)^p} \) for variable viscosity \( (\delta \neq 0) \).

**Corollary 6.7.9.** For RBCPM, we have \( R_c = 13.438735 \), for constant viscosity \( (\delta = 0) \) and \( a_2 = 0; \ P = 1 \).

In the following table we have presented the values of \( R_c \) for various values of the parameters for Case 1 of boundary conditions for each of the temperature profiles.

<table>
<thead>
<tr>
<th>( F(z) = 1 )</th>
<th>( F(z) = 2(1-z) )</th>
<th>( F(z) = 3(z-1)^2 )</th>
<th>( F(z) = 0.6 + 1.02(z-1)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
<td>( B=0.95 )</td>
<td>( B=1 )</td>
<td>( B=0.95 )</td>
</tr>
<tr>
<td>0.0</td>
<td>12.78</td>
<td>12.14</td>
<td>12.78</td>
</tr>
<tr>
<td>0.1</td>
<td>16.58</td>
<td>15.75</td>
<td>16.58</td>
</tr>
<tr>
<td>0.2</td>
<td>20.38</td>
<td>19.36</td>
<td>20.38</td>
</tr>
<tr>
<td>0.3</td>
<td>24.17</td>
<td>22.96</td>
<td>24.17</td>
</tr>
<tr>
<td>0.4</td>
<td>27.97</td>
<td>26.57</td>
<td>27.97</td>
</tr>
<tr>
<td>0.5</td>
<td>31.77</td>
<td>30.18</td>
<td>31.77</td>
</tr>
<tr>
<td>0.6</td>
<td>35.56</td>
<td>33.79</td>
<td>35.56</td>
</tr>
</tbody>
</table>

**Case 2. Both boundaries rigid and thermally Insulating**

Let us consider the trial functions
\[ w = z^4 - 2z^3 + z^2, \ \theta = 1, \]
which satisfies the boundary conditions
\[ w = 0 = Dw = D\theta \text{ at } z = 0 \text{ and } z = 1. \]

Now, using the above defined trial functions in expression (6.6.9) and obtain the values for Rayleigh number for each of the above defined temperature profiles.

(i). When \( F(z) = 1 \), the expression (6.6.9) for the given trial functions yields
\[
R = \frac{900((I_6 - 4I_7)a^2 + 2I_6(3a^2 + 8) - 4I_5(a^2 + 12) + I_4(5a^2 - 24I_3 + 4I_2))}{(1-T_0a_2)^p}.
\]
The minimum of \( R \) i.e. \( R_c \) exists at \( a^2 = 0 \), and is given by
\[
R_c = \frac{60(2+\delta)}{7(1-T_0a_2)^P}.
\]

We have now the following corollaries in view of Remarks 3;

**Corollary 6.7.10.** For MRBCPM, we have \( R_c = \frac{120}{(1-T_0a_2)^P} \) for constant viscosity (\( \delta = 0 \)).

**Corollary 6.7.11.** For RBCPM, we have \( R_c = \frac{60(2+\delta)}{7P} \) for variable viscosity (\( \delta \neq 0 \)).

**Corollary 6.7.12.** For RBCPM, we have \( R_c = 17.142857 \), for constant viscosity (\( \delta = 0 \)) and \( \alpha_2 = 0; P = 0 \), which is the same value obtained by Rudraiah et. al [1980] for Bénard Convection for this case of boundary conditions.

(ii) When \( (z) = 2(1 - z) \), the expression (6.6.9) for the given trial functions yields
\[
R = \frac{900((l_6-4l_7)a^2+2l_6(a^2+8)-4l_6(a^2+12)+l_4(52+a^2)-24l_5+4l_7)}{(1-T_0a_2)^P}.
\]
The minimum of \( R \) i.e. \( R_c \) exists at \( a^2 = 0 \), and is given by
\[
R_c = \frac{60(2+\delta)}{7(1-T_0a_2)^P}.
\]
which is the same value as obtained in case (i) of temperature profile, hence the results obtained for the former case are also valid for this case.

(iii) When \( F(z) = 3(z - 1)^2 \), the expression (6.6.9) for the given trial functions yields
\[
R = \frac{1050((l_6-4l_7)a^2+2l_6(a^2+8)-4l_6(a^2+12)+l_4(52+a^2)-24l_5+4l_7)}{(1-T_0a_2)^P}.
\]
The minimum of \( R \) i.e. \( R_c \) exists at \( a^2 = 0 \), and is given by
\[
R_c = \frac{10(2+\delta)}{(1-T_0a_2)^P}.
\]

We have the following corollaries in view of Remarks 3;

**Corollary 6.7.13.** For MRBCPM, we have \( R_c = \frac{20}{P(1-T_0a_2)} \) for constant viscosity (\( \delta = 0 \)).

**Corollary 6.7.14.** For RBCPM, we have \( R_c = \frac{10(2+\delta)}{P} \) for variable viscosity (\( \delta \neq 0 \)).

**Corollary 6.7.15.** For RBCPM, we have \( R_c = 20 \), for constant viscosity (\( \delta = 0 \)) and \( \alpha_2 = 0; P = 1 \).

(iv) When \( F(z) = 0.6 + 1.02(z - 1)^2 \), the expression (6.6.9) for the given trial functions yields
\[
R = \frac{875((l_6-4l_7)a^2+2l_6(3a^2+8)-4l_6(a^2+12)+l_4(52+a^2)-24l_5+4l_7)}{26(1-T_0a_2)^P}.
\]
The minimum of \( R \) i.e. \( R_c \) exists at \( a^2 = 0 \), and is given by
We have the following corollaries in view of Remarks 3;

**Corollary 6.7.16.** For MRBCPM, we have \( R_c = \frac{875(2+\delta)}{91(1-T_0a_2)^p} \) for constant viscosity \((\delta = 0)\).

**Corollary 6.7.17.** For RBCPM, we have \( R_c = \frac{875(2+\delta)}{91p} \) for variable viscosity \((\delta \neq 0)\).

**Corollary 6.7.18.** For RBCPM, we have \( R_c = 19.230769 \) for constant viscosity \((\delta = 0)\) and \( a_2 = 0; \; p = 1 \).

In the following table we have presented the values of \( R_c \) for various values of the parameters for Case 2 of boundary conditions for each of the temperature profiles.

**Table 6.7.2:** The values of \( R_c \) corresponding to different values of \( \delta \) and \( B = (1 - T_0a_2) \); \( p = 1 \) and for various temperature profiles, for both rigid boundaries, for linear viscosity variations.

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>( F(z) = 1 )</th>
<th>( F(z) = 2(1 - z) )</th>
<th>( F(z) = 3(z - 1)^2 )</th>
<th>( F(z) = 0.6 + 1.02(z - 1)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
<td>( R_c )</td>
<td>( R_c )</td>
<td>( R_c )</td>
<td>( R_c )</td>
</tr>
<tr>
<td>( B=0.95 )</td>
<td>( B=1 )</td>
<td>( B=0.95 )</td>
<td>( B=1 )</td>
<td>( B=0.95 )</td>
</tr>
<tr>
<td>0.0</td>
<td>18.05</td>
<td>17.14</td>
<td>18.05</td>
<td>17.14</td>
</tr>
<tr>
<td>0.1</td>
<td>18.95</td>
<td>18</td>
<td>18.95</td>
<td>18</td>
</tr>
<tr>
<td>0.2</td>
<td>19.85</td>
<td>18.86</td>
<td>19.85</td>
<td>18.86</td>
</tr>
<tr>
<td>0.3</td>
<td>20.75</td>
<td>19.71</td>
<td>20.75</td>
<td>19.71</td>
</tr>
<tr>
<td>0.4</td>
<td>21.65</td>
<td>20.57</td>
<td>21.65</td>
<td>20.57</td>
</tr>
<tr>
<td>0.5</td>
<td>22.56</td>
<td>21.43</td>
<td>22.56</td>
<td>21.43</td>
</tr>
<tr>
<td>0.6</td>
<td>23.46</td>
<td>22.29</td>
<td>23.46</td>
<td>22.29</td>
</tr>
</tbody>
</table>

**Case 3. Lower boundary rigid and upper free and both thermally insulating**

Let us consider the trial functions

\[ w = 2z^4 - 5z^3 + 3z^2, \; \theta = 1, \]

which satisfies the boundary conditions

\[ w = 0 = Dw = D\theta \text{ at } z = 0 \text{ and } w = 0 = D^2w = D\theta \text{ at } z = 1 \]

Now, using the above defined trial functions in expression (6.6.9) and obtain the values for Rayleigh number for each of the above defined temperature profiles.

