CHAPTER - I

INTRODUCTION
FLUID DYNAMICS

All materials exhibit deformation under the action of forces. If the deformation in the material increases continually without limit under the action of shearing forces, however small, the material is called 'fluid'. This continuous deformation under the action of forces compels the fluids to flow and this tendency of fluids is called 'fluidity'.

Fluid dynamics is the science which deals with the properties of fluids in motion. It is very difficult to trace the origin of the science of fluid dynamics because the excavations of the Indus Valley Civilization and Egyptian ruins show that even as long as four thousand years ago the principles of flow and resistance to flow were known. The drainage and irrigation systems of Mohanjo-daro, Egypt and China, the use of siphons and bellows and the construction of windmills and paddle wheels date from ancient times.

The study of fluid dynamics is of great practical importance. Some practical situations where fluid dynamics plays an important role are lubrication, flight of aeroplanes, ship science, meteorology, the influence of wind upon building structures, groundwater seepage, the extraction of oils from underground reservoirs, the use of pipelines, pumps and turbines. Even the swinging of the cricket ball is an example of the fluid forces used by the bowler, to deceive the batsman. The flow of fluids affects each one of us throughout our lives. The flow of blood in veins and pumping action of the heart are familiar examples.

It was only after the Euler’s discovery of the equations of motion of an inviscid fluid which started the systematic study of fluid dynamics. Earlier attempt to describe the effect of fluid motions is due to Newton, who conceived the idea that the
fluid consisted of a granulated structure of discrete particles. Later some contributions to this subject were given by the following. The concept of velocity potential and stream function was given by Lagrange. The principle of resistance to flow in capillary tubes was given by Poiseuille. The credit for the equations of motion of viscous fluids goes to Navier and Stokes. Reynolds discovered the equations of turbulent motion. Prandtl put forward the boundary layer theory. G.I. Taylor and Lord Rayleigh gave the theories of turbulence and stabilities. Later on, many more contributions were given by many famous scientists which include Bénard, Kutta, Prandtl, Lord Kelvin, Orr, Sommerfeld, Rayleigh, Zhukovski and Kármán etc. These days, fluid dynamics has become a very vast subject and has given birth to many other subjects like meteorology, gas dynamics, aerodynamics, non-Newtonian flows, magnetohydrodynamics etc.

Matter exists in the four forms namely (i) solid, (ii) liquid, (iii) gas and (iv) plasma. Liquids and gases taken together are classed as fluids. Liquids have strong intermolecular forces whereas the gases experience weak intermolecular forces. As a result of these, the liquids are incompressible fluids and the gases are highly compressible fluids. It should be mentioned that for velocities which are not comparable with the velocity of sound, the effect of compressibility on atmospheric air can be neglected and it may be considered to be a liquid and in this sense it is called incompressible air. There is no clear dividing line between solids and fluids, since there are many materials which in some respect behave like a solid and in other respect have dual character. However, a loose distinction can be made between solids and fluids. A solid mass has a definite shape, while a mass of fluid has no preferred shape and assumes the shape of the container more or less instantaneously. The deformation in the piece of solid is small even under the action of large external forces, whereas in the case of fluids
the deformation may be large under the suitable chosen forces, however small in magnitude.

The fourth state of matter is called plasma. Plasma is essentially a highly ionized matter. We have to take into account the charges on its particles and the associated electromagnetic phenomena in a plasma. We go to the plasma state when we deal with Earth’s molten core, ionosphere, stellar interiors and atmospheres.

In classical fluid dynamics, fluid molecules are considered electrically neutral. The study of water flowing in rivers, waves in ocean and the motion of aeroplane in the lower parts of Earth’s atmosphere is in the scope of classical fluid dynamics. The gross properties of various states of matter are directly related to the molecular structure and the nature of intermolecular forces that operate between the constituent molecules. In solids, the arrangement of molecules is virtually permanent and under normal conditions may have a simple periodic structure as in case of crystals, and molecules are acted upon by strong intermolecular forces. Our knowledge of the liquid state is incomplete, but it appears that the arrangement of molecules is partially ordered and are acted upon by medium intermolecular forces. In case of gases and plasmas, the particles are acted upon by weak short range intermolecular forces and molecular arrangements are disordered.

CONTINUUM HYPOTHESIS

In fluid dynamics, we make use of the continuum hypothesis though we know that matter is composed of atoms and molecules and therefore, has necessarily a discrete structure. In normal gases, the masses are concentrated in molecules which are widely separated with each other. When one is dealing with problems in which some characteristic length in the flow is very large compared with molecular distances, it is
convenient to think of a lump of fluid sufficiently small from macroscopic point of view but large enough at the microscopic level so as to contain a large number of molecules (for instance, at normal temperature and pressure a volume of $10^{-12}$ c.c. of a gas contains about $2.7 \times 10^7$ molecules) and to work with the average statistical properties of such a large number of molecules. When the fluid is viewed on microscopic scale so as to reveal the individual molecules, the properties of fluid such as composition, velocity and density have violently non-uniform distributions. Since we are generally concerned with the macroscopic behaviour of the fluids, therefore, we assume that the masses located at the mass centres are smeared out uniformly over a certain volume surrounding them and treat the matter as continuum. This is called the “continuum hypothesis”. There is an ample evidence that common real fluids, both liquids and gases move as if they were continuous under normal conditions and even under considerable departure from normal conditions. The hypothesis is justified when we consider only those systems in which the characteristic length is much larger than the mean free path of the fluid molecules. However, in large number of phenomena, such as the flow of highly rarefied gas at high temperature, motion of a gas through the pores of a catalyst pellet as is used in petroleum refining process, we have to take the molecular structure seriously into account. The continuum approach is simpler than the more rigorous kinematic one, because our hypothesis has made it possible to give meaning to such terms as density, pressure, temperature, momentum and angular momentum ‘at a point’. And, in general, the values of these quantities are continuous functions of position and time, thus permitting us the use of derivatives and differentials whenever they are needed.
NEWTONIAN MECHANICS

We shall restrict ourselves to those systems, where particle velocities are small compared with the velocity of light, so that relativistic effects are not prominent. In other words, we confine ourselves to Newtonian mechanics and shall not evoke the theory of relativity. Thus we shall not be concerned with such masses, velocities and temperatures for which Newtonian mechanics does not provide adequate description.

HYDROMAGNETICS

Hydromagnetics or magnetohydrodynamics is the name given to an area of study in which fluid dynamics and electrodynamics overlap. Hydromagnetics is the science which deals with the motion of electrically conducting fluids in the presence of a magnetic field. The study of the interaction between magnetic field and electrically conducting moving fluids is currently receiving considerable interest. This interest has been spurred primarily by astrophysical problems and by problems associated with the fusion reactor.

Fluid dynamics and electromagnetic theory were being developed independently of each other almost up to the first half of this century. The systematic study of MHD started only after 1942 when Alfvén combined the two subjects by considering the motion of conducting fluids in the presence of magnetic field. This study has now come to be known as Magnetohydrodynamics or Hydromagnetics. It is concerned with physical systems specified by the equations that result from the fusion of those of hydrodynamics and electromagnetic theory. It is a well known fact that when a conductor moves in a magnetic field, electric currents are induced in it. These currents experience a mechanical force called the Lorentz force, due to the presence of magnetic
field. This force tends to modify the initial motion of the conductor. Moreover, the induced currents generate their own magnetic field which is added on to the applied magnetic field. Thus there is coupling between the motion of the conductor and the electromagnetic field, which is exhibited in a more pronounced form in liquid and gaseous conductors. Lorentz force is generally small unless inordinately high magnetic fields are applied. Thus this force is incapable of altering the motion as a whole considerably, but if it acts for a sufficiently long period, the molecules of gases and liquids may get accelerated considerably to alter initial state of these types of conductors.

Alfvén (1942) proved his famous theorem that magnetic lines of force are glued to ideally conducting fluid. Every motion of the fluid perpendicular to the lines of force is forbidden because it can give infinite eddy currents. Thus the matter of the liquid is fastened to the lines of force. Alfvén (1942) also discovered the simplest example of coupling between the mechanical forces and the magnetic lines of force in a highly conducing fluid moving in an external magnetic field and showed that this interaction would produce a new kind of wave which he called a magnetohydrodynamic wave. The above discoveries of Alfvén led to a systematic study of MHD. The subject of MHD is, thus, comparatively of recent origin. It found its birth in attempts to explain certain phenomena in cosmic physics; for example, the generation and maintenance of original magnetic fields of Earth and Sun, the variability of magnetic stars and the production of sunspots which are associated with magnetic fields.

Magnetic field introduces anisotropy, elasticity and lateral pressure in the fluid. Anisotropy results in the difference of electrical conductivity and diffusion coefficients along and perpendicular to the magnetic field. Elasticity and lateral pressure
are responsible for the propagation of MHD waves. Attempts to show the existence of MHD waves in the laboratory were made by Lundquist (1951) and Lehnert (1954).

Bullard (1949) and Batchelor (1950) pointed out that the magnetic field imparts to the fluid a certain rigidity along with certain properties of elasticity which enables it to transmit disturbances by new modes of wave propagation. The experimental work of Lehnert (1952) concluded that the behaviour of a conducting fluid is very different in the absence and in the presence of a magnetic field. For example, there is a tendency for all motions to become uniform along the magnetic field, or in other words, a tendency towards two-dimensional motion. These are some of the interesting properties associated with the magnetic field. Generally, the magnetic field has a stabilizing effect on the instability. But a few exceptions are there. For example, Kent (1966) studied the effect of a horizontal magnetic field, which varies in the vertical direction, on the stability of parallel flows and showed that the system is unstable under certain conditions, while in the absence of magnetic field the system is known to be stable.

Although the continuum approach is much simpler than the more rigorous gas kinematic one, it is not without difficulty. This is due to the reason that the hydrodynamic equations are basically non-linear even though the electrodynamic equations are linear. The coupling of the two systems of equations, hydrodynamic and electrodynamic causes the non-linear aspects to be carried over into the resulting MHD equations. The new phenomena which arise are interesting. For example, the coupling between longitudinal and transverse fields provides the possibility of energy transfer between the longitudinal and transverse modes of oscillations. This is of interest in astrophysics as well as in technology.
THE BASIC HYDRODYNAMICAL EQUATIONS

The fundamental equations of the flow of viscous compressible fluid are

(i) Equation of state, (one)
(ii) Equation of continuity, (one)
(iii) Equations of motion, (three) and
(iv) Equation of energy, (one)

These equations are to be considered for the stability or instability of the systems. These equations are mathematical expressions of basic physical laws. These are six in number and therefore, determine the six unknowns of the fluid motion viz., the three components of velocity, $u_i(u,v,w)$, the temperature $T$, the pressure $p$ and the density $\rho$, which are functions of both space coordinates and time.

EQUATION OF STATE

Variables that depend only upon the state of a system are called variables of state. The variables of state are the pressure $p$, the density $\rho$ and the temperature $T$. It is an experimental fact that a relationship between these three thermodynamic variables exist and can be written as

$$F(p,\rho,T) = 0, \quad (1.1)$$

which is commonly called the 'Equation of state'. For substances with which we shall be principally concerned, we can write the equation of state as

$$\rho = \rho_0[1 + \alpha(T_0 - T)], \quad (1.2)$$

where $\alpha$ is the coefficient of volume expansion and $T_0$ is the temperature at which $\rho = \rho_0$. 
EQUATION OF CONTINUITY - CONSERVATION OF MASS

This equation expresses that the rate of generation of mass within a given volume is entirely due to the net inflow of mass through the surface enclosing the given volume (assuming that there are no internal sources). It amounts to the basic physical law that the matter is conserved, it is neither being created nor destroyed. For viscous compressible fluids, the equation of continuity is

\[ \frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = -\rho \frac{\partial u_j}{\partial x_j}, \]  

(1.3)

where \( u_j \) is the jth component of velocity.

For an incompressible fluid

\[ \frac{\partial \rho}{\partial t} + u_j \frac{\partial \rho}{\partial x_j} = 0, \]  

(1.4)

so that equation (1.3) reduces to

\[ \frac{\partial u_j}{\partial x_j} = 0. \]  

(1.5)

EQUATION OF MOTION (NAVIER-STOKES' EQUATIONS) - CONSERVATION OF MOMENTUM

The equations of motion are derived from Newton's second law of motion which states that

\[ \text{Rate of change of linear momentum} = \text{Total force}. \]

For viscous compressible fluids, Navier-Stokes’ equations can be expressed as

\[ \rho \frac{\partial u_j}{\partial t} + p u_j \frac{\partial u_j}{\partial x_j} = \rho X_j - \frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_i} \left[ \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \mu \frac{\partial u_k}{\partial x_k} \delta_{ij} \right], \]  

(1.6)
where $X$, $(0,0,-g)$ is the external force, $\mu$ is the coefficient of viscosity and $\delta_\eta$ is the Kronecker delta.

In case of incompressible fluid flow, the equation of continuity is

$$\frac{\partial u_i}{\partial x_i} = 0,$$

and if $\mu$ is also regarded as constant, the equation of motion (1.6) can be simplified to

$$\rho \left[ \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right] = \rho X_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2},$$

(1.7)

keeping in view that

$$\frac{\partial}{\partial x_j} \left( \frac{\partial u_i}{\partial x_i} \right) = \frac{\partial}{\partial x_i} \left( \frac{\partial u_i}{\partial x_j} \right) = 0 .$$

EQUATION OF ENERGY - CONSERVATION OF ENERGY

To obtain the energy equation we have to apply the law of conservation of energy which requires that, the difference in the rate of supply of energy to a controlled surface $S$ enclosing a volume $V$ in the region occupied by a moving fluid and the rate at which the energy goes out through $S$ must be equal to the net rate of increase of energy in the enclosed volume $V$.

For viscous compressible fluids, the equation of energy is

$$\frac{\partial}{\partial t} (\rho c_v T) + \frac{\partial}{\partial x_j} (\rho c_v T u_j) = \frac{\partial}{\partial x_j} \left( k_T \frac{\partial T}{\partial x_j} \right) - \rho \frac{\partial u_j}{\partial x_j} + \Phi ,$$

(1.8)

where $\Phi = 2\mu e_0^2 - \frac{2}{3} \mu (e^2_\mu )^2$,  

(1.9)
is the ‘rate of viscous dissipation’ (which gives the rate at which energy is dissipated irreversibly by viscosity in each element of volume of the fluid),

\[ e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]  
(1.10)

is the ‘rate of strain tensor’, \( c_v \) is the specific heat at constant volume and \( k_r \) is the coefficient of heat conduction.

For an incompressible fluid \( e_{ij} = 0 \) and the corresponding expression for \( \Phi \) is given by

\[ \Phi = 2\mu e_{ij}^2. \]

Thus, for an incompressible fluid, the equation of energy (1.8) takes the form

\[ \rho \frac{\partial}{\partial t} (c_v T) + \rho u_j \frac{\partial}{\partial x_j} (c_v T) = \frac{\partial}{\partial x_j} \left( k_r \frac{\partial T}{\partial x_j} \right) + 2\mu e_{ij}^2. \]  
(1.11)

THE BOUSSINESQ APPROXIMATION

In solving the above hydrodynamical equations, we have difficulties regarding their non-linear character and the variable nature of the various coefficients with the variation in temperature. Due to these difficulties, the need for introducing some mathematical approximation which simultaneously points out to the appropriate physical situation is strongly felt in order to simplify the basic equations. One of the contributions of Boussinesq (1903) in these problems of thermal instability is precisely at this point in the form of an approximation which is after his name. This approximation has also gained a wide recognition in other problems of non-homogeneous fluids, for example, the problems of Kelvin-Helmholtz instability type. In fact, while dealing with homogeneous fluids, one observes that density variations, mainly, have a two-fold effect
on the stability of the problem:

(a) When coupled with the inertial terms in the equations of motion, namely 
\[ u_i \frac{\partial u_i}{\partial x_j}, \]
it gives an inertial acceleration to the system, and

(b) it interacts with the external forces acting on the system, which is gravity here, to produce a gravitational acceleration.

Boussinesq pointed out that there are situations in the domain of meteorology and oceanography where one is justified in neglecting the inertial effects of density variations as compared to its gravitational effects. This is as if \( \rho \) is taken as constant everywhere in the equations of motion except in the term with external force. Consequently, we replace \( \rho_0 \left[1 + \alpha (T_0 - T) \right] \) by \( \rho_0 \) everywhere in the equations of motion except the term representing the external body force \( X_i \). One can easily verify that for the classical Bénard problem, the perturbation equations derived in the usual way and using Boussinesq approximation are one and the same because of the smallness of the coefficient of volume expansion, whose range is \( 10^{-3} \) to \( 10^{-4} \).

On the basis of foregoing remarks the equation of continuity becomes

\[ \frac{\partial u_j}{\partial x_j} = 0. \] (1.12)

Equation of momentum, namely (1.6) becomes

\[ \frac{\partial u_j}{\partial t} + u_j \frac{\partial u_j}{\partial x_j} = - \frac{1}{\rho_0} \frac{\partial \rho}{\partial x_i} + \left(1 + \frac{\delta \rho}{\rho_0} \right) X_i + \nu \nabla^2 u_i. \] (1.13)

where \( \nu = \frac{\mu}{\rho_0} \) represents the coefficient of kinematic viscosity and

\[ \delta \rho = \rho_0 \alpha (T_0 - T). \] (1.14)
Since we are assuming temperature variations to be small, we shall take $\alpha$ constant with value corresponding to $T_0$. It is to be noted at this point that while simplifying equation (1.6), $\mu$ has been treated as constant and hence, taken outside the differentiation sign on account of the fact that for small temperature variations as mentioned above, the variations in $\mu$ are small, of the order of density variations and hence, can be ignored.

Similarly, in the equation of heat conduction (1.8), we can treat $c_v$ and $k_T$ as constants and hence, take them outside the sign of differentiation. The term $-\rho \frac{\partial u_j}{\partial x_j}$ does not contribute anything because of the equation of continuity (1.12). Further, viscous dissipation $\Phi$ can also be neglected because, according to equations (1.12) and (1.13), the prevailing velocities are of the order $[\alpha\Delta T|X|d]^1/2$ and hence, the term $\Phi$ is of the order

$$\frac{\mu\alpha|X|d}{k_T},$$

relative to the term arising from the conduction of heat. But, this ratio for ordinary liquids (such as water and mercury) is $10^{-7}$ or $10^{-8}$ for $d \equiv 1 \text{ cm}$ and $|X| \equiv g$ (the acceleration due to gravity). In the above situations, the equation of heat conduction reduces to

$$\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} = \kappa \nabla^2 T . \quad (1.15)$$

where $\kappa = \frac{k_T}{\rho_0 c_v}$.

Thus, equations (1.12)-(1.15) are the basic hydrodynamical equations under the Boussinesq approximation.
HYDRODYNAMIC AND HYDROMAGNETIC STABILITY

‘Stability’ can be defined as the quality of being immune to small disturbances. Thus, by stability we mean permanent type of equilibrium state. An equilibrium state or steady flow, to be of permanent type, should not only satisfy the mechanical equations, but must be stable against arbitrary perturbations.

The equations of hydrodynamics, inspite of their complexity, allow some simple patterns of flow (such as between parallel planes or rotating cylinders) as stationary solutions. These patterns of flow can, however, be realized only for certain ranges of the parameters characterizing them. They cannot be realized outside these ranges. The reason for this lies in their inherent instability, i.e., in their inability to sustain themselves against small perturbations to which any physical system is subjected. It is in the differentiation of the stable from the unstable patterns of permissible flows that the problems of hydrodynamic stability originate.

In recent years, the class of such problems of stability has been enlarged by the interest in hydrodynamic flow of electrically conducting fluids in the presence of magnetic fields. This is the domain of hydromagnetics, as we have discussed earlier; and there are problems of hydromagnetic stability even as there are problems of hydrodynamic stability.

We consider a hydrodynamic or hydromagnetic system in a stationary state, that is in which none of the variables describing the configuration is a function of time. To investigate the stability, we have to determine the reactions of the system to arbitrary small perturbations. If the system is disturbed and the disturbance gradually dies down or if the system never departs appreciably from this stationary state, the system is said to
be stable with respect to that particular disturbance. If the disturbance grows in amplitude in such a way that the system progressively departs from the initial state and never reverts to it, the system is called unstable with respect to that particular disturbance. A system must be considered as unstable even if there is only one special mode of disturbance with respect to which it is unstable. And a system cannot be considered as stable unless it is stable with respect to every possible disturbance to which it can be subject.

Consider a hydromagnetic system which in accordance with the equations governing it is in a stationary state. Let \( X_1, X_2, \ldots, X_j \) be a set of parameters which define the system. While considering the stability of such a system, we seek to determine the reaction of the system to small disturbances.

If all the initial states are classified as stable or unstable, according to the criteria stated, then in the space of parameters; \( X_1, X_2, \ldots, X_j \), the locus which separates the two classes of states defines the state of 'marginal stability' of the system. This definition implies that a marginal state is a state of 'neutral stability'. The locus of the marginal states in the \( (X_1, X_2, \ldots, X_j) \) space will be defined by an equation of the form

\[
\Sigma (X_1, X_2, \ldots, X_j) = 0.
\]

The determination of this locus is one of the prime objects of an investigation on hydrodynamic stability. If the amplitude of a small disturbance can grow or be damped aperiodically, the transition from stability to instability takes place via a marginal state exhibiting a stationary pattern of motions. If the amplitude of a small disturbance can grow or be damped by oscillations of increasing or decreasing amplitude, the transition takes place via a marginal state exhibiting oscillatory motions with a certain definite characteristic frequency.
If at the onset of instability a stationary pattern of motions prevails, then one says that the ‘principle of the exchange of stabilities’ is valid and that instability sets in as stationary cellular convection or secondary flow. On the other hand, if at the onset of instability, oscillatory motions prevail, then it is called the case of ‘overstability’.