1. When \( F(z) = 1 \), the expression (6.6.9) for the given trial functions yields

\[
R = \frac{400(4l_6-20l_7)a^2+l_6(64+37a^2)-30l_5(a^2+8)+l_4(321+9a^2)-180l_3+36l_2}{9(1-T_0a_2)^p}.
\]

The minimum of \( R \) i.e. \( R_c \) exists at \( a^2 = 0 \), and is given by
We have the following corollaries in view of Remarks 3;

**Corollary 6.7.19.** For MRBCPM, we have \( R_c = \frac{320 (24 + 15\delta)}{21 (1 - \theta_0 a_2)} \) for constant viscosity \((\delta = 0)\).

**Corollary 6.7.20.** For RBCPM, we have \( R_c = \frac{40 (24 + 15\delta)}{63 \rho} \) for variable viscosity \((\delta \neq 0)\).

**Corollary 6.7.21.** For RBCPM, we have \( R_c = 15.238095 \) for constant viscosity \((\delta = 0)\) and \( a_2 = 0; \rho = 1 \), which is the same value obtained by Rudraiah et. al [1980] for Bénard Convection for this case of boundary conditions.

2. When \( F(z) = 2(1 - z) \), the expression (6.6.9) for the given trial functions yields
\[
R = \frac{50 ((4l_0 - 20l_2) a^2 + l_0 (64 + 37 a^2) - 30l_4 (a^2 + 8) + l_4 (321 + 9a^2) - 180l_5 + 36l_2)}{(1 - \theta_0 a_2) \rho}.
\]
The minimum of \( R \) i.e. \( R_c \) exists at \( a^2 = 0 \), and is given by
\[
R_c = \frac{5 (24 + 15\delta)}{7 (1 - \theta_0 a_2)}.
\]
We have the following corollaries in view of Remarks 3;

**Corollary 6.7.22.** For MRBCPM, we have \( R_c = \frac{120}{7 (1 - \theta_0 a_2) \rho} \) for constant viscosity \((\delta = 0)\).

**Corollary 6.7.23.** For RBCPM, we have \( R_c = \frac{5 (24 + 15\delta)}{7 \rho} \) for variable viscosity \((\delta \neq 0)\).

**Corollary 6.7.24.** For RBCPM, we have \( R_c = 17.142857 \) for constant viscosity \((\delta = 0)\) and \( a_2 = 0; \rho = 1 \).

3. When \( F(z) = 3(z - 1)^2 \), the expression (6.6.9) for the given trial functions yields
\[
R = \frac{560 ((4l_0 - 20l_2) a^2 + l_0 (64 + 37 a^2) - 30l_4 (a^2 + 8) + l_4 (321 + 9a^2) - 180l_5 + 36l_2)}{9 (1 - \theta_0 a_2) \rho}.
\]
The minimum of \( R \) i.e. \( R_c \) exists at \( a^2 = 0 \), and is given by
\[
R_c = \frac{8 (24 + 15\delta)}{9 (1 - \theta_0 a_2) \rho}.
\]
We have the following corollaries in view of Remarks 3;

**Corollary 6.7.25.** For MRBCPM, we have \( R_c = \frac{192}{9 (1 - \theta_0 a_2)} \) for constant viscosity \((\delta = 0)\).

**Corollary 6.7.26.** For RBCPM, we have \( R_c = \frac{8 (24 + 15\delta)}{9 \rho} \) for variable viscosity \((\delta \neq 0)\).

**Corollary 6.7.27.** For RBCPM, we have \( R_c = 21.3333 \) for constant viscosity \((\delta = 0)\) and \( a_2 = 0; \rho = 1 \).

4. When \( F(z) = 0.6 + 1.02(z - 1)^2 \), the expression (6.6.9) for the given trial functions yields
The minimum of $R$ i.e. $R_c$ exists at $\alpha^2 = 0$, and is given by

$$R_c = \frac{400(24+155)}{531(1-T_0\alpha_2)^P}.$$ 

We have the following corollaries in view of Remarks 3;

**Corollary 6.7.28.** For MRBCPM, we have $R_c = \frac{9600}{531(1-T_0\alpha_2)^P}$ for constant viscosity ($\delta = 0$).

**Corollary 6.7.29.** For RBCPM, we have $R_c = \frac{400(24+155)}{531^P}$ for variable viscosity ($\delta \neq 0$).

**Corollary 6.7.30.** For RBCPM, we have $R_c = 18.079096$, for constant viscosity ($\delta = 0$) and $\alpha_2 = 0; P = 1$.

In the following table we have presented the values of $R_c$ for various values of the parameters for Case 3 of boundary conditions for each of the temperature profiles.

**Table 6.7.3:** The values of $R_c$ corresponding to different values of $\delta$ and $B = (1 - T_0\alpha_2)$; $P = 1$ and for various temperature profiles, for both Lower rigid and upper free boundary, for linear viscosity variations.

<table>
<thead>
<tr>
<th>$F(z)$</th>
<th>$R_c$</th>
<th>$R_c$</th>
<th>$R_c$</th>
<th>$R_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F(z) = 1.$</td>
<td>$B=0.95$</td>
<td>$B=1$</td>
<td>$B=0.95$</td>
<td>$B=1$</td>
</tr>
<tr>
<td>$F(z) = 2(1-z).$</td>
<td>$B=0.95$</td>
<td>$B=1$</td>
<td>$B=0.95$</td>
<td>$B=1$</td>
</tr>
<tr>
<td>$F(z) = 3(z-1)^2$</td>
<td>$B=0.95$</td>
<td>$B=1$</td>
<td>$B=0.95$</td>
<td>$B=1$</td>
</tr>
<tr>
<td>$F(z) = 0.6 + 1.02(z-1)^2$</td>
<td>$B=0.95$</td>
<td>$B=1$</td>
<td>$B=0.95$</td>
<td>$B=1$</td>
</tr>
</tbody>
</table>

6.8. **EXPONENTIAL VARIATION OF VISCOSITY**

Let us consider exponential variation of viscosity defined in (6.6.7). The corresponding expression for the Rayleigh number for this type of viscosity variations is given by the expression (6.6.8), which is valid for all cases of boundary conditions (6.5.13)-(6.5.15).

In the following analysis, we shall now treat each of the cases of boundary conditions separately and derive the values of Rayleigh numbers for each of the temperature gradients profiles (i)-(iv).
Case 1. Both boundaries are dynamically free and thermally insulating

Let us consider the trial functions
\[ w = z^4 - 2z^3 + z, \quad \theta = 1, \]
which satisfies the boundary conditions
\[ w = 0 = D^2w = D\theta \text{ at } z = 0 \text{ and } z = 1. \]

Now, using the above defined trial functions in expression (6.6.8) and obtain the values for Rayleigh number for each of the above defined temperature profiles.

(i) When \( F(z) = 1 \), the expression (6.6.8) for the given trial functions yields
\[
R = \frac{25[(l_8 - 4l_7)a^2 + 4l_6(a^2 + 4) - 2l_5(a^2 - 24) - 4I_4(a^2 - 9) + 8I_3 + l_2(a^2 - 12) + I_0]}{(1 - T_0\alpha_2)^p};
\]
where, \( I_n = \int_0^1 e^{\delta z} z^n \, dz ; \ n = 0, 1, 2, ..., 8. \)

(ii) When \( F(z) = 2(1 - z) \), the expression (6.6.8) for the given trial functions yields
\[
R = \frac{25[(l_8 - 4l_7)a^2 + 4l_6(a^2 + 4) - 2l_5(a^2 - 24) - 4I_4(a^2 - 9) + 8I_3 + l_2(a^2 - 12) + I_0]}{(1 - T_0\alpha_2)^p};
\]
which is the same as obtained in case (i) of temperature profile \( F(z) = 1 \).