For nearly a century now, hydrodynamic stability has been recognized as one of the central problems of fluid mechanics. It is concerned with when and how laminar flows break down, their subsequent development and their eventual transition to turbulence. It has many applications in engineering, in meteorology and oceanography and in astrophysics and geophysics. Due to the paramount importance in the quest for thermonuclear fusion, the subject of hydromagnetic stability is a large one and much has been written about it. The first major contribution to the study of hydrodynamic stability can be found in the theoretical papers of Helmholtz (1868). Lord Rayleigh (1880) developed a general linear stability theory for inviscid plane-parallel shear flows, which was mathematically tractable and had intuitively sensible results. The combined efforts of Reynolds (1883), Kelvin (1880, 1887) and Rayleigh [1879,1880, 1892(a), 1892(b), 1913, 1914, 1916(a), 1916(b)] produced a rich harvest of knowledge. Reynolds (1883) predicted that Reynolds number was a crude measure of the relative importance of inertial (non-linear) effects relative to viscous processes in determining the evolution of a flow. He discovered the first experimental evidence of ‘sinuous’ motions in water and is generally credited for a first description of random or ‘turbulent’ flow. He pointed out that disorder begins when Reynolds number exceeds a critical value and that special stresses must be taken into account. The founder of hydrodynamical stability is Lord Rayleigh, who, published a great number of papers (as cited above), regarding the
importance of inflection points in the velocity profile and the instability of rotating flows between cylinders.

Early in this century, studies on hydrodynamic stability were connected with the Bénard experiments on thermal convection in thin liquid layers. Around 1907, it became apparent that the existence of critical Reynolds number could not be explained easily and that the problem involved both the effect of the second derivative of the mean flow and of the viscous forces. The key equation was arrived at independently by Orr (1907) and by Sommerfeld (1908). This Orr-Sommerfeld equation remained unsolved for twenty two years, until Tollmien (1929) calculated the first neutral eigen-value and obtained a critical Reynolds number. The work of Taylor (1923) on vortices between concentric rotating cylinders was the principal and best known contribution. Indeed this was a dual effort where theory and experiment were matched simultaneously. The analysis of Heisenberg (1924) was more abstract and points towards the possibility of resistive instability. Jeffreys demonstrated in 1928, the mathematical equivalence of the two stability problems of convection and flow between rotating cylinders. A large body of theoretical work by Lin (1955), Joseph (1976), Chandrasekhar (1981), Gershuni and Zhukhovitskii (1976), and Drazin and Reid (1981) has been developed in an attempt to understand and predict phenomena of stability or instability.

Only recently have computer-extended series been applied to problems of hydrodynamic stability. Von Kerczek (1982) has examined the linear stability of the oscillatory plane Poiseuille flow produced by a pressure gradient that is the sum of a steady term plus one varying sinusoidally in time. After reducing the time-dependent Orr-Sommerfeld equation to a system of ordinary differential equations, using a
Galerkin-like method, he expanded each solution in powers of a parameter that measures the ratio of oscillatory to steady velocities. He found that the imposed oscillations stabilize the flow except at very low and very high frequencies.

Finally, the theory of non-linear processes was set-up by Meksyn and Stuart (1951). Some other good works in non-linear theory which need mention are by Coles (1965), Segel (1966), Büssé (1969), Reynolds and Potter (1967), Kirchgässner and Sorger (1969), Stewartson and Stuart (1971) and Weissman (1979).

**METHODS DETERMINING INSTABILITY**

Generally, the instability of a system is determined by the following methods:

(a) **Perturbation Method**

This is the most suitable method for establishing instability of a system. In this method, the hydrodynamic system whose instability we wish to establish, is supposed to undergo a specific small trial displacement and the effect of the additional forces brought into play is considered. If these forces thus produced tend to increase the displacement, thereby enhancing the deformation of the system still further, the system is unstable.

(b) **Energy Method**

The second and more general method to discuss stability is the energy method. It was first used by Rayleigh (1877) in the calculation of the frequencies of vibrating systems. Reynolds (1895) and Orr (1907) used this method in their early works. The calculation procedures and the results associated with the use of this older method were summarized by Bateman, Dryden and Murnaghan (1932). In fact, this is the oldest method of stability analysis which can accommodate finite disturbances also. It has also
been applied with some success to oscillating flow between rotating cylinders by Conard and Criminals (1965).

In this method we make use of the energy principle. In a mechanical system for which there exists a potential energy function $V'$, the system in a stationary state will be stable if $V'$ is strict minimum and unstable if $V'$ is strict maximum. In such a system, when dissipation forces are neglected,

$$T' + V' = \text{constant},$$

where $T'$ denotes the kinetic energy of the system.

For a stationary configuration, let $V'$ has a strict minimum $V_0'$. When the system is disturbed, $V' > V_0'$ in a neighbouring configuration. If $T_0'$ is the initial kinetic energy of the system generated by the small disturbance. Then, we have

$$T' + V' = T_0' + V_0',$$

which gives

$$T' = T_0' - (V' - V_0') < T_0'.$$

Thus the system does not tend to deviate further from the stationary configuration but remains in its proximity. The system is, therefore, stable.

If $V_0'$ is a strict maximum, then $T' > T_0'$ and the system will tend to depart further and further from its initial state. The system is thus unstable. So we calculate the change $V' - V_0'$ in the potential energy of the system, when it is given a small displacement satisfying the boundary conditions. The system is stable if this change is positive for all possible infinitesimal displacements and is unstable if $V' - V_0'$ can be shown negative for any one particular trial displacement.
(c) **Normal Mode Analysis Method**

We now discuss the normal mode analysis method. Normal mode analysis method is used to determine the stability of a stationary state of a hydrodynamic or a hydromagnetic system. This method is quite general and has found extensive applications. The beauty of this method is that it gives complete information about instability including the rate of growth of any unstable perturbation. This method has been used throughout by Chandrasekhar in his book *Hydrodynamic and Hydromagnetic Stability* [Dover Publication, New York (1981)] while discussing the various instability problems.

We start from an initial flow which represents a stationary state of the system. We assume that the various physical variables describing the flow suffer small infinitesimal perturbations and obtain the equations governing these perturbations. Also, we make use of the linear theory by retaining only linear terms in the equations governing perturbations. To study these equations, we assume further that perturbed quantities have time variations proportional to $e^{int}$. The parameter ‘$n$’ is, in general, a function of $k$ (the wave number) and other parameters defining the system.

If the value of $n$ determined by the dispersion relation is:

(i) Real and negative, the system is stable;

(ii) Real and positive, the system is unstable;

(iii) Complex, say,

\[ n = n_r + in_i, \]  
where $n_r$ and $n_i$ are real. Then we have the following cases:

(a) If $n_r < 0$, the system is stable;

(b) If $n_r > 0$, the system is unstable,
(c) if \( n_r = 0 \), the system is oscillatory.

(iv) Further, if \( n_r = 0 \), implies that \( n_i = 0 \), then the stationary (cellular) pattern of flow prevails on the onset of instability. In other words, the 'principle of exchange of stabilities' is valid.

(v) If \( n_r = 0 \), does not imply that \( n_i = 0 \), then overstability occurs.

From this, it follows that if \( n \) is real, then \( n = 0 \) will separate the stable and unstable modes and we will always have exchange of stabilities.

**SCOPE OF METHODS**

The formulations of mathematics of energy method is rigorous and relatively simple. But this does not diminish its capacity for truth. However, the potentials of the linear theory of stability and of the energy method are complementary. The linear theory of hydrodynamic stability has the drawback that one cannot, in general, make judgement regarding the growth potential of finite disturbances. It cannot be claimed with certainty that a given system will remain stable if disturbed under conditions judged favourable by linear theory. This and other questions about the effects of finite disturbances are in the domain of non-linear theory. The recent review has been given by Segel (1966) and very recent one by Drazin and Reid (1981). Comparison of stability limits as given by energy and linear theory yields the range of values of relevant stability parameters in which subcritical instabilities (which means that the system first becomes unstable to steady finite amplitude before it becomes unstable to infinitesimal disturbances) of the hydrodynamic system are possible. Joseph and Shir (1966) pointed out that energy method provides mathematically rigorous and sometimes physically precise theory of subcritical convective instability. There are cases, for example, plane Couette flow,
where both the theories cannot depict real state of affairs as obtained experimentally. Energy theory errs in giving a safe Reynolds number which is far too conservative and linear theory errs in giving the flow as always stable. However, there are cases, as in case of Bénard problem in which the Boussinesq approximation is made and fluid is heated from below, where both the theories coincide.

The energy method has not the potential of the linear theory for fine discrimination of the limits of stability. The potential of energy method has not been fully utilized and it needs more exploration in its field. The normal mode analysis method is quite general to determine the stability of a system. It also gives the rate of growth of the perturbation. But this method is based on the linearized stability theory and therefore, it has all the defects of linear theory. Further, in some problems the dispersion relation becomes so much complicated that it is not possible to draw any meaningful conclusions from it.

Nevertheless, in the present thesis, we shall confine ourselves to the normal mode analysis and shall not be making use of the energy method. In the present time, the normal mode analysis method is most commonly used and has found extensive use as it gives complete information about instability.

**SOME INSTABILITY PROBLEMS**

In the present thesis, we have worked on various hydrodynamic and hydromagnetic instability problems. Therefore, for the clear and better understanding of our work, we feel it necessary to explain and also review, in brief, some fundamental contributions relating to these instabilities. We take them one by one below:
THERMAL INSTABILITY (OR BÉNARD PROBLEM)

Consider a horizontal layer of fluid of uniform density which is subjected to an adverse temperature gradient by heating it from below. Then the fluid at the bottom becomes lighter than the fluid at the top and thus it becomes a top-heavy arrangement, which is potentially unstable. As a consequence of this, there will be a natural tendency on the part of the fluid to redistribute itself to make up the deficiency in the arrangement. But this redistribution is prevented to a certain extent by its own viscosity and therefore instability can set in only when the adverse temperature gradient exceeds certain critical value.

The origin of the problem of the onset of thermal instability in liquid layers heated from below, lies in the experimental works of Bénard (1900). He carried out his experiment on a very thin layer of non-volatile liquid (1 mm in depth), placed on a carefully leveled metallic plate maintained at a constant temperature. The upper surface of the layer was kept in contact with the free air. It was found that the layer resolved itself into a number of cells, known as Bénard cells. The principal facts established by the experiments of Bénard and others may be summarized as:

i) A certain critical adverse temperature gradient must be exceeded before the instability sets in.

ii) The motions that follow on exceeding the critical adverse temperature gradient have a stationary cellular convection.

The formation of this cellular convection takes place in two phases. The first phase is quite rapid, lasting for a second or two for less viscous liquids like alcohol. For heavy oils, especially when the upward flux of heat is small, this phase may be characterized by 'semi-regular regime'. The cells are nearly identical taking form of
approximately regular convex polygons. The second phase is, however, of permanent nature. Experimentally it is found that it is difficult to maintain a constant flow of heat, but if one succeeds in doing so with extreme care, the cells take the form of identical regular hexagons.

On earlier occasions, Count Rumford (1870) and James Thomson (1882) have recognized the phenomenon of thermal convection. The instability of the Bénard model has been a subject of interest till today and an excellent review of this work up to 1957, with special reference to its possible fields of application has been given by Ostrach (1957). For mathematical details one may be referred to "Hydrodynamic and Hydromagnetic Stability" by S. Chandrasekhar (1981), Saltzmann (1962) and Spiegel (1971).

It was not so easy to find a mathematical theory which could give a correct interpretation of these experimental facts. As many as sixteen years were lapsed after Bénard's experiments when Rayleigh (1916a) could succeed in laying down the theoretical foundations of the subject for the first time with his pioneering paper dealing with cellular convection in a fluid heated uniformly from below. Rayleigh showed that there is a nondimensional number, that represents the physical factors entering the problem. It is now called the Rayleigh number and is given by the expression

\[ R = \frac{g \alpha \beta d^4}{\nu \kappa} \]

(1.16)

Here \( R \) denotes the Rayleigh number, \( g \) the acceleration due to gravity, \( \alpha \) the coefficient of volume expansion, \( \beta = |dT/dz| \) the uniform adverse temperature gradient which is maintained, \( d \) the depth of the fluid layer, \( \kappa \) the thermal diffusivity and \( \nu \) the kinematic
viscosity. Rayleigh further showed that instability must set in when \( R \) exceeds a certain critical value \( R_c \) and a stationary pattern of motions must prevail when \( R \) just exceeds \( R_c \).

Jeffrey's (1926, 1928) discussed the theoretical aspects of the Benard problem and modified the Rayleigh's criterion for a number of boundary conditions. Pellew and Southwell (1940) gave intermediate steps and also confirmed that oscillatory motions are always damped whereas non-oscillatory motions are always manifested. Chandrasekhar (1954) has studied the Benard problem as a characteristic value problem and has determined the critical Rayleigh number

\[
R_c = \frac{g \beta_c d^4}{\nu \kappa},
\]

which yields the critical adverse temperature gradient at which the thermal instability sets in. Here \( \beta_c \) is the critical adverse temperature gradient.

Chandrasekhar (1958) has reconsidered the Benard problem in the presence of vertical magnetic field and has obtained the critical Rayleigh number and the corresponding wave numbers of unstable modes at marginal stability in the three cases, namely, boundary surfaces both free, both rigid and lower boundary rigid and upper boundary free.

Eltayeb (1972) considered the linear stability of a rotating, electrically conducting viscous fluid layer, heated from below and cooled from above, lying in a uniform magnetic field and used the Boussinesq approximation. If \( \tilde{H} \) and \( \tilde{\Omega} \) denote, respectively, the magnetic field strength and the rotation, the models which Eltayeb considered were:

(i) \( \tilde{H} \) vertical, \( \tilde{\Omega} \) vertical,
(ii) \( \vec{H} \) horizontal, \( \vec{\Omega} \) vertical,

(iii) \( \vec{H} \) horizontal, \( \vec{\Omega} \) horizontal and \( \phi \) the angle between them,

(iv) \( \vec{H} \) vertical, \( \vec{\Omega} \) horizontal.

All the above cases were considered only for large Taylor number and large Hartmann numbers which are non-dimensional measures of rotation rate and magnetic field strength respectively.

Eltayeb (1975) also considered the above four cases under the variety of different boundary conditions and extended his work [Eltayeb (1972)] to include the possibility of overstability for non-zero values of the magnetic Prandtl number \( p_2 \).

**THERMOSOLUTAL INSTABILITY (OR DOUBLE-DIFFUSIVE CONVECTION)**

In classical thermal instability problems, it has been assumed that the driving density differences are produced by the spatial variation of single diffusing property i.e. heat. Recently, it has been shown that a new phenomenon occurs when the simultaneous presence of two or more components with different diffusivities is considered. Keeping ocean in mind, as heat and salt (or some dissolved substance) are important there, the problem has been proved. This problem has been termed as ‘thermosolutal convection’ (or thermohaline convection). Related effects have now been observed in other contexts, and the name double-diffusive convection has been used to cover this wide range of phenomena.

The problem of thermohaline convection in a layer of fluid heated from below and subjected to a stable salinity gradient has been studied by Veronis (1965). The physics is quite similar to Veronis (1965) thermohaline configuration, in the stellar case where helium acts like salt in raising the density but in diffusing more slowly than heat.
Nield (1967) has studied the problem of thermohaline convection in a horizontal layer of viscous fluid heated from below and salted from above.

This problem of the onset of thermal instability in the presence of a solute gradient is of great importance because of its application to oceanography, atmospheric physics and astrophysics. The heat and solute being two diffusing components, double-diffusive convection is the general term dealing with such phenomena.

It was shown by Turner (1973, 1974) that the form of resulting motions depends on whether the driving energy comes from the component having the higher or lower diffusivity. When one layer of fluid is placed above another (denser) layer having different diffusive properties, two basic types of convective instabilities arise, in the 'diffusive’ and ‘finger’ configurations. In both the cases, the double-diffusive fluxes can be much larger than the vertical transport in a single component fluid because of the coupling between diffusive and convective processes.

With the introduction of uniform temperature gradient $\beta (=|dT/dz|)$, uniform solute gradient $\beta'(=|dC/dz|)$, thermal diffusivity $\kappa$ and solute diffusivity $\kappa'$, four non-dimensional parameters are required to specify the system. In addition to the ordinary thermal Rayleigh number

$$R = \frac{g\alpha \beta d^4}{\nu \kappa}$$

and the Prandtl number

$$Pr = \frac{\nu}{\kappa},$$

where $\alpha$ is the coefficient of volume expansion, $d$ is the depth, $\nu$ is the kinematic viscosity. One can use the ratio
\[ \tau'_i = \frac{\kappa'}{\kappa} \quad \text{(with } \kappa > \kappa') \]

and the density ratio

\[ R_p = \frac{\alpha' \beta'}{\alpha \beta}, \]

where \( \alpha' \) is the corresponding coefficient of expansion relating salinity to density variation.

Linear calculations have also been made for a variety of boundary conditions by Nield (1967) and for an unbounded fluid by Walin (1964). A study of the onset of convection in a layer of sugar-solution, with a stabilizing concentration gradient, when the layer is heated from below, has been made by Shirtcliffe (1967). He found that the first stage of the development of convection layers similar to those described by Turner and Stommel (1964) is the appearance in a thin bottom layer of a cellular oscillatory motion which initially has a very definite period. When the solute gradient is stabilizing, Veronis (1965) and Sani (1965) have found that finite amplitude subcritical instability (convection at a thermal Rayleigh number less than that given by the linear theory) is possible. Sharma and Kumari (1992) have studied the effect of rotation and magnetic field on thermosolutal convection in porous medium.

RAYLEIGH-TAYLOR INSTABILITY

Rayleigh-Taylor instability arises from the character of equilibrium of an incompressible heavy fluid of variable density (i.e., of a heterogeneous fluid). The simplest, nevertheless important, example demonstrating the Rayleigh-Taylor instability is when we consider two fluids of different densities superposed one over the other (or accelerated towards each other), the instability of the plane interface between the two
fluids, if it occurs, is known as ‘Rayleigh-Taylor instability’. Rayleigh (1900) was the first to investigate the character of equilibrium of an inviscid, non-heat conducting as well as incompressible heavy fluid of variable density which is continuously stratified in the vertical direction. The cases of (i) two uniform fluids of different densities superposed one over the other and (ii) an exponentially varying density distribution were also treated by him. The main result in all such cases is that the configuration is stable or unstable with respect to infinitesimal small perturbations according as the higher density fluid underlies or overlies the lower density fluid.

Taylor (1950) carried out this theoretical investigation further and studied the instability of liquid surfaces when accelerated in a direction perpendicular to their planes. The experimental demonstration of the development of the Rayleigh-Taylor instability is described by Lewis (1950). Chandrasekhar (1955) studied the effect of variable viscosity upon the above problem and established that if the original density stratification is monotonically increasing upwards everywhere in the flow domain then there cannot exist any oscillatory modes. He also showed that a variational procedure of solving for the characteristic values is possible. The most detailed consideration of the effects arising from surface tension is due to Reid (1961). Surface tension is generally found to have a stabilizing effect on the Rayleigh-Taylor instability. It was also proved that the wave numbers which are stabilized by surface tension are independent of viscosity. The problem of waves in a heavy incompressible viscous electrically conducting fluid in the presence of a uniform magnetic field directed in a direction parallel to the force of gravity was considered by Hide (1955). Hide concluded that a transverse magnetic field stabilizes the unstable arrangement considerably. The effect of rotation has been
considered by Hide (1956a, 1956b). However, Hide’s treatment by including the effects of viscosity and of inclination between the direction of $\tilde{\Omega}$ and $\tilde{g}$ tends to obscure the essential elements of the problem. Perhaps for this reason, the inviscid case has been treated ‘de novo’ by Chandrasekhar (1981). In the case of two uniform fluids separated by a horizontal layer, it was established that rotation does not affect the instability or stability as such. However, in the case of exponentially varying density \( \rho = \rho_0 e^{\beta z} \), rotation stabilizes the potentially unstable arrangement for all wave numbers less than

\[
k_{\text{min}} = \left[ \frac{4\Omega^2}{g\beta d^2 \left( \frac{1}{4} \beta^2 d^2 + \pi^2 \right)} \right]^{1/2}.
\]

The effect of a horizontal magnetic field on the development of the Rayleigh-Taylor instability was considered by Kruskal and Schwarzschild (1954). They established the stabilizing nature of magnetic field for all perturbations except those at right angles to the magnetic field. Gupta (1963) has investigated the stability of a horizontal layer of a perfectly conducting fluid, with continuous density and viscosity stratifications. He has shown that contrary to the usual role of viscosity as a damping factor, it may sometimes act as a destabilizing agent. Jukes (1963) investigated the effect of finite resistivity on the Rayleigh-Taylor problem in the presence of a horizontal magnetic field and concluded that finite resistivity introduces unstable modes. The effects of Hall currents on the Rayleigh-Taylor instability in the presence of a horizontal magnetic field have been considered by Hosking (1965) and Agarwal (1969). Talwar and Kalra (1967) considered the combined Taylor and Helmholtz instability in hydromagnetics including Hall effect. Sharma and Srivastava (1968) studied the effect of horizontal and vertical magnetic fields on the Rayleigh-Taylor instability. The effect
of finite Larmor radius on the Rayleigh-Taylor instability of a rotating plasma in the presence of a horizontal magnetic field was studied by Ariel and Bhatia (1970).