(iii) When \( F(z) = 3(z - 1)^2 \), the expression (6.6.8) for the given trial functions yields
\[
R = \frac{28[(l_8 - 4l_7)a^2 + 4l_6(a^2 + 4) - 2l_5(a^2 - 24) - 4I_4(a^2 - 9) + 8I_3 + l_2(a^2 - 12) + I_0]}{(1 - T_0\alpha_2)^p};
\]

(iv) When \( F(z) = 0.6 + 1.02(z - 1)^2 \), the expression (6.6.8) for the given trial functions yields
\[
R = \frac{7000[(l_8 - 4l_7)a^2 + 4l_6(a^2 + 4) - 2l_5(a^2 - 24) - 4I_4(a^2 - 9) + 8I_3 + l_2(a^2 - 12) + I_0]}{253(1 - T_0\alpha_2)^p};
\]

It is important to note here that the above expressions obtained for each case of temperature profiles has a common factor put in square brackets. Thus, the minimum value for each of the above expressions for \( R \) shall depend upon this common factor, as the other terms in each of these expressions are constant.

To find the minimum value of \( R \) in each of the above expressions, let us write the above said common factor namely:
\[
(l_8 - 4l_7)a^2 + 4l_6(a^2 + 4) - 2l_5(a^2 - 24) - 4I_4(a^2 - 9) + 8I_3 + l_2(a^2 - 12) + I_0
\]
in the following convenient form
\[
(l_8 - 4l_7 + 4l_6 - 2l_5 - 4I_4 + l_2)a^2 + (16l_6 + 48l_5 + 36l_4 + 8I_3 - 12I_2 + I_0).
\]

It is clear from the above expression that the definite integral \( I_0 \) to \( I_6 \) shall have real values, therefore the minimum of this expression shall depend upon the value of \( a^2 \) only.
Hence, the minimum of the expression and consequently the minimum values of $R$ exist at $a^2 = 0$.

In the following tables we have presented the values of $R_c$ for different values various parameters for $a^2 = 0$.

**Table 6.8.1:** The values of $R_c$ corresponding to different values of $\delta$ and $B = (1 - T_0a^2)$ and for various temperature profiles, for both dynamically free boundaries, for exponential viscosity variations.

<table>
<thead>
<tr>
<th>Viscosity Variations</th>
<th>Linear $F(z) = 1.$</th>
<th>Inverted parabolic $F(z) = 2(1 - z)$</th>
<th>Cubic-1 $F(z) = 3(z - 1)^2$</th>
<th>Cubic-2 $F(z) = 0.6 + 1.02(z - 1)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>$B=0.95$</td>
<td>$B=0.95$</td>
<td>$B=0.95$</td>
<td>$B=0.95$</td>
</tr>
<tr>
<td>$R_c$</td>
<td>$R_c$</td>
<td>$R_c$</td>
<td>$R_c$</td>
<td>$R_c$</td>
</tr>
<tr>
<td>0.0</td>
<td>12.78</td>
<td>12.14</td>
<td>12.78</td>
<td>12.14</td>
</tr>
<tr>
<td>0.1</td>
<td>13.45</td>
<td>12.77</td>
<td>13.45</td>
<td>12.77</td>
</tr>
<tr>
<td>0.2</td>
<td>14.17</td>
<td>13.46</td>
<td>14.17</td>
<td>13.46</td>
</tr>
<tr>
<td>0.3</td>
<td>14.94</td>
<td>14.19</td>
<td>14.94</td>
<td>14.19</td>
</tr>
<tr>
<td>0.4</td>
<td>15.78</td>
<td>14.99</td>
<td>15.78</td>
<td>14.99</td>
</tr>
<tr>
<td>0.5</td>
<td>16.69</td>
<td>15.86</td>
<td>16.69</td>
<td>15.86</td>
</tr>
<tr>
<td>0.6</td>
<td>17.68</td>
<td>16.79</td>
<td>17.68</td>
<td>16.79</td>
</tr>
</tbody>
</table>

**Case 2. Both boundaries are rigid and thermally insulating**

The boundary conditions are

$w = 0 = Dw = D\theta$ at $z = 0$ and $z = 1$ and these are satisfied by $w = z^4 - 2z^3 + z^2, \theta = 1$. Now, using the above defined trial functions in expression (6.6.8) and obtain the values for Rayleigh number for each of the above defined temperature profiles.

**(i)** When $F(z) = 1$, the expression (6.6.8) for the given trial functions yields

$$R = \frac{900(1_6 - 41_7)a^2 + 21_6(3a^2 + 8) - 41_6(a^2 + 12) + 1_6(52 + a^2) - 241_3 + 41_2}{(1 - 7_0a^2)}.$$  

**(ii)** When $F(z) = 2(1 - z)$, the expression (6.6.8) for the given trial functions yields

$$R = \frac{900(1_6 - 41_7)a^2 + 21_6(3a^2 + 8) - 41_6(a^2 + 12) + 1_6(52 + a^2) - 241_3 + 41_2}{(1 - 7_0a^2)};$$

which is same as obtained in (i) for $F(z) = 1$.

**(iii)** When $F(z) = 3(z - 1)^2$, the expression (6.6.8) for the given trial functions yields

$$R = \frac{1050(1_6 - 41_7)a^2 + 21_6(3a^2 + 8) - 41_6(a^2 + 12) + 1_6(52 + a^2) - 241_3 + 41_2}{(1 - 7_0a^2)};$$
(iv) When \( F(z) = 0.6 + 1.02(z - 1)^2 \), the expression (6.6.8) for the given trial functions yields

\[
R = \frac{13125(\log a^2 + 2\log_6(3a^2 + 8) - 4\log_6(a^2 + 12) + a_4(52a^2 - 24a + 4))}{13(1 - T_0 a_2)^P}.
\]

It is important to note here that the expressions obtained above for each case of temperature profiles have a common factor put in square brackets. Thus, the minimum value for each of the above expressions for \( R \) shall depend upon this common factor, since the other factor in each of these expressions is constant. As shown earlier, the minimum value of \( R \) exits at \( a^2 = 0 \) for each of the above mentioned expressions for \( R \).

**Table 6.8.2:** The values of \( R_c \) corresponding to different values of \( \delta \) and \( B = (1 - T_0 a_2) \); \( P = 1 \) and for various temperature profiles, for both rigid boundaries, for exponential viscosity variations.

<table>
<thead>
<tr>
<th>Viscosity Variations</th>
<th>Linear ( F(z) = 1 )</th>
<th>Inverted parabolic ( F(z) = 2(1 - z) )</th>
<th>Cubic-1 ( F(z) = 3(z - 1)^2 )</th>
<th>Cubic-2 ( F(z) = 0.6 + 1.02(z - 1)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta )</td>
<td>( B = 0.95 )</td>
<td>( B = 0.95 )</td>
<td>( B = 0.95 )</td>
<td>( B = 0.95 )</td>
</tr>
<tr>
<td>( R_c )</td>
<td>( R_c )</td>
<td>( R_c )</td>
<td>( R_c )</td>
<td>( R_c )</td>
</tr>
<tr>
<td>0.0</td>
<td>18.05</td>
<td>17.14</td>
<td>21.05</td>
<td>20.24</td>
</tr>
<tr>
<td>0.1</td>
<td>18.98</td>
<td>18.03</td>
<td>22.14</td>
<td>21.03</td>
</tr>
<tr>
<td>0.2</td>
<td>19.98</td>
<td>18.97</td>
<td>23.31</td>
<td>22.14</td>
</tr>
<tr>
<td>0.3</td>
<td>21.04</td>
<td>19.99</td>
<td>24.55</td>
<td>23.32</td>
</tr>
<tr>
<td>0.4</td>
<td>22.19</td>
<td>21.08</td>
<td>25.89</td>
<td>24.59</td>
</tr>
<tr>
<td>0.5</td>
<td>23.41</td>
<td>22.24</td>
<td>27.32</td>
<td>25.95</td>
</tr>
<tr>
<td>0.6</td>
<td>24.73</td>
<td>23.49</td>
<td>28.85</td>
<td>27.40</td>
</tr>
</tbody>
</table>

**Case 3. Lower boundary is rigid and upper boundary is free.**

The boundary conditions are \( w = 0 = Dw = D\theta \) at \( z = 0 \) and \( z = 1 \) and these are satisfied by \( w = 2z^4 - 5z^3 + 3z^2, \theta = 1 \).

Now, using the above defined trial functions in expression (6.6.8) and obtain the values for Rayleigh number for each of the above defined temperature profiles.

\( (i) \) When \( F(z) = 1 \), the expression (6.6.8) for the given trial functions yields

\[
R = \frac{400(4a^2 - 20a^2) + a_6(64 + 37a^2) - 30a_5(a^2 + 8) + a_4(321 + 9a^2) - 180a_3 + 36a_2)}{(1 - T_0 a_2)^P}.
\]

\( (ii) \) When \( F(z) = 2(1 - z) \), the expression (6.6.8) for the given trial functions yields

\[
R = \frac{50(4a^2 - 20a^2) + a_6(64 + 37a^2) - 30a_5(a^2 + 8) + a_4(321 + 9a^2) - 180a_3 + 36a_2)}{(1 - T_0 a_2)^P}.
\]
When $F(z) = 3(z - 1)^2$, the expression (6.6.8) for the given trial functions yields
\[
R = \frac{560\left((4/8-20/7)a^2/6(64+37a^2)-30/s(a^2+8)/4(321+9a^2)-180/3++36/2\right)}{9(1-T_0a_2)^P}.
\]

When $F(z) = 0.6 + 1.02(z - 1)^2$, the expression (6.6.8) for the given trial functions yields
\[
R = \frac{28000\left((4/8-20/7)a^2/6(64+37a^2)-30/s(a^2+8)/4(321+9a^2)-180/3++36/2\right)}{531(1-T_0a_2)^P}.
\]

It is important to note here that the expressions obtained above for each case of temperature profiles have a common factor put in square brackets. As also pointed out in the above cases of boundary conditions, the minimum values for each of the above expressions for $R$ exit at $a^2 = 0$ for each of the above mentioned expressions for $R$.

Table 6.8.3 The values of $R_c$ corresponding to different values of $\delta$ and $P = 1$ and $B = (1 - T_0a_2)$ for various temperature profiles, for both Lower rigid and upper free boundary, for exponential viscosity variations.

<table>
<thead>
<tr>
<th>Viscosity Variations</th>
<th>Linear $F(z) = 1.$</th>
<th>Inverted parabolic $F(z) = 2(1-z)$.</th>
<th>Cubic-1 $F(z) = 3(z - 1)^2$</th>
<th>Cubic-2 $F(z) = 0.6 + 1.02(z - 1)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>$B=0.95$ $B=1$</td>
<td>$B=0.95$ $B=1$</td>
<td>$B=0.95$ $B=1$</td>
<td>$B=0.95$ $B=1$</td>
</tr>
<tr>
<td>$R_c$</td>
<td>$R_c$</td>
<td>$R_c$</td>
<td>$R_c$</td>
<td>$R_c$</td>
</tr>
<tr>
<td>0.0</td>
<td>16.04</td>
<td>15.24</td>
<td>18.05</td>
<td>17.14</td>
</tr>
<tr>
<td>0.1</td>
<td>17.08</td>
<td>16.23</td>
<td>19.22</td>
<td>18.26</td>
</tr>
<tr>
<td>0.2</td>
<td>18.21</td>
<td>17.30</td>
<td>20.49</td>
<td>19.46</td>
</tr>
<tr>
<td>0.3</td>
<td>19.43</td>
<td>18.46</td>
<td>21.86</td>
<td>20.77</td>
</tr>
<tr>
<td>0.4</td>
<td>20.76</td>
<td>19.72</td>
<td>23.35</td>
<td>22.19</td>
</tr>
<tr>
<td>0.5</td>
<td>22.19</td>
<td>21.08</td>
<td>24.97</td>
<td>23.72</td>
</tr>
<tr>
<td>0.6</td>
<td>23.75</td>
<td>22.56</td>
<td>26.72</td>
<td>25.38</td>
</tr>
</tbody>
</table>

6.9 RESULTS AND DISCUSSION

In this section, we have investigated the effect of non uniform temperature gradient on the onset of Rayleigh–Bénard convection in a saturated porous medium with variable viscosity, for all types of boundary conditions. The analysis proves that the PES is valid in this general configuration when $T_0a_2 < 1$ and $A = M[1 - T_0(\alpha_1 + \alpha_3) + (1 - \alpha_2T_0)]$ is positive. Thus, we conclude that under these physically acceptable conditions, the stationary modes of onset of convection are the only mode of instability. Form the numerical analysis; we conclude that the among the four basic state temperature profile, the Cubic 1 temperature profile is the most stabilizing profile. It is also observed from the above analysis, that the
onset of stationary convection can be postponed by the application of non uniform
temperature gradients. As in the previous cases discussed in earlier chapters, the critical
Rayleigh number increases for positive $\delta$ (temperature dependent viscosity factor) for all
types of non-uniform temperature gradient and for all types of boundary conditions. The
results show that the variable viscosity can suppress the Rayleigh-Bénard convection with
porous medium and thus has a stabilizing effect on the system. Further, it is clear from the
values presented in each of the tables that the medium porosity has destabilizing effect on the
onset of stationary convection. It is observed that values of the critical Rayleigh numbers for
thermally insulating boundaries are lower than those for the corresponding isothermal cases,
thereby establishing that the fluid saturated porous layer bounded by thermally insulating
surfaces is convectively less stable than the one bounded by the isothermal surfaces. When
the temperature gradient is uniform i.e. $F(z) = 1$ and viscosity is constant, the PES is valid
in the present framework. Further, the value of the critical Rayleigh number for uniform
temperature gradient and for constant viscosity can easily found from equation (6.6.4)-(6.6.5)
by substituting the proper solution $w = A \sin \pi z$, for the case of both dynamically
free boundaries, which is equal to $\frac{4\pi^2}{\beta}$; and this is the same value of the critical Rayleigh
number as obtained by Lapwood [1948] and Pathania [1986].
(SECTION-B)
STABILITY ANALYSIS OF RAYLEIGH-BENARD CONVECTION IN POROUS MEDIUM IN PRESENCE OF VARIABLE VISCOSITY WITH SMALL THROUGHFLOW EFFECT
6.10 INTRODUCTION

Natural convection in fluid saturated porous medium is of fundamental interest due to its importance in various fields such as geothermal energy extraction, oil recovery process in petroleum industry, insulation of reactor vessels and many others where porous media occur in natural situations. Due to these applications, several studies have been undertaken to analyze the effects of different phenomenon connected with such media. Review of most of these studies has been reported by D.A. Nield [2006].

In the above mentioned applications, control of instability plays an important role and one such effective mechanism to control the convective instability is that of maintaining a non-linear temperature gradient. Often, the non-linearity of the temperature profile is due to rapid heating (or cooling) at a boundary, by a suitable thermal modulation, by radiative heat transfer or by volumetric distribution of internal heat sources. The experiments of Graham [1933] and Chandra [1938] attracted the attention of many researchers due to the fact that the value of the Rayleigh number is lower than the critical value predicted by the classical theory. Sutton [1950] explained this phenomenon on the basis of non-uniformity of the temperature gradient in such layers.

In many practical problems, like coal gasification and packed bed reactors involve the non-isothermal flow of fluids through porous media which is called through flow. The effect of throughflow is in general complex and not only the basic temperature profile changes, but in the perturbation equations contributions arise from the convection of both temperature and velocity, and there is an interaction between all of these contributions.