Ariel (1970a, 1970b) considered the character of equilibrium of an inviscid, infinitely conducting fluid of variable density, separately, in the presence of horizontal and vertical magnetic fields with Hall currents. The Hall currents are found to destabilize the wave number range which is stable otherwise. Sharma (1972b) considered the effect of rotation and a general oblique magnetic field on the Rayleigh-Taylor instability. The effect of rotation and surface tension on the Rayleigh-Taylor instability in the presence of a variable horizontal magnetic field has been studied by Sharma (1973).

**KELVIN-HELMHOLTZ INSTABILITY**

The Kelvin-Helmholtz instability occurs when we consider the character of the equilibrium of a stratified heterogeneous fluid in which the different layers are in relative motion. The most important case is when two superposed fluids flow one over the other with a relative horizontal velocity, the instability of the plane interface between the two fluids when it occurs in this instance, is known as "**Kelvin-Helmholtz Instability**". Helmholtz (1868) and Kelvin (1910) were primarily interested in the stability of superposed fluids in a state of differential streaming. When two uniform fluids in relative horizontal motion are separated by a horizontal boundary, Kelvin (1910) established that instability occurs when we have $k > k_{\text{min}}$ where

$$k_{\text{min}} = \frac{g(\alpha_1 - \alpha_2)}{\alpha_1 \alpha_2 (U_1 - U_2)^2}, \quad \alpha_{1,2} = \frac{\rho_{1,2}}{\rho_1 + \rho_2},$$

(1.18)
is the acceleration due to gravity and \( \rho_1, U_1 \) and \( \rho_2, U_2 \) are the densities and velocities of the lower and the upper fluids, respectively. The striking aspect of this instability is that it occurs no matter how small \( |U_1 - U_2| \) is. Helmholtz (1868) stated the same result in the following form “Every perfect geometrically sharp edge by which a fluid flows must tear it asunder and establish a surface of separation however slowly the rest of the fluid may move”.

Kelvin pointed out the stabilizing effect of surface tension and concluded that surface tension \( T \), will suppress the Kelvin-Helmholtz instability if

\[
(U_1 - U_2)^2 < \frac{2}{\alpha_1 \alpha_2} \sqrt{\frac{T \alpha_1 \alpha_2}{\rho_1 + \rho_2}} \tag{1.19}
\]

From this result he derived the critical wind velocity to be 650 cm/sec for the first time. The significance of this particular velocity for the generation of waves by wind has been a subject of much discussion. Later on, Munk (1947) pointed out that the wind velocity of 650 cm/sec is associated with several phenomena observed on the surface of the seas. The work of Munk, in fact, is a recognition that Kelvin’s critical velocity, indeed, has a bearing on the phenomena which are observed in nature. The experimental observation of the Kelvin-Helmholtz instability has been given by Francis (1954).

The source of Kelvin-Helmholtz instability clearly lies in the energy stored in the kinetic energy of relative motion of different layers, the tendency towards mixing and instability will be greater, the larger the prevailing shear as measured by \( \frac{dU}{dz} \). The only counteracting force damping this tendency is derived from inertia; and as long as inertia
can maintain a sufficient pressure gradient and prevent the mixing, instability will not occur. From the quantitative expression of this condition it can be shown that a sufficient condition for stability is

$$\frac{1}{4} \rho (\delta U)^2 < -g(\delta \rho)(\delta z).$$

This condition is equivalent to

$$\left(\frac{dU}{dz}\right)^2 < -\frac{4g}{\rho} \frac{dp}{dz}.$$

Therefore a necessary condition for stability is that the Richardson number

$$J = -\frac{g}{\rho} \frac{dp}{dz} \frac{\frac{dU}{dz}^2}{\rho},$$

be everywhere greater than $$\frac{1}{4}$$. The above argument does not, of course, justify one’s inferring that its violation must lead to instability, because, it can very well happen that although stored energy is available for initiating instability, no mechanism exists for transforming the energy into possible hydrodynamic modes.

Some other fundamental works in this field of knowledge are those of Taylor (1931), Goldstein (1931), Dyson (1960), Case (1960), Miles (1961) and Howard (1961). Chandrasekhar (1981) described the effect of rotation on the development of Kelvin-Helmholtz instability and showed that it is least uninhibited for perturbations in the direction of streaming. Michael (1955) has studied the stability of a combined current and a vortex sheet in a perfectly conducting fluid, while the effect of a magnetic field transverse to the direction of streaming on the Kelvin-Helmholtz instability has been
considered by Northrop (1956). Talwar (1962) investigated the magnetohydrodynamical stability of two superposed fluids in relative tangential motion at the horizontal interface when the whole system is pertaking to a uniform rotation about vertical axis. Both the magnetic field and rotation have a stabilizing influence on the configuration, the former for disturbances of short wave length and the later for long wave length. Drazin (1974) considered the Kelvin-Helmoltz instability of a slowly varying flow. He considered the model of instability when air is blown over water in a wide long channel. Such problems are important in many applications as it is rare in practice that a flow is both steady and depends on one space coordinate only.

The non-linear development of the Kelvin-Helmholtz instability has been studied by Drazin (1970), Nayfeh and Saric (1971, 1972), Weissman (1979) and many others. Recently, Sharma and Rana (1999) have studied the instabiity of streaming Rivlin-Ericksen fluids in porous medium in hydromagnetics.

GRAVITATIONAL INSTABILITY

The problem of gravitational instability of a static infinite homogeneous medium was first discussed by Jeans (1902) in connection with the fragmentation of interstellar matter in star formation. He started from an infinite homogeneous medium at rest and considered the velocity of propagation of a small disturbance in density. If the gravitational effects of the disturbance are ignored, the problem reduces to the classical one of the propagation of sound, and as is well known, the velocity of sound \( c_s \) is independent of the wave number and is given by

\[
c_s^2 = \frac{\gamma p}{\rho},
\]  

(1.21)
where $\gamma$, $\rho$ and $\rho$ are, respectively, the ratio of specific heats, the pressure and the density.

However, if the change in gravitational potential consequent to the disturbance $\delta \rho$ in density is taken into account, Jeans showed that the velocity of wave propagation $V_j$ is given by

$$V_j = c_s \sqrt{1 - \frac{4\pi G \rho}{k^2 c_s^2}},$$

which depends on the wave number $k$, and is, indeed, imaginary for all wave numbers less than a certain value $k_j$. The instability which follows for $k < k_j$ where

$$k_j = \frac{1}{c_s} \sqrt{4\pi G \rho}$$

is the gravitational instability discovered by Jeans and this condition for instability is often referred to as Jeans’ criterion. The origin of the gravitational instability associated with the long waves lies in the circumstance that, while the generation of a pressure wave requires the expenditure of energy proportional to $c_s^2$, it simultaneously results in a release of gravitational potential energy of an amount proportional to $4\pi G \rho k^2$; and when Jeans’ criterion is satisfied, the system can spontaneously go to states of lower energy with the liberation of thermal energy and this, of course, means instability.

The effect of rotation on Jeans’ criterion was considered by Chandrasekhar (1955). Chandrasekhar and Fermi (1953) and Chandrasekhar (1953, 1960) studied the gravitational instability in the presence of magnetic field. The simultaneous effect of rotation and magnetic field has also been considered by Chandrasekhar (1954). In the above references, it was shown that Jeans’ criterion for the gravitational instability of an
infinite homogeneous medium is unaffected by the presence, separately or simultaneously, of a uniform rotation and a uniform magnetic field. Kalra and Talwar (1964) studied the gravitational instability of an infinite homogeneous medium in the presence of Hall current and found that the Jeans’ criterion remains unaltered. Sharma (1975b) has further extended the gravitational instability of an infinite homogeneous gas-particle medium to include the effects of suspended particles. It has been shown in the above studies that the Jeans’ criterion remains unaffected.

The effect of uniform rotation on the gravitational instability of a self-gravitating medium in the presence of suspended particles has been studied by Raghawachar (1979). In this way many authors have studied the effect of rotation on the gravitational instability with other parameters. Alfvén and Carlqvist (1978) have studied influence of magnetic field and dust on gravitational accretion and gravitational collapse of dense clouds and provided a possible mechanism of star formation. Goodman and Narayan (1988) have investigated the problem of self-gravity on the stability of accretion tori and discussed the Jeans’ instability. Nayyar (1961) has studied the magneto-gravitational instability of infinite medium with finite electrical and thermal conductivity and showed that finite electrical conductivity removes the influence of the magnetic field on critical wave length and the effect of thermal conductivity is to replace adiabatic sound velocity by isothermal one in Jeans’ expression. The effect of thermal conductivity on magnetogravitational instability has been studied by Kato and Kumar (1960) and Kumar (1961).

Sharma (1982) has studied the effect of Hall current and rotation on the gravitational instability of a self-gravitating medium in the presence of suspended
particles. Chhajlani and Vyas (1990) have studied the magnetogravitational instability of a thermally conducting rotating plasma through a porous medium but they have not taken electrical conductivity and Hall current. Recently, Sharma and Kumar (2000) have studied the magnetogravitational instability of a thermally conducting rotating Rivlin-Ericksen fluid with Hall current.

**VARIOUS EFFECTS ON INSTABILITY PROBLEMS**

**EFFECT OF ROTATION**

Rotation has a profound effect on the onset of thermal instability in a given system. It introduces a number of new elements in the fluid dynamics, for example, under certain circumstances, the role of viscosity is inverted. In the case of an inviscid fluid in which external forces are derivable from a potential function $V''$, the Taylor-Proudman theorem states that “all steady slow motions in a rotating inviscid fluid are necessarily two-dimensional”. It forbids variation of motion parallel to axis of rotation, though the transverse wave propagation is possible. Convection implies that motions which occur have necessarily a three-dimensional character, but Taylor-Proudman theorem does not allow it for an inviscid fluid so long as the non-linear terms in the equations of motion are neglected. The first of these conditions is certainly met in linear stability theory and second obtains for steady motions. Therefore, in contrast to non-rotating fluids, an inviscid fluid in rotation is expected to be thermally stable for all adverse temperature gradients. In fact thermal instability can arise only in the presence of viscosity. While Taylor-Proudman theorem forbids any variation of the velocity in the direction of $\Omega$, oscillatory motions are possible, $\Omega$ being the angular velocity of rotation. Chandrashekhar (1981) has obtained the following important conclusions of the effects of rotation on the onset of thermal instability:

(i) Rotation inhibits the onset of instability. The extent of inhibition depends on the Taylor number
and the thermal Prandtl number
\[ p_1 = \frac{u}{\kappa} \]  \hspace{1cm} (1.24)

(ii) The onset of instability is stationary convection so long as the thermal Prandtl number \( p_1 \) exceeds a certain critical value \( p_1^* \). The precise value of \( p_1^* \) depends on the nature of the bounding surfaces.

(iii) Onset of instability will be as overstable oscillations if \( p_1 < p_1^* \) and Taylor number exceeds a certain value \( T_A[p_1] \) depending on \( p_1 \).