The present study is motivated by the above mentioned importance of the flow of fluids through porous media and the modulation of temperature gradient to control the onset of convection (through flow).

In this section, we have investigated the effects of small throughflow and temperature dependent viscosity on the onset Rayleigh-Bénard convection in a saturated porous medium.

6.11 THE PHYSICAL CONFIGURATION

Consider an infinite, viscous quasi-incompressible (Boussinesq) saturated porous fluid layer in the force field of gravity heated from below. The fluid layer is confined between two horizontal boundaries $z = 0$ and $z = d$, maintained respectively at constant
temperature $T_0$ and $T_1 (T_0 > T_1)$. A constant vertical throughflow of magnitude $w_0$ is superimposed parallel to the gravity. The porous material is composed of well packed distribution of particles completely surrounded by Boussinesq fluid (Darcy's model).

Our objective is to examine the onset of instability of this otherwise stable configuration, when the viscosity is temperature dependent in presence of small throughflow. It is assumed that the saturated fluid and the porous layer are incompressible and the porosity of the medium is constant. Let the origin be taken on the lower boundary $z = 0$ with $z$-axis perpendicular to it so that $xy$ - plane constitutes the horizontal plane $z = 0$.

Proceeding exactly as in Part -1 of Chapter 6 and following Nield [1987] and Shivakumara [1998], the basic hydrodynamic equations governing the above physical problem are given by;

\[
\rho_0 \frac{\partial u_i}{\partial t} + \rho_0 u_j \frac{\partial u_i}{\partial x_j} = \rho x_i - \frac{\partial p}{\partial x_i} - \mu \varepsilon \frac{u_i}{K_2} \quad (6.11.1)
\]

\[
\frac{\partial u_j}{\partial x_j} = 0 \quad (6.11.2)
\]

\[
M \frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = \kappa \nabla^2 T. \quad (6.11.3)
\]

The various symbols occurring in the above equations have the same meanings as defined in Part 1 of Chapter 6.

### 6.12. INITIAL STATIONARY STATE AND SOLUTION

Since the equilibrium state under consideration is a static one, so it is clearly characterized by the following equations;

\[
u_i = (u, v, w) = (0, 0, w_0); \quad T = T_b(z); \quad p = p_b(z); \quad \mu = \mu_0 f(z); \quad \text{and} \quad \rho = \rho_b(z) \quad (6.12.1)
\]

where, the subscript 'b' indicates basic state values.

For this equilibrium state, the hydrodynamical equations (6.11.1) require that the pressure distribution for $X_i = (0, 0, -g)$; $g$ is the acceleration due to gravity, is governed by

\[
\frac{\partial p}{\partial x} = -\rho_b g - \frac{\mu e}{K_2} w_0 \quad (6.12.2)
\]

Also, equation (6.11.3) implies that the basic temperature distribution is governed by

\[
\Box \frac{d^2 T_b}{dz^2} = w_0 \frac{dT_b}{dz}.
\]

The solution to this equation is

\[
\theta_b(z) = T_b - T_0 = (T_1 - T_0) \frac{(1-e^{-w_0 z})}{w_0 (1-e^{-w_0})}, \quad (6.12.3)
\]

which is non-linear in $z$. 
The density distribution is given by;
\[ \rho_b = \rho_0 [1 - \alpha(T_b - T_0)]. \] (6.12.4)
where, \( \rho_0 \) represents the density at the lower boundary \( z = 0 \).

6.13. THE PERTURBATION EQUATIONS AND BOUNDARY CONDITIONS

Let the initial state described by equations (6.12.1) be slightly perturbed, so that the perturbed state is given by;
\( (u_1', u_2', u_3') = (0,0, w_0) + (u,v,w) = (u,v,w_0 + w); \quad p' = p_b + \delta p; \)
\[ T' = T_b + \theta; \quad \rho' = \rho_0 [1 - \alpha(T_b + \theta - T_0)] \] (6.13.1)
where, \( (u,v,w) \), \( \delta p \) and \( \theta \) are respectively the perturbations in the initial velocities, pressure \( p \) and temperature \( T \).

Substituting (6.13.1) in equations (6.11.1)-(6.11.3), using the initial state solutions given in (6.12.1)-(6.12.4), neglecting the products of perturbations and higher order terms containing perturbations, we have the following linearized perturbation equations representing the equations of motion, continuity and heat conduction;
\[ \rho_0 \frac{\partial u}{\partial t} = - \frac{\partial (\delta p)}{\partial x} - \mu_0 f(z) \frac{e}{k_1} u \] (6.13.2)
\[ \rho_0 \frac{\partial v}{\partial t} = - \frac{\partial (\delta p)}{\partial y} - \mu_0 f(z) \frac{e}{k_1} v \] (6.13.3)
\[ \rho_0 \left( \frac{\partial w}{\partial t} + w_0 \frac{\partial w}{\partial z} \right) = - \frac{\partial (\delta p)}{\partial z} - \mu_0 f(z) \frac{e}{k_1} w + \rho_0 g \alpha \theta \] (6.13.4)
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \] (6.13.5)
\[ \frac{\partial \theta}{\partial t} + w_0 \frac{\partial \theta}{\partial z} = - \frac{\partial \theta_b}{\partial z} w + \kappa \nabla^2 \theta \] (6.13.6)

We shall consider the same boundary conditions as discussed in Chapter 2 and Part-1 of Chapter 6. Thus, the boundaries conditions on \( w \) and \( \theta \) which must be satisfied are given by;
\[ w = 0 = \frac{\partial \theta}{\partial z} = \frac{\partial w}{\partial z} \quad \text{on a rigid surface} \] (6.13.7)
and \[ w = 0 = \frac{\partial \theta}{\partial z} = \frac{\partial^2 w}{\partial z^2} \quad \text{on a free surface}. \] (6.13.8)

6.14 THE ANALYSIS IN TERMS OF NORMAL MODES

We shall now investigate the stability of the system by analyzing an arbitrary perturbation into a complete set of normal modes individually. Proceeding exactly as in Chapter 2, we consider the following dependence of the perturbations \( u, v, w, \delta p \) and \( \theta \);
\[ g(x,y,z,t) = g(z) e^{i[(k_x x + k_y y) + i\omega t]} \] (6.14.1)
For the perturbations with this dependence on \( x, y \) and \( t \), we have
The linearized perturbation equations (6.13.2)-(6.13.6), in view of the above dependence yield

\[ \frac{\partial}{\partial t} = \alpha; \quad \frac{\partial^2}{\partial t^2} \equiv -k^2; \quad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} - k^2. \quad (6.14.2) \]

The linearized perturbation equations (6.13.2)-(6.13.6), in view of the above dependence yield

\[ \rho_0 n u = -ik_x \delta p - \mu_0 f(z) \frac{\epsilon}{k_1} u \quad (6.14.3) \]
\[ \rho_0 n v = -ik_y \delta p - \mu_0 f(z) \frac{\epsilon}{k_1} v \quad (6.14.4) \]
\[ \rho_0 n w + \rho_0 n \frac{dw}{dz} = -\frac{d}{dz} \delta p - \mu_0 f(z) \frac{\epsilon}{k_1} w + \rho_0 \alpha g \theta \quad (6.14.5) \]
\[ \frac{dw}{dz} = -i(k_x u + k_y v) \quad (6.14.6) \]
\[ M \theta + w_0 \frac{\partial \theta}{\partial z} = -wF_1(z) + \kappa \left( \frac{d^2}{dz^2} - k^2 \right) \theta \quad (6.14.7) \]

where, \( u, v, w, \delta p \) and \( \theta \) are now functions of \( z \) only.

Multiplying equation (6.14.3) by \( ik_x \) and equation (6.14.4) by \( ik_y \), adding the resulting equations and making use of (6.5.10), we have

\[ \rho_0 n \frac{dw}{dz} = k^2 \delta p - \mu_0 f(z) \frac{\epsilon}{k_1} Dw \quad (6.14.8) \]

Eliminating \( \delta p \) between equations (6.14.5) and (6.14.8), we have

\[ [\rho_0 n + \mu_0 f(z) \frac{\epsilon}{k_1}] \left( \frac{d^2}{dz^2} - k^2 \right) w + \frac{\mu_0 \epsilon}{k_1} DfDw - k^2 \rho_0 w_0 Dw = -\rho_0 (\alpha \theta) k^2 g \quad (6.14.9) \]

Using the following non-dimensional quantities

\[ z^* = \frac{z}{\alpha}; \quad D = \frac{d}{dz}; \quad \alpha^* = kd; \quad P = \frac{k_1}{\epsilon d^2}; \]
\[ p^* = \frac{\rho d^2}{\kappa}; \quad w^* = w; \quad \theta^* = \frac{\kappa \theta}{\alpha}; \quad \sigma^* = \frac{\sigma}{\alpha}; \quad (6.14.10) \]

in equations (6.14.7) and (6.14.9), dropping the star for convenience in writing and following the usual steps used in Chapter 2 and Part-1 of Chapter 6 we obtain the following linearized perturbation equations, for small values of \( Q ( \text{small through flow}) \), governing the present problem

\[ \left[ \frac{D^2 - \alpha^2}{p} \right] (D^2 - \alpha^2)w - \alpha^2 NDw + \frac{DfDw}{p} = -Ra^2 \theta \quad (6.14.11^*) \]
\[ (D^2 - \alpha^2 - p - Q'D) \theta = F_1(z)w \quad (6.14.12^*) \]

In the above equations, \( \alpha^2 \) represents the square of the wave number, \( p = p_r + ip_i \) is the complex growth rate; \( f(T) \) is the temperature dependent viscosity factor, \( R = \frac{a \alpha \sigma d}{\kappa \theta_0} \) is the thermal Rayleigh number, \( Q' = \frac{w_0 d}{\alpha} \) is the throughflow-dependent Peclet number, \( \sigma = \frac{\theta_0}{\alpha} \) is the Prandtl number, \( \theta_0 = \frac{\mu_0}{\rho_0} \) is the kinematic viscosity, \( \kappa \) is the thermal diffusivity, and, \( N = \frac{Q'}{\sigma} = \frac{w_0 d}{\theta_0} \) is a constant and
\[ \frac{\partial \theta}{\partial z} = F_1(z) = \frac{-Q'e^{Q'z}}{(e^{Q'} - 1)}. \]

It is important to note from the above expression that for small values of \(Q'(\text{small through flow})\), \(F_1(z)\) approaches to the conduction limit \(-1\). In view of this limit equations (6.14.11*) (6.14.12*) now take the following forms;

\[
\left[ \frac{p}{\sigma} + \frac{f}{p} \right] (D^2 - a^2)w - a^2NDw + \frac{DfDw}{p} = -Ra^2\theta \\
(D^2 - a^2 - p - Q'D)\theta = -w
\]


The boundary conditions (6.13.7)-(6.13.8) in view of (6.14.1)-(6.14.2) and (6.14.10) assume the following forms;

\[
w = 0 = D\theta = D^2w \quad \text{on a free surface} \\
w = 0 = D\theta = Dw \quad \text{on a rigid surface.}
\]

The solution of the above equations must be sought subject to certain boundary conditions as stated in Part-1 of Chapter 6. As in the previous Part, we have considered the following cases of boundary conditions;

**Case 1:** Both boundaries dynamically free and thermally insulating

\[
w = 0 = D\theta = D^2w \quad \text{at } z = 0 \text{ and } z = 1 \\
\]

(6.14.13)

**Case 2:** Both boundaries rigid and thermally insulating

\[
w = 0 = D\theta = Dw \quad \text{at } z = 0 \text{ and } z = 1
\]

(6.14.14)

**Case 3:** Lower rigid and upper boundary free

\[
w = 0 = D\theta = Dw \quad \text{at } z = 0 \text{ and } w = 0 = D\theta = D^2w \quad \text{at } z = 1
\]

(6.14.15)


Rayleigh-Bénard Convection in porous medium (RBCPM) with variable viscosity, if we take \(Q' = 0\) which implies \(N = 0\).

Further, when the temperature dependent viscosity \(f = 1\), the above problem refers to the respective configurations with constant viscosity.

### 6.15 MATHEMATICAL ANALYSIS

System of equations (6.14.11)-(6.14.12) together with either of the boundary conditions (6.14.13)-(6.14.15) constitutes an eigenvalue problem for \(R\) for given values of other parameters; namely \(\sigma, p\ and\ a\). A given state of the system is stable, neutral or
unstable according as \( p_r \) (real part of \( p \)) is negative, zero or positive respectively. Further, if \( p_r = 0 \) implies \( p_l = 0 \) for every wave number \( a \), then the principle of exchange of stability (PES) is valid, which means that instability sets in as stationary convection, otherwise we shall have overstability at least when instability sets in as certain modes.

In the following theorem, we shall check the validity of PES for the problem.

**Theorem 6.15.1** If \((p, w, \theta)\) is a solution of equations (6.14.11)-(6.14.12) together with either of the boundary conditions (6.14.13)-(6.14.15), \( Q' \geq 0 \), then \( p_l = 0 \).

In particular, \( p_r = 0 \) implies \( p_l = 0 \) i.e. PES is valid.

Proof. Equation (6.14.11) can be written as

\[
\frac{1}{p} D[f Dw] - \frac{f}{p} a^2 w + \frac{p}{\sigma} (D^2 - a^2) w + Na^2 Dw = -Ra^2 \theta
\]

(Multiplying equation (6.15.1) by \( w^* \), integrating the resulting equation by parts over the vertical range of \( z \), using boundary conditions (6.14.13)-(6.14.15), we get

\[
\frac{1}{p} \int_0^1 w^* D(f Dw) dz - \frac{f}{p} a^2 \int_0^1 f w^* w dz + \frac{p}{\sigma} \int_0^1 (|Dw|^2 + a^2|w|^2) dz - Na^2 \int_0^1 w^* Dw dz = -Ra^2 \int_0^1 w^* \theta dz
\]

Using equation (6.14.12) on the right hand side the above equation and integrating by parts and using boundary conditions (6.14.13)-(6.14.15), we have

\[
\frac{1}{p} \int_0^1 f |Dw|^2 dz + \frac{f}{p} a^2 \int_0^1 f |w|^2 dz + \frac{p}{\sigma} \int_0^1 (|Dw|^2 + a^2|w|^2) dz + Na^2 \int_0^1 w^* Dw dz
= Ra^2 \int_0^1 (|D\theta|^2 + a^2|\theta|^2 + Mp^*|\theta|^2) dz
\]

(6.15.2)

Now, since we have considered \( Q \) to be small (i.e. small through flow), which in turns implies that \( N = \frac{Q'}{\sigma} \) is also small, hence the term \( Na^2 \int_0^1 w^* Dw dz \) in the above equation can be neglected.

Now, comparing the imaginary parts of the above equation, we have

\[
\frac{p_l}{\sigma} \left[ \int_0^1 (|Dw|^2 + a^2|w|^2) dz + Ra^2 M \int_0^1 |\theta|^2 dz \right] = 0
\]

(6.15.3)

which clearly implies that \( p_l = 0 \).

Hence, in particular, PES is valid.

We observe that for small through flow the stationary convection is the only mode of onset of convection, whether the viscosity is variable or constant (classical Bénard Problem). Further, when the temperature gradient is non uniform and is monotonically decreasing upward (in the vertical direction \( z \)), the PES is still valid. Thus, the onset of convection is through stationary convection.
NUMERICAL ANALYSIS

Now applying Galerkin method as described by Finlayson [1972] and we have considered the boundaries on which the basic heat flux is kept constant, so that \( D\theta = 0 \) on each boundary.

Since for the present problem, the instability sets in as stationary convection, therefore by putting \( p = 0 \), in equations (6.14.11)-(6.14.12) and after little simplification reduce to the following forms;

\[
\frac{1}{p}\int p D[f Dw] - \frac{\ell}{p} a^2 w - a^2 NDw = -Ra^2 \theta \quad (6.15.4)
\]

\[
(D^2 - a^2 - Q'D)\theta = -w \quad (6.15.5)
\]

The system of equations (6.15.4)-(6.15.5) together with either of the boundary conditions (6.14.13)-(6.14.15) constitutes an eigenvalue problem for \( R \) for the onset of stationary convection.