The above results can be understood as follows:

"Thermal instability as stationary convection will set in at the minimum (adverse) temperature gradient which is necessary to maintain a balance between the rate of dissipation of energy by viscosity and the rate of liberation of the thermodynamically available energy by the buoyancy force acting on the fluid. Likewise, the onset of thermal instability will be as overstable oscillations if it is possible (at a lower adverse temperature gradient) to balance in a synchronous manner, the periodically varying amounts of kinetic energy with similarly varying amounts of dissipation and liberation of energy.

**EFFECT OF UNIFORM / VARIABLE MAGNETIC FIELD**

Consider a fluid to be electrically conducting and be under the influence of a magnetic field. The electrical conductivity of the fluid and the prevalence of magnetic fields contribute to effects of two kinds: First, by the motion of the electrically conducting fluid across the magnetic lines of force, electric currents are generated and the associated magnetic fields contribute to changes in the existing fields; and second, the fact that the fluid elements carrying currents transverse magnetic lines of force contributes to additional forces acting on the fluid elements. It is this two fold
interaction between the motion of the fields that is responsible for patterns of behaviour which are often striking and unexpected.

The interactions between the fluid motions and magnetic fields are contained in Maxwell’s equations. As a consequence of Maxwell’s equations, equations of hydrodynamics are modified suitably.

Magnetic field has the effect of inhibiting the onset of thermal instability by convection. When a magnetic field is impressed on an electrically conducting fluid there will also be the dissipation by viscosity. As is expected, the effect of inhibiting the onset of instability becomes more pronounced as the strength of the magnetic field increases and elongates the cells which appear at the marginal stability for certain ranges of the concerned parameters. In the case of an inviscid fluid of zero resistivity when external forces are derivable from the potential function $V''$, the analogue of the Taylor-Proudman theorem states that “all steady slow motions in the presence of a uniform magnetic field are necessarily two dimensional”. It forbids variation of motion in the direction of magnetic field. Convection implies that motion which occur have necessarily a three dimensional character. But such motions are forbidden for fluids of zero resistivity as slow steady two-dimensional motions are only allowed. A fluid of zero resistivity should therefore, be thermally stable for all adverse temperature gradients.

In a steady state, the energy released by the buoyancy acting on the fluid must balance the energy dissipated by both means, and this can be achieved only at higher adverse temperature gradients than are sufficient in the absence of Joule heating, showing thereby the stabilizing effect of the magnetic field. In the presence of a magnetic field, disturbances can be propagated as Alfvén waves.
Chandrasekhar’s (1981) analysis predicted another interesting point that the marginal state could either be stationary or oscillatory in character. It was found in agreement with the experimental results of Nakagawa (1955, 1957) and others. It was shown that when the magnetic Prandtl number \( p_2 = \frac{v}{\eta} \) is less than the Prandtl number \( p_1 = \frac{v}{\kappa} \), which is a requirement met by a large margin under most terrestrial conditions, overstability cannot occur and the principle of exchange of stabilities is valid. Here \( \kappa = \frac{k'}{\rho c_v} \) and \( \eta = \frac{1}{4\pi \mu_\sigma} \) denote the thermal diffusivity and electrical resistivity respectively. Moreover, for \( p_2 < p_1 \) there exists a value of Chandrasekhar number \( Q = \frac{\mu_e H^2 d^2}{4\pi \rho \eta} \),

say \( Q' \), depending on \( p_1 \) and \( p_2 \) such that for \( Q \leq Q' \), the onset of instability will be as stationary convection, while for \( Q > Q' \), it will be as overstability. Here \( \mu_e \), \( H \) and \( d \) denote respectively, the magnetic permeability, the magnetic field and the depth of the fluid layer.

**EFFECT OF HALL CURRENTS**

It was Hall in 1879 who pointed out the Hall effect, while working as a graduate student at Johns Hopkins University. Hall’s original experiments were limited to solid metallic conductors. A thin, flat strip of width \( b \) and thickness \( d \) was traversed by a current \( I \). Two fine wires were connected at equipotential points on opposite edges of the strip and in turn joined to the terminals of a sensitive galvanometer. When the
magnetic field $\vec{H}$ was introduced at right angles to the face of the strip, the galvanometer gave a steady deflection. The voltage indicated by the galvanometer is known as the Hall voltage and is directly proportional to both current and magnetic field.

It is a well known fact that if the mean free path is much larger than the electron Larmor radius, electrons will be able to gyrate freely round the magnetic lines of force several times before suffering collisions. Consequently, the electrons and ions appear to be tied with the lines of force in a way and this reduces their mobility transverse to the magnetic field. In the continuum picture this is manifested by the gyrotropic properties of the medium. Thus, when an electric field be applied at right angles to the magnetic field, the whole current will not flow along the electric field. This tendency of the electric current to flow across an electric field in the presence of a magnetic field is called 'Hall effect'. The Hall effect is more pronounced in the strong magnetic field or in the case of ionized gas (degree of ionization is small). It has dual effect on the stability like viscosity and magnetic field. In some circumstances, it does not affect the stability.

The Hall effect may be best analysed in terms of the current density,

$$J' = N'eq_1,$$  \hspace{1cm} (1.26)

where $N'$ is the number density per unit volume, $e$ is the charge carried by a particle and $q_1$, is the velocity of drift. The magnetic field exerts a force (Lorentz force) on these current carrying particles given by the cross-product $eq_1 \times \vec{H}$. When equilibrium is released, there is an electric field $\vec{E}$ which gives a counter balancing force $e\vec{E}'$ to that magnetic field. Thus

$$\vec{E}' = -\vec{q}_1 \times \vec{H} = -\vec{J}' \times \vec{H} / N'e = -R'(\vec{J}' \times \vec{H}),$$  \hspace{1cm} (1.27)
where $R' = 1/N'e$ is known as Hall coefficient and is reciprocal of the current carrying charge per unit volume.

Ware (1961) included this effect in his study of stability waves in magnetically confined plasma. Sato (1961) and Tani (1962) have considered the Hall effect in an incompressible viscous flow of an ionized gas with tensor conductivities in channels. They found that the inclusion of Hall currents gives rise to cross flow i.e. a flow at right angle to the primary flow in a channel in the presence of a transverse magnetic field. Taylor (1962) pointed out that the Hall current has a strong stabilizing effect on the low density plasma. Coppi (1964) pointed out that as long as the resistivity of the fluid is neglected, Hall term has no effect on the stability, which was further confirmed by Buti et al. (1965). Hosking (1965) pointed out that it has a destabilizing influence on the stability of the system. Tasso and Schram (1966) discussed in the context of microscopic theory the instability and stability effects and concluded that Hall effect has a destabilizing effect. Gupta (1967) has shown that the Hall currents have a destabilizing effect on the thermal instability of a horizontal layer of a conducting fluid in the presence of a uniform magnetic field. He showed that Hall current introduces a vertical component of vorticity and this may be the reason for destabilizing influence.

The effect of the magnetic field on the flow of plasma at a constant pressure gradient is weakened as the Hall parameter increases. This is due to a decrease in the conductivity in the direction of the induced electric field with an increase in the Hall parameter. The study of these effects is of great importance because of its application to the physics of atmosphere and astrophysics. In particular, Hall effects are likely to be present in the case of ionosphere and outer layers of the solar atmosphere.
EFFECT OF SUSPENDED PARTICLES

Motivation for the study of certain effects of particles immersed in the fluid such as: particle heat capacity, particle mass fraction and thermal force is due to the fact that the knowledge concerning fluid-particle mixtures is not commensurate with their industrial and scientific importance. Further motivation is provided by recalling decades-old contradiction between the theory for the onset of convection and experiment. The theory agrees with experimental determinations of the onset of convection in liquid layers confined between two horizontal rigid surfaces. Chandra (1938) observed a contradiction between the theory and his experiments for the onset of convection in fluids heated from below. He performed his experiment in an air layer and found that the instability depended on the depth of the layer.

A Bénard-type cellular convection with fluid descending at the cell centre was observed when the predicted gradients were imposed, for layers deeper than 10 mm. However, if the layer depth was less than 7 mm, convection which was different in character from that in deeper layers occurred at much lower gradients than predicted. Chandra called this motion ‘Columnar instability’. A complete survey of subsequent experimental studies, which confirm Chandra’s result can be found in a report by Jones (1962) on the effect of different aerosols on stability. These effects which he felt may be important are thermal forces, electrostatic charges, evaporations, condensation and buoyancy forces. Jones concluded that columnar instability is not an example of single phase natural convection and that it is most likely due to the unique properties of aerosol suspensions.
Theoretical discussions of columnar instability have been given by Sutton (1950) and Segel and Stuart (1962). Motivated by interest in fluid-particle mixtures, generally and columnar instability in particular, Scanlon and Segel (1973) investigated some of the continuum effects of particles on Bénard convection. They have found that the critical Rayleigh number was reduced solely because the heat capacity of the pure gas was supplemented by that of the particles. The effect of suspended particles was found to destabilize the layer i.e. to lower the critical temperature gradient. The Rayleigh-Taylor instability of two superposed conducting fluids in the presence of suspended particles has been studied by Sharma and Sharma (1979). Recently, Sharma and Sharma (1998) have studied the stability of stratified Rivlin-Ericksen fluid in porous medium in the presence of suspended particles and variable magnetic field whereas Sharma and Rana (1998) have studied the stability of stratified Walters’ (model B’) fluid in porous medium in the presence of suspended particles and variable magnetic field.

**GRAVITATIONAL FIELD**

The importance of convection in our environment can scarcely be overestimated; we need only look to our immediate environment and note that the circulation of the Earth’s atmosphere could not be explained without reference to convective motions induced by solar heating. However, the idealization of uniform gravity assumed in theoretical investigation, although valid for laboratory purposes, can scarcely be justified for large-scale convection phenomena occurring in the atmosphere, the ocean or the mantle of the Earth. It then becomes imperative to consider gravity as variable quantity varying with distance from a reference point or surface.

Although Gresho and Sani (1970) have considered a time-varying gravitational field in the convection problem of a spatially-varying gravitation field in the plane layer
case still remains open. They prompted Pradhan and Samal (1987) to consider the plane layer problem under a spatially varying gravity field, neglecting viscosity. They found among other things, that if gravity remained downward (upward) throughout the flow domain, neutral modes did not exist. A sufficient condition for stability of layer heated from above was that gravity remain directed downward over a sufficiently large part of the flow domain. However, a sufficient condition for instability of a layer heated from below was that, besides remaining directed downwards, the gravity profile must have concave curvature throughout the flow domain. Pradhan, Samal and Tripathy (1989) investigated the instability of a heated layer of a viscous fluid confined between two horizontal planes and subjected to a variable gravitational field varying spatially with height and found that (i) ‘principle of exchange of stabilities’ is valid when the layer is heated from above, (ii) the layer is stable when it is heated from above, irrespective of whether gravitational acceleration is increasing or decreasing with height.