Now, we shall use the Galerkin’s method to find the value of Rayleigh numbers for each case of boundary combinations by taking a single term in the expansions for \( w \) and \( \theta \). Proceeding exactly as in Part-1 of Chapter-6, we obtain the following expression for Rayleigh number as;

\[
R = \frac{\{\int \int [(D\theta)^2 + a^2(\theta)^2 + \frac{Q'}{2}]dz\} - \{\int \int [(D\theta)^2 + a^2(\theta)^2 + \frac{Q'}{2}]dz\}}{Pa^2 (\int \int w\theta dz) (\int \int w\theta dz)} \quad (6.15.6)
\]

It is remarkable to note here that the expression (6.15.6) is valid for all cases of boundary conditions and for arbitrary function of viscosity variation.

We shall compute the values of Rayleigh numbers for four different profiles of temperature gradients, as mentioned in Part-1 of Chapter-6. Further, the Rayleigh numbers and consequently the values of the critical Rayleigh numbers \( R_c \) for different cases of boundaries conditions for above profiles of temperature gradients shall be respectively obtained for the exponential and linear non-dimensional viscosity variations given by;

\[
f_1 = e^\delta z \quad \text{and} \quad f_2 = (1 + \delta z), \quad (6.15.7)
\]

where, \( \delta \) is the temperature dependent viscosity factor and the respective expressions for Rayleigh number for linear and exponential types of viscosity variation becomes

\[
R = \frac{\{\int \int [(1+\delta z)(D\theta)^2 + a^2(\theta)^2 + \frac{Q'}{2}]dz\} - \{\int \int [(D\theta)^2 + a^2(\theta)^2 + \frac{Q'}{2}]dz\}}{Pa^2 (\int \int w\theta dz) (\int \int w\theta dz)} \quad (6.15.8)
\]

and

\[
R = \frac{\{\int \int [e^{\delta z}(D\theta)^2 + a^2(\theta)^2 + \frac{Q'}{2}]dz\} - \{\int \int [(D\theta)^2 + a^2(\theta)^2 + \frac{Q'}{2}]dz\}}{Pa^2 (\int \int w\theta dz) (\int \int w\theta dz)} \quad (6.15.9)
\]
6.16. LINEAR VARIATION OF VISCOSITY

Let us consider linear variation of viscosity defined in (6.6.7). The corresponding expression for the Rayleigh number for this type of viscosity variations is given by the expression (6.15.8), which is valid for all cases of boundary conditions (6.14.13)-(6.14.15).

In the following analysis, we shall now treat each of the cases of boundary conditions separately and derive the values of Rayleigh numbers.

Case 1. Both boundaries dynamically free and thermally Insulating

Let us consider the trial functions

\[ w = z^4 - 2z^3 + z, \quad \theta = 1, \]

which satisfies the boundary conditions

\[ w = 0 = D^2 w = D \theta \text{ at } z = 0 \text{ and } z = 1. \]

Now using the above defined trial functions in expression (6.15.8), we have the following expression for Rayleigh numbers;

\[ R = \frac{25(a^2 + \delta a^2 + 4\alpha (a^2 + 4) - 2\alpha_5 (a^2 - 24) + 8\alpha_3 + 2(a^2 - 12) + a)}{a^2 P}, \]

where \( I_n = \int_0^1 (1 + \delta z)z^n \, dz ; n = 0,1,2, \ldots, 8. \)

The minimum value \( R \) i.e. \( R_c \) exits for some \( a^2 = a_c \) for different values of \( \delta \) and \( Q' \).

The various values of \( a_c \) are computed for different values of \( \delta, P \) and \( Q' \) and are shown in Table 6.16.1. The values of \( R_c \) corresponding to these obtained values of \( a_c \) are computed for different values of \( \delta \) and \( Q' \) are shown in the following tables. It is clear from the expression (6.16.8) that \( P \), the porosity parameter occurs as factor in the denominator, hence the value of \( R \) corresponding to the variation of \( P \) are inversely proportional to \( P \), therefore we have used \( P = 1 \), following the analysis of Rudraiah [1982]. Further, the values of \( R \) corresponding to the different values of \( P \) can be easily analyzed.

Table 6.16.1: The values of \( R_c \) corresponding to different values of \( \delta, Q' \) and \( P = 1 \), for both dynamically free boundaries, with linear viscosity variations.

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>( Q' )</th>
<th>( a_c )</th>
<th>( R_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>12.14</td>
</tr>
<tr>
<td>0.5</td>
<td>1</td>
<td>1.5709</td>
<td>16.32</td>
</tr>
<tr>
<td>1</td>
<td>2.2216</td>
<td>18.22</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0</td>
<td>0</td>
<td>15.75</td>
</tr>
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<td>1</td>
<td>1.5709</td>
<td>17.13</td>
</tr>
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<td>1</td>
<td>2.2216</td>
<td>19.14</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>0</td>
<td>0</td>
<td>22.96</td>
</tr>
<tr>
<td>0.5</td>
<td>1</td>
<td>1.5709</td>
<td>18.76</td>
</tr>
<tr>
<td>1</td>
<td>2.22159</td>
<td>20.96</td>
<td></td>
</tr>
</tbody>
</table>
Case 2. Both boundaries rigid and thermally Insulating

Let us consider the trial functions
\[ w = z^4 - 2z^3 + z^2, \quad \theta = 1, \]
which satisfies the boundary conditions
\[ w = 0 = Dw = D\theta \text{ at } z = 0 \text{ and } z = 1. \]
Now using the above defined trial functions in expression (6.15.8), we have the following expression for Rayleigh numbers;
\[ R = \frac{900((A_{12} - 4I_{17})a^2 + 2I_{12}(3a^2 + 8) - 4I_{12}(a^2 + 12) + I_{12}(52 + a^2) - 24I_3 + 4I_2)(a^2 + Q')}{pa^2}. \]
The minimum value \( R \) i.e. \( R_c \) exits for some \( a^2 = a_c \) for different values of \( \delta, B, \) and \( Q' \) and are shown in Table 6.16.2. The values of \( R_c \) from the above expression for different values of \( \delta \) and \( a_c \) at which \( R \) attains its minima is computed by using Mathematica\textsuperscript{(R) 5.2.}

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>( Q' )</th>
<th>( a_c )</th>
<th>( R_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>17.14</td>
</tr>
<tr>
<td>0.5</td>
<td>1.7321</td>
<td>22.45</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.4495</td>
<td>25.39</td>
<td></td>
</tr>
</tbody>
</table>

Case 3. Lower boundary rigid and upper free and both thermally insulating

Let us consider the trial functions
\[ w = 2z^4 - 5z^3 + 3z^2, \quad \theta = 1, \]
which satisfies the boundary conditions
\[ w = 0 = Dw = D\theta \text{ at } z = 0 \text{ and } w = 0 = D^2w = D\theta \text{ at } z = 1 \]
Now using the above defined trial functions in expression (6.15.8), we have the following expression for Rayleigh numbers;
\[ R = \frac{400((A_{12} - 20I_{17})a^2 + I_{12}(64 + 37a^2) - 30I_{12}(a^2 + 8) + I_{12}(321 + 9a^2) - 180I_3 + 36I_2)(a^2 + Q')}{9pa^2}. \]
The minimum value \( R \) i.e. \( R_c \) exits for some \( a^2 = a_c \) for different values of \( \delta, B, \) and \( Q' \) and are shown in Table 6.16.3. The values of \( R_c \) from the above expression for different values of \( \delta \) and \( a_c \) at which \( R \) attains its minima is computed by using Mathematica\textsuperscript{(R) 5.2.}

Table 6.16.3: The values of \( R_c \) corresponding to different values of \( \delta \) and \( Q' \) and for various temperature profiles, for both rigid boundaries, for linear viscosity variations.