NEWTONIAN AND NON-NEWTONIAN FLUIDS

In a simple rectilinear motion of fluid, if the two neighbouring fluid layers are moving with the velocities \( u \) and \( u + \delta u \) and are at a distance \( \delta y \) then the shearing stress

\[
\tau \propto \frac{\delta u}{\delta y} \quad \text{or} \quad \tau = \mu \frac{du}{dy},
\]

where \( \mu \) is called coefficient of viscosity of the fluid and \( \frac{du}{dy} \) is called strain rate of the fluid. In other words, the stress on a fluid is linearly proportional to the strain rate of the fluid. The fluid satisfying linear relation between stress and rate-of-strain is called ‘Newtonian Fluid’. The constitutive equation for Newtonian fluid in terms of shear stress tensor \( T_{ij} \) is

\[
T_{ij} = -\rho \delta_{ij} + 2\mu \epsilon_{ij} - \frac{2}{3} \mu \epsilon_{kk} \delta_{ij},
\]

(1.28)
where $e_{ij}$ is given by equation (1.10) and $\delta_{ij}$ is Kronecker delta.

Due to emergence of polymers in recent years, it has been found that fluid shows different types of response to an applied shearing stress. We can use the nature of response of fluid to applied stresses to classify the fluids. All those fluids, which show non-linear relationship between stress and rate-of-strain are called 'non-Newtonian fluids'. There is a vast variety of non-Newtonian fluids.

**VISCOELASTIC FLUIDS**

The problem of convection in a horizontal layer of fluid heated from below (Bénard convection) has been extended to the case of electrically conducting fluid by Thompson (1951) and Chandrasekhar (1952). It was shown that a uniform magnetic field inhibits the onset of thermal convection, the extra stabilization being due to dissipation by Joule heating. A comprehensive account of thermal instability of a layer of fluid heated from below both in hydrodynamics and hydromagnetics has been given in a treatise by Chandrasekhar (1981).

Idealized viscoelastic liquids are those whose behaviour at sufficiently small variable shear stresses can be characterized by three constants (i.e. a coefficient of viscosity, a relaxation time and a retardation time) and whose invariant differential equations of state for general motion are linear in the stresses and include terms of no higher degree than the second in the stresses and velocity gradients together. The original work on viscoelastic fluids appears to be that of Herbert (1963) on plane Couette flow heated from below.

Several important viscoelastic fluids considered in the present thesis, are described as under:
OLDROYD FLUIDS

Oldroyd (1950) proposed a non-linear theory of a class of isotropic incompressible elastico-viscous fluids with the constitutive relations

\[ T_{ij} = -p\delta_{ij} + \tau_{ij}, \]  

\[ \left(1 + \lambda \frac{d}{dt}\right)\tau_{ij} = 2\mu\left(1 + \lambda_0 \frac{d}{dt}\right)e_{ij}, \]  

where \( \frac{d}{dt} \) is the convective derivative, \( p \) is the isotropic pressure, \( \delta_{ij} \) is the Kronecker delta, \( \lambda \) is stress relaxation time and \( \lambda_0 \) is strain retardation time. \( T_{ij} \) and \( \tau_{ij} \) denote respectively, the stress tensor and viscous stress tensor. These relations were first proposed by Jeffreys for Earth and studied by Oldroyd (1958). These fluids have been shown to explain the rheological behaviour of some polymer solutions at small rates of shear by Toms and Strawbridge (1953). There are many possible sets of rheological equations of state, with the right invariance properties for general validity under the conditions of motion and stress [Oldroyd (1950)] which reduce to equations (1.29) and (1.30) when the velocity gradients and shear stresses are sufficiently small for their squares and products to be neglected. None of the possible general forms of the equations of state is linear in velocity gradients \( \frac{\partial u_i}{\partial x_j} \) and the stresses \( T_{ij} \). The simplest one linear in stresses alone, include, terms of the second degree in the stresses and velocity gradients together [Oldroyd (1951)].

MAXWELLIAN FLUIDS

Maxwell proposed a non-linear theory of a class of isotropic incompressible elastico-viscous fluids with constitutive relations
\[ \left( 1 + \lambda \frac{d}{dt} \right) \tau_{ij} = 2 \mu e_{ij} \]  \hspace{1cm} (1.31) 

together with equation (1.29).

Bhatia and Steiner (1972, 1973) have studied the problem of thermal instability of a Maxwellian viscoelastic fluid in the presence of rotation and magnetic field, separately, and have found that the rotation has a destabilizing effect in contrast to the stabilizing effect on Newtonian fluid whereas the magnetic field has stabilizing effect on viscoelastic fluid just as in the case of Newtonian fluid.

The Oldroyd fluid reduces to Maxwellian fluid if \( \lambda_0 = 0 \) and for stationary convection, the Maxwellian visco-elastic fluid behaves like a Newtonian fluid.

**RIVLIN-ERICKSEN FLUIDS**

Rivlin-Ericksen (1955) proposed a non-linear theory of a class of isotropic incompressible elastico-viscous fluids with the constitutive relations

\[ \tau_{ij} = 2(\mu + \mu' \frac{d}{dt}) e_{ij} \]  \hspace{1cm} (1.32) 

together with equation (1.29).

where \( \mu \) is viscosity and \( \mu' \) is viscoelasticity of the fluid.

Such elastico-viscous fluids have relevance and importance in chemical technology and industry. Sisodia and Gupta (1984) have studied the unsteady flow of a dusty elastico-viscous Rivlin-Ericksen fluid through channel of different cross-sections in the presence of time dependent pressure gradient. A problem of unsteady flow of a dusty conducting Rivlin-Ericksen fluid through a uniform pipe with sector of circle as cross-section was investigated by Yadav and Singh (1987). Garg et al. (1994) have
studied the drag on sphere oscillating in conducting dusty Rivlin-Ericksen elastico-viscous liquid. Sharma and Kumar (1997a) have also studied hydromagnetic stability of two Rivlin-Ericksen elastico-viscous superposed conducting fluids.

WALTERS’ (MODEL B') FLUIDS

Walters' (1962) proposed another important class of elastico-viscous fluids with the constitutive relations

\[ \tau_{ij} = 2(\mu - \mu' \frac{d}{dt})\delta_{ij}, \]  

(1.33)

together with equation (1.29).

Thomas and Walters' (1964) have studied the motion of an elastico-viscous liquid due to a sphere rotating about its diameter. Yadav and Ray (1991) have studied the unsteady flow of n-immiscible viscoelastic Walters' (model B') fluid through a porous medium between two parallel plates in the presence of a transverse magnetic field. Chakraborty and Sengupta (1994) have studied the MHD flow of unsteady viscoelastic (Walters' liquid B') conducting fluid between two porous concentric circular cylinders. Sharma and Kumar (1995a) have studied the steady flow and heat transfer of Walters' (model B') fluid through porous pipe of uniform circular cross-section with small suction. Sharma and Kumar (1997b) have also studied the stability of two superposed Walters' (model B') viscoelastic liquids.

FLOW THROUGH POROUS MEDIUM

The increasing demands for oil, water and flood produced in an environmentally sound manner have placed emphasis on the manner of their production, a major part of which is concerned with flow through porous medium. Flow through
porous media is also of interest in chemical engineering (adsorption, filtration, flow in packed columns), in petroleum engineering, in hydrology, in soil physics, in biophysics and in geophysics.

A porous medium is a solid with holes in it. The manner in which the holes are imbedded, how they are interconnected and the description of their location, shape and interconnection characterize the porous medium. Accordingly, we have different classes of porous media. Porous media are classified as unconsolidated or consolidated and as ordered or random. Examples of unconsolidated media are beach sand, glass beads, catalyst pellets, soil, gravel etc. Examples of consolidated media are most of the naturally occurring rocks such as sandstones, limestones and so forth. In addition concrete, cement, bricks, paper, cloth etc. are man-made consolidated media. Ordered porous media are regular packing of various types of materials such as spheres, column packing etc. Random media are media without any particular correlating factor.

The experimental work [Wallace, Pierce and Sawyer (1969] on the flow of mercury in a porous medium, in the presence of a transverse magnetic field, has revealed that the behaviour of fluids in petroleum reservoir rock depends to a large extent on the properties of the rock. Techniques of course study that yield new or additional information on the characteristics of the rock, would contribute to a better understanding of the petroleum reservoir performance. Wallace, Pierce and Sawyer (1969) have investigated only the flow pattern but not the stability of the flow. Lapwood (1948), using Rayleigh’s procedure has studied the stability of convective flow in hydrodynamics in a porous medium and has shown that the criterion for the convective flow is

\[ R_c = 4 \pi^2 \], where \( R_c \) is the critical Rayleigh number.
MACROSCOPIC PROPERTIES OF NON-IDEAL POROUS MEDIA

Two macroscopic properties of non-ideal porous media which may be used to describe fluid flow are described as follows:

POROSITY

It is the ratio of void to total volume. This is denoted by $\varepsilon$ (in this thesis), and is expressed either as a fraction of one or in percent. Mathematically it is given by

$$\varepsilon = \frac{1}{V} \int \alpha(\delta) \, dV',$$

where void distribution function

$$\alpha(\delta) = \begin{cases} 
1 & \text{if } \delta \text{ is the pores} \\
0 & \text{if } \delta \text{ is the matrix.}
\end{cases}$$

If the calculation of porosity is based upon the inter-connected pore space instead of on the total pore space, the resulting quantity is termed as effective porosity, which is given by

$$e = \frac{\varepsilon}{1 - \varepsilon}.$$ (1.34)

Porosity macroscopically characterizes the effective pore volume of the medium. The porosity is directly related to the size of pores related to the matrix. When porosity is substituted, we loose the details of the structure.

PERMEABILITY

The conductance of the medium is defined with direct reference of Darcy’s law as the seepage velocity of the percolating water per unit drop of hydraulic head. The permeability is related to pore-size distribution since the distribution of the sizes of entrances, exits and lengths of the pore walls make up the major resistance to flow. The
permeability is the single parameter that reflects the conductance of a given pore-
structure. The dimensions of the permeability are length squared. In oil industry it is
measured in ‘darcy’ with

\[ 1 \text{ darcy} = 9.87 \times 10^{-9} \text{ cm}^2. \]

The permeability and porosity are related since if the porosity is zero the
permeability is zero. Although there may be correlation between porosity and
permeability, permeability cannot be predicted from porosity alone since we need
additional parameters which obtain more information about pore-structure.

GOVERNING EQUATIONS OF HYDRDYNAMICS IN POROUS MEDIA

EQUATION OF MOTION IN POROUS MEDIUM

The theory of porous flow is largely based on a generalization of empirical
observations of Darcy [1856]. Darcy’s law, to steady slow flow in homogeneous isotropic
materials is given by

\[-\rho \ddot{\mathbf{g}} + \nabla p = -\frac{\mu}{k} \mathbf{q}.\]  \hspace{1cm} (1.35)

Comparing with the Navier-Stokes equations linearized for slow flow,

\[-\rho \ddot{\mathbf{g}} + \nabla p = \mu \nabla^2 \mathbf{V},\]  \hspace{1cm} (1.36)

it is seen that Darcy’s law replaces the term \( \mu \nabla^2 \mathbf{V} = \text{div} \) (viscous part of the stress
tensor) with the term \(-\frac{\mu}{k} \mathbf{q}\). This term is sometimes regarded as giving the force which
the porous solid exerts on the fluid. The term \( \mu \nabla^2 \mathbf{V} \) is unaffected by the addition to \( \mathbf{V} \)
of a rigid body velocity field. It depends only on relative velocities. Analogously, we
interpret \( \mathbf{q} \) to be a velocity measured relative to axes fixed in the porous solid. In the
same way equation (1.29) may be said to represent equation (1.30) for slow steady flow, it is useful to assume that,

$$\rho \frac{d\tilde{V}}{dt} = -\nabla p + \rho \mathbf{g} - \frac{\mu}{k_1} \tilde{q}. \quad (1.37)$$

The pre-average velocity $\tilde{V}$ and the seepage velocity $\tilde{q}$ are assumed to be defined at each point and are related by the equation

$$\tilde{q} = \varepsilon \tilde{V}. \quad (1.38)$$

**EQUATION OF CONTINUITY**

The equation of continuity in porous medium is given by (for compressible fluid)

$$\varepsilon \frac{\partial \rho}{\partial t} + (\nabla \cdot \rho \tilde{q}) = 0, \quad (1.39)$$

and for incompressible fluid it is given by

$$\nabla \cdot \tilde{q} = 0.$$

**EQUATION OF HEAT CONDUCTION OR ENERGY EQUATION**

The energy equation in porous medium is obtained from an enthalpy balance over the fluid and over the solid (Bear, 1972). The enthalpy balance over the fluid is given by

$$\int_{\mathcal{V}_f} \left( \rho c_p \right)_f \left( \frac{\partial T}{\partial t} + \tilde{V} \cdot \nabla T \right) d\mathcal{V}_f = \int_{\mathcal{V}_f} \nabla \cdot (k_{\tau} \nabla T) d\mathcal{V}_f \quad (1.40)$$

and energy balance over the solid is given by

$$\int_{\mathcal{V}_s} \left( \rho c_p^{\prime} \right)_s \frac{\partial T}{\partial t} d\mathcal{V}_s = \int_{\mathcal{V}_s} \nabla \cdot (k_{\tau} \nabla T) d\mathcal{V}_s, \quad (1.41)$$
where $c_p$, $T$ and $k_T$ denote the specific heat at constant pressure, temperature and thermal conductivity. The subscripts ‘f’ or ‘s’ stand for fluid and solid respectively. $V_f$ and $V_s$ are volume occupied by fluid and solid.

The integration is transferred to common volume

$$V' = V_f \cup V_s$$

by the Jacobian relation

$$\frac{dV_f}{dV'} = e, \quad \frac{dV_s}{dV'} = 1 - e. \quad (1.42)$$

Then upon adding equations (1.40) and (1.41), we find

$$\int \left[ (\rho c_p)_f \left( \frac{\partial T}{\partial t} + \bar{V} \cdot \nabla T \right) + \int (\rho c_p)_s \frac{\partial T}{\partial t} (1 - e) dV' \right] dV'$$

$$= \int \nabla \cdot (k_{T_f} \nabla T) dV' + \int \nabla \cdot (k_{T_s} \nabla T) (1 - e) dV'$$

or

$$\int \left[ (\rho c_p)_f e + (1 - e)(\rho c_p)_s \frac{\partial T}{\partial t} + (\rho c_p)_s q \cdot \nabla T - \nabla \cdot (k_{T_s} \nabla T) \right] dV' = 0. \quad (1.45)$$

where $k_{T_s} = e k_{T_f} + (1 - e) k_{T_s}$ is called effective thermal conductivity. Since $V'$ is an arbitrary volume

$$\left[ (\rho c_p)_f e + (1 - e)(\rho c_p)_s \frac{\partial T}{\partial t} + (\rho c_p)_s q \cdot \nabla T - \nabla \cdot (k_{T_s} \nabla T) \right]. \quad (1.46)$$

Equation (1.46) is the required equation of heat conduction.

**CONTRIBUTIONS OF THE PRESENT THESIS**

The work embodied in the present thesis is divided into seven chapters. The problem and summary of the work done is presented below chapterwise.
CHAPTER-I

It is an introductory chapter which reviews the essential, existing literature on the various stability problems, relevant to the thesis e.g., hydrodynamics, hydromagnetics, stability of a system, methods and scopes determining stability etc. The thermal instability, thermosolutal instability, Rayleigh-Taylor instability self-gravitating instability and Kelvin-Helmholtz instability problems have been described and the effects of various factors like rotation, variable gravity field, uniform/variable magnetic field, suspended particles, gravitational field, Hall currents, porous medium and viscoelasticity have been discussed.

CHAPTER-II

This chapter deals with the instability of elastico-viscous Walters' (model B') fluid heated from below. The chapter is divided into two sections.

In Section A of this chapter, a study has been made of the thermal instability of Walters’ (model B') elastico-viscous fluid in the presence of variable gravity field and rotation in the porous medium. The principle of exchange of stabilities is valid in the regimes $T_A P' < e^2$ and $P_r > \frac{\nu'}{d^2}$; where $T_A$ is the Taylor number, $P_r$ is the dimensionless medium permeability, $\nu'$ is the kinematic viscoelasticity, $e$ is the medium porosity and $d$ is the depth of the fluid layer. For the stationary convection, the Walters' (model B') elastico-viscous fluid behaves like a Newtonian fluid. It is found that rotation has stabilizing effect as gravity increases upwards and destabilizing effect as gravity decreases upwards; medium permeability has stabilizing/destabilizing effects depending on the rotation parameter, gravity is considered to be increasing upward from its value $g$ (i.e., $\lambda > 0$). The sufficient conditions for the non-existence of overstability
are also obtained. The effects of rotation and medium permeability have also been shown graphically.

In Section B, the thermosolutal instability of Walters’ (model B’) viscoelastic rotating fluid permeated with suspended particles and variable gravity field in porous medium is considered. The stable solute gradient, rotation, gravity field and viscoelasticity introduce oscillatory modes which were non-existent in their absence. For the stationary convection, the stable solute gradient has a stabilizing effect, whereas rotation has stabilizing/destabilizing effects when gravity increases/decreases upwards. The suspended particles are found to have destabilizing effect on the system, whereas the medium permeability has destabilizing/stabilizing effect on the system under certain conditions as the gravity increases upwards. The above results have also been shown graphically.

CHAPTER-III

In this chapter, the thermosolutal instability of Rivlin-Ericksen visco-elastic fluid in the presence of rotation in porous medium has been considered to include the effects of magnetic field and variable gravity field. For the stationary convection, the stable solute gradient has a stabilizing effect, whereas rotation has stabilizing/destabilizing effects when gravity increases/decreases upwards. The magnetic field has stabilizing effect in the absence of rotation, whereas in the presence of rotation, magnetic field has stabilizing effect when

\[ T_{\Delta_{\text{m}}} < \left(1 + x + \frac{PQ}{\varepsilon}\right)^{2} \left[(1 + x)P^{2}\right]^{-1}. \]  

The medium permeability has destabilizing effect in the absence of rotation, whereas in the presence of rotation medium permeability has stabilizing effect when
$T_{\lambda_i} \leq \epsilon^2 \left( 1 + x + \frac{PQ_i}{\epsilon} \right)^2 \left[ (1 + x)P^2 \right]^{-1}$ increases upwards (i.e., $\lambda > 0$). The case of overstability is also considered wherein the sufficient conditions for the existence of overstability are derived. The effect of stable solute gradient, rotation, magnetic field and medium permeability have also been shown graphically.

CHAPTER-IV

This chapter is devoted to the instability which derives from the character of equilibrium of a fluid of variable density known as Rayleigh-Taylor instability.

A study has been made of the stability of Rivlin-Ericksen superposed fluids in porous medium. In this chapter, a study has also been made of the stability of Rivlin-Ericksen superposed fluids in the presence of uniform rotation and horizontal magnetic fields, separately. The system is found to be stable/unstable for bottom-heavy/top-heavy configuration density wise as in Newtonian viscous fluid. For exponentially varying density, viscosity, viscoelasticity, medium porosity and medium permeability, the system is found to be stable for all wave numbers for stable stratifications and unstable for unstable stratifications. The behaviour of growth rates with respect to fluid kinematic viscosity, viscoelasticity and medium permeability, have been examined analytically and the same results have been shown graphically. The system can be completely stabilized by uniform magnetic field for a certain wave number range, which was unstable in the absence of magnetic field.

CHAPTER-V

In this chapter, the stability of stratified Walters’ (model B’) elastico-viscous fluid in the presence of suspended particles and variable horizontal magnetic field in
porous medium, separately is studied. The system is stable for stable configuration and unstable for unstable configuration. The system can be completely stabilized by large enough magnetic field, which was unstable in the absence of magnetic field; provided the initial configuration is top-heavy density wise. The suspended particles number density has damping as well as enhancing effects on the growth rates depending on certain conditions.

CHAPTER-VI

This chapter is devoted to the instability of streaming Rivlin-Ericksen fluid. The chapter is divided into two sections.

In section A, a study has been made of the instability of streaming Rivlin-Ericksen fluids in porous medium. The configuration is taken to be bottom-heavy. In the absence of surface tension, perturbations transverse to the direction of streaming are found to be unaffected by the presence of streaming if perturbations in the direction of streaming are ignored, whereas for perturbations in all other directions, there exists instability for a certain wave number range. The surface tension is able to suppress this Kelvin-Helmholtz instability for small wave length perturbations and the medium porosity and the viscoelasticity reduce the stability range given in terms of a difference in streaming velocities. For the top-heavy configurations, the surface tension stabilizes a certain wave number range.

In section B, the instability of counter-streaming Rivlin-Ericksen fluids in porous medium is considered to include a horizontal magnetic field. As in Section A, the configuration is taken to be bottom-heavy. In the absence of surface tension, perturbations transverse to the direction of streaming are ignored, whereas for perturbations in all other directions, there exists instability for a certain wave number
range. The magnetic field and surface tension are able to suppress this Kelvin-Helmholtz instability for small wave length perturbations and the medium porosity and the viscoelasticity reduces the stability range given in terms of a difference between the streaming velocities and the Alfvén velocities. The instability of the system is postponed by the presence of magnetic field.

CHAPTER VII

In this chapter, magnetogravitational instability of a thermally conducting, rotating Rivlin-Ericksen fluid through a porous medium with finite conductivity in the presence of Hall current is considered. The wave propagation is considered for both parallel and perpendicular axes of rotation. It is observed that condition of instability is determined by the Jeans' criterion in its modified form. For transverse wave propagation for rotational axis parallel to the magnetic field, rotation produces stabilizing effect. For transverse wave propagation (rotation perpendicular to the magnetic field), electrical conductivity reduces the magnetic field effect in Jeans' condition. For longitudinal wave propagation in both the direction of rotation, the adiabatic velocity of sound is replaced by isothermal one. The Jeans' condition is independent of rotation.