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>( Q' )</th>
<th>( a_c )</th>
<th>( R_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>18</td>
</tr>
<tr>
<td>0.5</td>
<td>1.7321</td>
<td>23.60</td>
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</tr>
<tr>
<td>1</td>
<td>2.4495</td>
<td>26.10</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0</td>
<td>0</td>
<td>19.71</td>
</tr>
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<td>1.7321</td>
<td>23.60</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.4495</td>
<td>28.58</td>
<td></td>
</tr>
</tbody>
</table>
Table 6.16.3: The values of $R_c$ corresponding to different values of $\delta$, $Q$ and $P = 1$ and for various temperature profiles, for both Lower rigid and upper free boundary, for linear viscosity variations.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$Q'$</th>
<th>$a_c$</th>
<th>$R_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>15.24</td>
</tr>
<tr>
<td>0.5</td>
<td>1.6906</td>
<td>20.09</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.3842</td>
<td>22.30</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0</td>
<td>0</td>
<td>16.22</td>
</tr>
<tr>
<td>0.5</td>
<td>1.6906</td>
<td>20.09</td>
<td></td>
</tr>
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<td>1</td>
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<td>19</td>
</tr>
<tr>
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</tr>
<tr>
<td>1</td>
<td>2.4022</td>
<td>26.41</td>
<td></td>
</tr>
</tbody>
</table>

6.17. EXPONENTIAL VARIATION OF VISCOSITY

Let us consider exponential variation of viscosity defined in (6.6.7). The corresponding expression for the Rayleigh number for this type of viscosity variations is given by the expression (6.15.9), which is valid for all cases of boundary conditions (6.14.13)-(6.14.15).

In the following analysis, we shall now treat each of the cases of boundary conditions separately and derive the values of Rayleigh numbers.

**Case1. Both boundaries are dynamically free and thermally insulating**

Let us consider the trial function

$$w = z^4 - 2z^3 + z, \ \theta = 1,$$

which satisfies the boundary conditions

$$w = 0 = D^2w = D\theta \text{ at } z = 0 \text{ and } z = 1.$$

Now using the above defined trial functions in expression (6.15.9), we have the following expression for Rayleigh numbers;

$$R = \frac{25(a^2+Q)(l_6-4l_2)a^2+4l_6(a^2+4)-2l_3(a^2-24)-4l_4(a^2-9)+8l_3+l_2(a^2-12)+l_0)}{a^2};$$

where, $I_n = \int_0^1 e^{\delta z}z^n \, dz \ ; n = 0, 1, 2, ..., 8$.

Table 6.17.1: The values of $R_c$ corresponding to different values of $\delta$, $Q$ and $P = 1$, for both dynamically free boundaries, for exponential viscosity variations.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$Q'$</th>
<th>$a_c$</th>
<th>$R_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>12.14</td>
</tr>
<tr>
<td>0.5</td>
<td>1.5709</td>
<td>16.32</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.2216</td>
<td>18.22</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0</td>
<td>0</td>
<td>12.77</td>
</tr>
<tr>
<td>0.5</td>
<td>1.5713</td>
<td>17.16</td>
<td></td>
</tr>
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</tr>
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</tr>
<tr>
<td>1</td>
<td>2.2267</td>
<td>21.28</td>
<td></td>
</tr>
</tbody>
</table>
Case 2. Both boundaries are rigid

The boundary conditions are \( w = 0 = Dw = D\theta \) at \( z = 0 \) and \( z = 1 \) and these are satisfied by \( w = z^4 - 2z^3 + z^2, \; \theta = 1 \).

Now using the above defined trial functions in expression (6.15.9), we have the following expression for Rayleigh numbers;

\[
R = \frac{900(a^2 + 9')((l_9-a^2) + 2l_6(3a^2 + 8) - 4l_5(a^2 + 12) + l_4(52 + a^2) - 24l_3 + 4l_2)}{a^2}
\]

The minimum value \( R \) i.e. \( R_c \) exits for some \( a^2 = a_c \) for different values of \( \delta \) and \( Q' \) and are shown in Table 6.17.2.

Table 6.17.2: The values of \( R_c \) corresponding to different values of \( \delta \), \( Q \) and \( P = 1 \) and for various temperature profiles, for both rigid boundaries, for exponential viscosity variations.

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>( Q' )</th>
<th>( a_c )</th>
<th>( R_c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>17.14</td>
</tr>
<tr>
<td>0</td>
<td>1.7321</td>
<td>22.45</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>2.4495</td>
<td>26.16</td>
<td></td>
</tr>
<tr>
<td>0.1</td>
<td>0</td>
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</tr>
<tr>
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<td>2.4528</td>
<td>28.97</td>
<td></td>
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</tbody>
</table>

Case 3. Lower boundary is rigid and upper boundary is free

The boundary conditions are \( w = 0 = Dw = D\theta \) at \( z = 0 \) and \( z = 1 \) and these are satisfied by \( w = 2z^4 - 5z^3 + 3z^2, \; \theta = 1 \).

Now using the above defined trial functions in expression (6.15.9), we have the following expression for Rayleigh number;

\[
R = \frac{400(a^2 + 9')((4l_9-20l_7)a^2 + l_6(64 + 37a^2) - 30l_5(a^2 + 8) + l_4(321 + 9a^2) - 180l_3 + 36l_2)}{9pa^2}
\]

The minimum value \( R \) i.e. \( R_c \) exits for some \( a^2 = a_c \) for different values of \( \delta \), \( Q' \) and \( P = 1 \), are shown in Table 6.17.3.
Table 6.17.3: The values of $R_c$ corresponding to different values of $\delta$ and $P = 1$ and for various temperature profiles, for both Lower rigid and upper free boundary, for exponential viscosity variations.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$Q'$</th>
<th>$a_c$</th>
<th>$R_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>15.24</td>
</tr>
<tr>
<td>0</td>
<td>0.5</td>
<td>1.6906</td>
<td>20.09</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2.3842</td>
<td>22.30</td>
</tr>
<tr>
<td>0.1</td>
<td>0</td>
<td>0</td>
<td>16.23</td>
</tr>
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<td>1</td>
<td>2.4093</td>
<td>26.92</td>
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</table>

6.18 RESULTS AND DISCUSSION

In the present part of Chapter 6, we have investigated the effect of temperature dependent viscosity and small through flow on the onset of Rayleigh-Bénard convection in a saturated porous medium for all types of boundary conditions. We have shown that for small values of $Q'$ and consequently for small $N$, PES is valid. Thus, the onset of convection is through stationary convection. From the numerical analysis, we observe that the onset of Rayleigh-Bénard convection in a saturated porous with small through flow can also be delayed by the application of temperature dependent viscosity for positive $\delta$. The various results presented in the tables also show that the variable viscosity considered suppress the Bénard convection in porous medium and it contributes to stabilize the system. Tables (6.17.1)-(6.17.3); Tables (6.16.1)-(6.16.3); show the critical values of the critical Rayleigh numbers for all cases of boundaries, these values increase with increase in temperature dependent viscosity factor $\delta$, for all values of throughflow dependent Peclet number $Q'$. The corresponding effect of variable viscosity on Rayleigh-Bénard convection can be observed as it is clear that the critical Rayleigh number for all boundary conditions is a monotonically increasing function of $\delta$. When there is no throughflow and viscosity is constant, the PES is valid in the present framework. Further, the value of the critical Rayleigh number in the absence of through flow and for constant viscosity can easily found from equation (6.15.4)-(6.15.5) by substituting the proper solution $w = A \sin \pi z$, for the case of both dynamically free boundaries, which is equal to $\frac{4\pi^2}{P}$; and this is the same value of the critical Rayleigh number as obtained by Lapwood [1948] and Pathania [1986].

It is also observed from analysis of the numerical data that both viscosity variations and throughflow dependent Peclet number stabilize the system under the assumed
conditions. But this may not be true if for large values of $Q'$. Further, various consequences in view of Remarks 4 can be easily worked out, which are in good agreement with the results earlier obtained in this thesis.